GEOMETRICAL REPRESENTATION THEOREMS FOR CYLINDRIC-TYPE ALGEBRAS

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ABSTRACT. In this paper, we give new proofs of the celebrated Andréka-Resek-Thompson representability results of certain axiomtaized cylindric-like algebras. Such representability results provide completeness theorems for variants of first order logic, that can also be viewed as multi-modal logics. The proofs herein are combinatorial and we also use some techniques from game theory.

1. INTRODUCTION

Stone's representation theorem for Boolean algebras can be formulated in two, essentially equivalent ways. Every Boolean algebra is isomorphic to a field of sets, or the class of Boolean set algebras can be axiomatized by a finite set of equations. As is well known, Boolean algebras constitute the algebraic counterpart of propositional logic. Stone's representation theorem, on the other hand, is the algebraic equivalent of the completeness theorem for propositional logic.

Throughout, fix a finite ordinal $n \geq 2$. Cylindric algebras of dimension n were introduced by A. Tarski as the algebraic counterpart of first order logic restricted to n-many variables. Unfortunately, not every abstract cylindric algebra is representable as a field of sets, where the extra non-Boolean operations of cylindrifiers and diagonal elements are faithfully represented by projections and equalities. This is basically a reflection of the essential incompleteness of the finite variable fragments of first order logics.

In the present paper, we consider some variants of these cylindric algebras that were shown to have representation theorems. Thus, the corresponding logics are variants of first order logic that have completeness theorems. Such logics can be also viewed as multi dimensional modal logics, c.f. [6] and [7]. The following axioms are essentially due to D. Resek and R. Thompson.

Definition 1.1. The class RC_n is defined to be the class of all algebras of the form $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, d_{ij}\rangle_{i,j\in n}$ that satisfy the axioms (Ax0)

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through (Ax7) below. Note that for each $i, j \in n$ with $i \neq j$, we have $s^i_i x \stackrel{\text{def}}{=} x$ and $s^i_j x \stackrel{\text{def}}{=} c_i(x \cdot d_{ij})$.

(Ax0) $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra.
(Ax1) $c_i 0 = 0$, for each $i \in n$.
(Ax2) $x \leq c_i x$, for each $i \in n$.
(Ax3) $c_i(x \cdot c_i y) = c_i x \cdot c_i y$, for each $i \in n$.
(Ax4) $d_{ii} = 1$, for each $i \in n$.
(Ax5) $d_{ik} \cdot d_{kj} \leq d_{ij} = d_{ji} = c_k d_{ji}$, for each $i, j, k \in n$ such that $k \neq i, j$.
(Ax6) $c_i(x \cdot d_{ij}) \cdot d_{ij} \leq x$, for each $i, j \in n$ such that $i \neq j$.
(Ax7) $s^{i_m}_{j_m} c_{k_m} \cdots s^{i_1}_{j_1} c_{k_1} x \cdot \prod\{d_{l\tau(l)} : l \in K\} \leq c_i x$,
for each finite $m \geq 1$ and $i_1, \ldots, i_m \in n$,$j_1, \ldots, j_m \in n$, $k_1, \ldots, k_m \in n, i \in n$ such that $k_{t+1} \notin ([i_t/j_t] \circ \cdots \circ [i_1/j_1])^* K$, $t < m$, where $\tau = [i_m/j_m] \circ \cdots \circ [i_1/j_1]$ and $K = \{i_1, \ldots, i_m, k_1, \ldots, k_m\} \setminus \{i\}$.

Remark 1.2. Let ψ be any function. We denote its range by $Rng(\psi)$. Moreover, for any X subset of the domain of ψ, by $\psi^* X$ we mean the set $\{\psi(x) : x \in X\}$. Let $i, j \in n$. Then, $[i/j]$ denotes the transformation on n that fixes everything except that it sends i to j, while $[i, j]$ is the transformation that fixes everything except that $[i, j](i) = j$ and $[i, j](j) = i$.

The following axioms are due to R. Thompson and H. Andréka.

Definition 1.3. We define $\mathrm{DC}_n = \{\mathfrak{A} \in \mathrm{RC}_n : \mathfrak{A} \models \text{(Ax8), (Ax9),(Ax10)}\}$, $\mathrm{SC}_2 = \{\mathfrak{A} \in \mathrm{DC}_2 : \mathfrak{A} \models \text{(Ax11)}\}$ and $\mathrm{SC}_n = \{\mathfrak{A} \in \mathrm{DC}_n : \mathfrak{A} \models \text{(Ax12)}\}$ if $n \geq 3$, where:

(Ax8) $c_i c_j x \geq c_j c_i x \cdot d_{jk}$, for each $i, j, k \in n$ and $k \neq i, j$.
(Ax9) $d_{ij} = c_k(d_{ik} \cdot d_{kj})$, for each $i, j, k \in n$ such that $k \neq i, j$.
(Ax10) $s^k_i s^i_j s^j_m s^m_k c_k x = s^k_m s^m_i s^i_j s^j_k c_k x$, for each $i, j, k, m \in n$, $k \neq i, j, m$ and $m \neq i, j$.
(Ax11) $x \cdot -d_{01} \leq c_0 c_1(-d_{01} \cdot s^0_1 c_1 x \cdot s^1_0 c_0 x)$.
(Ax12) $x \leq c_i c_j (s^i_j c_j x \cdot s^j_i c_i x \cdot \prod_{k \in n, k \neq i, j} s^k_i s^i_j s^j_k c_k x)$, for each $i, j \in n$.

In [4], D. Resk and R. Thompson proved a representation theorem for the class RC_n. Despite the novelty of their proof, it was quite long and requires a deep knowledge of the literature of algebraic logic. A simpler proof was then provided by H. Andréka (this proof can be found in [8, Theorem 9.4]). Andréka's method could also suggest an elegant proof for the representation of the algebras of DC_n [5].

The representability of the DC_n-algebras originally is due to R. Thompson, but his original proof was never published. In [9], H. Andréka proved the representability of the algebras in SC_n by reducing the problem to the case of DC_n and then applying [5]. Andréka's representing structures were build using the step-by-step construction, which consists of treating defects one by one and then taking a limit where the contradictions disappear.

In the present paper, we provide new proofs for the representation theorems of the classes defined above. We use games (and networks) as introduced to algebraic logic by R. Hirsch and I. Hodkinson, c.f. [10] and [11]. Our proofs are relatively shorter than all the known proofs. We give direct constructions for all classes, even for SC_n, unlike its original proof in [9].

The translation from step-by-step techniques to games is not a purely mechanical process. This transfer can well involve some ingenuity, in obtaining games that are transparent, intuitive and easy to grasp. The real advantage of the game technique is that games do not only build representations, when we know that such representations exist, but they also tell us when such representations exist, if we do not know a priori that they do.

Now, we give the formal definition of the representing concrete algebras. We start with the following basic notions. For every $i \in n$ and every two sequences f, g of length n, we write $g \equiv_i f$ if and only if $g = f(i/u)$, for some u, where, $f(i/u)$ is the sequence which is like f except that it's value at i equals u. Let V be an arbitrary set of sequences of length n. For each $i, j \in n$ and each $X \subseteq V$, we define

$$C_i^{[V]}X = \{f \in V : (\exists g \in X) f \equiv_i g\}.$$
$$D_{ij}^{[V]} = \{f \in V : f(i) = f(j)\}.$$

When no confusion is likely, we omit the superscript $[V]$.

Definition 1.4. The class of all relativized cylindric set algebras of dimension n, denoted by RCs_n, is defined to be the class that consists of all subalgebras of the (full) algebras of the form,

$$\mathfrak{P}(V) \stackrel{\text{def}}{=} \langle \mathcal{P}(V), \cup, \cap, \setminus, \emptyset, V, C_i^{[V]}, D_{ij}^{[V]} \rangle_{i,j\in\alpha},$$

where V is a non-empty set of sequences of length n and $\mathcal{P}(V)$ is the family of all subsets of V. For every $\mathfrak{A} \subseteq \mathfrak{P}(V)$, the set V is called the unit of $\mathfrak{A}$, while the smallest set U that satisfies $V \subseteq {}^nU$ is called the base of $\mathfrak{A}$.

Definition 1.5. The class DCs_n of diagonalizable cylindric set algebras, of dimension n, consists of all algebras $\mathfrak{A} \in \mathrm{RCs}_n$ whose units V are diagonalizable sets, i.e. for every $f \in V$ and every $i, j \in n$ we have $f \circ [i/j] \in V$.

Definition 1.6. The class SCs_n of locally squares cylindric set algebras, of dimension n, consists of all algebras $\mathfrak{A} \in \mathrm{DCs}_n$ whose units V are permutable sets, i.e. for every $f \in V$ and every $i, j \in n$ we have $f \circ [i, j] \in V$.

In contrast with the literature, other notations for the classes RCs_n, DCs_n and SCs_n are Crs_n, D_n and G_n, respectively. Let $\Lambda_n \stackrel{\text{def}}{=} {}^n n$ be the set of all transformations on n. Let $\Omega_n \stackrel{\text{def}}{=} \{\tau \in \Lambda_n : |Rng(\tau)| < n\}$, the set of all transformations on n that are not permutations. Let V be a set of sequences of length n. The following characterization is well known and can be verified easily using some simple facts of transformations.

- V is diagonalizable if and only if $f \circ \tau \in V$, for every $f \in V$ and every $\tau \in \Omega_n$.
- V is diagonalizable and permutable if and only if $f \circ \tau \in V$, for every $f \in V$ and every $\tau \in \Lambda_n$.

For any class K of algebras, **I**K is the class that consists of all isomorphic copies of the members of K. As we mentioned before, we aim to reprove the following theorem.

Main Theorem 1. *Let $n \geq 2$ be a finite ordinal and let $\mathrm{K} \in \{\mathrm{RC}, \mathrm{DC}, \mathrm{SC}\}$. Then,* $\mathrm{K}_n = \mathbf{I}\mathrm{Ks}_n$.

2. Preliminary lemmas

Recall the basic concepts of Boolean algebras with operators (BAO) from the literature, see e.g. [1]. For any $\mathfrak{B} \in \mathrm{RC}_n$, let $At(\mathfrak{B})$ be the set of all atoms in $\mathfrak{B}$.

Note that RC_n (similarly DC_n and SC_n) is defined by positive equations, the negation does not appear in a non-Boolean axiom. Also note that the cylindrifications are defined to be normal and additive operators. Thus, by [1, Theorem 2.18], every algebra in RC_n can be embedded into a complete and atomic algebra in RC_n. Therefore, it is enough to prove that every complete and atomic algebra is representable. We need to prove some auxiliary lemmas that may be interesting in their own.

Lemma 2.1. *Let $\mathfrak{A} \in \mathrm{RC}_n$, let $i \in n$ and let $x, y \in At(\mathfrak{A})$. Then,*

$$x \leq c_i y \iff c_i x = c_i y.$$

Proof. We prove the non-trivial direction only. The assumption $x \leq c_i y$ and [2, Theorem 1.2.9] imply that $c_i x \leq c_i y$. Moreover,

$$\begin{aligned}
x \leq c_i y \quad &\Longrightarrow \quad x \cdot c_i y \neq 0 \\
&\Longrightarrow \quad c_i(x \cdot c_i y) \neq 0 \qquad \text{by [2, Theorem 1.2.1]} \\
&\Longrightarrow \quad c_i x \cdot c_i y \neq 0 \qquad \text{by axiom (Ax3)} \\
&\Longrightarrow \quad c_i(y \cdot c_i x) \neq 0 \qquad \text{by axiom (Ax3)} \\
&\Longrightarrow \quad y \cdot c_i x \neq 0 \qquad \text{by [2, Theorem 1.2.1]} \\
&\Longrightarrow \quad y \leq c_i x \qquad \text{by the fact that } b \in At(\mathfrak{A}) \\
&\Longrightarrow \quad c_i y \leq c_i x \qquad \text{by [2, Theorem 1.2.9].}
\end{aligned}$$

Note that all the cited theorems uses only the axioms (Ax0) - (Ax3). □

To represent an atomic algebra $\mathfrak{A}$, we roughly represent each atom $a \in \mathfrak{A}$ by a sequence f. Then, we show that $\mathfrak{A}$ can be embedded into the full algebra whose unit consists of all sequences representing atoms. The real challenge now is to arrange that the unit has the desired properties, by adding the substitutions $f \circ [i/j]$ or the transpositions $f \circ [i, j]$ whenever it is necessary. Such new sequences need to be associated to some atoms to keep the claim that each atom is represented by some sequences (maybe more than one), and there are no irrelevant sequences. For example, the following Lemma defines the substitutions of an atom, if any exists.

Notation: Let $i, j \in n$ be such that $i \neq j$. Define $t^i_i x \stackrel{\text{def}}{=} x$ and $t^i_j x \stackrel{\text{def}}{=} c_i x \cdot d_{ij}$

Lemma 2.2. *Suppose that $\mathfrak{A} \in \mathrm{RC}_n$ is an atomic algebra. Let $x \in At(\mathfrak{A})$ and let $i, j \in n$. Then, $t^i_j x \neq 0 \implies t^i_j x \in At(\mathfrak{A})$. Moreover, if $x \leq d_{ij}$ then we have $t^i_j x = x$.*

Proof. The statement is obvious if $i = j$, so we may assume that $i \neq j$. Suppose that $t^i_j x \neq 0$. Since $\mathfrak{A}$ is atomic, one can find an atom $y \in At(\mathfrak{A})$ such that $y \leq t^i_j x = c_i x \cdot d_{ij}$. Thus, by Lemma 2.1, we have $c_i x = c_i y$. Note also that $y \leq d_{ij}$, so axiom (Ax6) implies

$$t^i_j x = c_i x \cdot d_{ij} = c_i y \cdot d_{ij} = c_i(y \cdot d_{ij}) \cdot d_{ij} = y.$$

Therefore, $t^i_j x$ is an atom in $\mathfrak{A}$ as desired. The remaining part is obvious. □

Definition 2.3. Let $\mathfrak{A} \in \mathrm{DC}_n$ and let $\tau = [i_1/j_1] \circ \cdots \circ [i_m/j_m] \in \Omega_n$. For each $x \in \mathfrak{A}$, we define $\tau^{\mathfrak{A}} x \stackrel{\text{def}}{=} t^{i_m}_{j_m} \cdots t^{i_1}_{j_1} x$. This is well defined by the following Lemma.

Lemma 2.4. *Let $\mathfrak{A} \in \mathrm{DC}_n$ and let $\tau, \sigma \in \Omega_n$. Then*

$$(\forall i \in n)\ \tau(i) = \sigma(i) \implies (\forall x \in \mathfrak{A})\ \tau^{\mathfrak{A}} x = \sigma^{\mathfrak{A}} x.$$

Proof. See [5] proof of Lemma 1 therein. Axiom (Ax10) is used here. We note that Andreka's proof of this lemma is long, but using fairly obvious results on semigroups a much shorter proof can be given, c.f. . □

Similarly, we need to define the transpositions of an atom, if $\mathfrak{A} \in \mathrm{SC}_n$. This can be done using the axioms (Ax11) and (Ax12). By Lemma 2.6, there is at least one choice (maybe many) to define these transpositions

Definition 2.5. Assume that $\mathfrak{A} \in \mathrm{SC}_n$. Let $i, j \in n$ be such that $i \neq j$ and let $x \in \mathfrak{A}$. We define $p_{ii}x \stackrel{\text{def}}{=} x$ and

$$p_{ij}x \stackrel{\text{def}}{=} s^i_j c_j x \cdot s^j_i c_i x \cdot \prod_{k \in n, k \neq i,j} s^k_i s^i_j s^j_k c_k x.$$

If the product is empty (i.e. if $n = 2$) then it is defined to be 1.

Lemma 2.6. *Let $\mathfrak{A} \in \mathrm{SC}_n$, let $i, j \in n$ and let $x \in At(\mathfrak{A})$. Then, $p_{ij}x \neq 0$.*

Proof. If $i = j$ then the statement is trivial. So we assume that $i \neq j$. If $n \geq 3$ then we are done by axiom (Ax12). Suppose that $n = 2$. If $x \leq d_{01}$ then by axiom (Ax6) it follows that $p_{01}x = c_0x \cdot c_1x$ and we are done. If $x \leq -d_{01}$ then the desired follows by axiom (Ax11). □

We will use the following Lemma in the proceeding sections.

Lemma 2.7. *Suppose that $\mathfrak{A} \in \mathrm{SC}_n$ is complete and atomic algebra. Let $x, y, z \in At(\mathfrak{A})$ be some atoms and let $i, j, k \in n$ be such that $k \neq i, j$.*

(1) $y \leq s^k_i s^i_j s^j_k c_k x$ and $z = t^i_k t^j_i t^k_j x \implies c_k y = c_k z$.
(2) $y \leq s^i_j c_j x$ and $z = t^j_i x \implies c_i z = c_i y$.

Proof. Suppose that $\mathfrak{A} \in \mathrm{SC}_n$ is complete and atomic algebra. Hence, by [2, Theorem 1.2.6 and Theorem 1.5.3], the operations c_i's and s^i_j's are completely additive.

(1) Suppose that $y \leq s^k_i s^i_j s^j_k c_k x$. Thus, by the assumptions on $\mathfrak{A}$, there are some atoms $a_0, a_1, a_2 \in At(\mathfrak{A})$ such that

$$a_0 \leq c_k x,\ a_1 \leq c_j a_0,\ a_2 \leq c_i a_1,\ y \leq c_k a_2,$$

$$a_0 \leq d_{jk},\ a_1 \leq d_{ij}\ a_2 \leq d_{ki}.$$

Inductively, Lemma 2.2 implies that

$$a_0 = t^k_j x,\ a_1 = t^j_i a_0 = t^j_i t^k_j x,\ a_2 = t^i_k a_1 = t^i_k t^j_i t^k_j x.$$

Therefore, $z = a_2$ and hence $y \leq c_k z$. The desired follows by Lemma 2.1.

(2) Again, by assumptions, there is an atom a such that $y \leq c_i a$, $a \leq d_{ij}$ and $a \leq c_j x$. Hence, by Lemma 2.2, we have $a = t_i^j x = z$. Therefore, $y \leq c_i a \leq c_i z$ and we are done by Lemma 2.1. □

3. Networks and mosaics

Throughout, let $\mathrm{K} \in \{\mathrm{RC}, \mathrm{DC}, \mathrm{SC}\}$ and let $\mathfrak{A} \in \mathrm{K}_n$ be arbitrary but fixed. Suppose that $\mathfrak{A}$ is complete and atomic. The networks and mosaics we define here are approximations of the desired representation of $\mathfrak{A}$.

Definition 3.1. A pre-network is a pair $N = (N_1, N_2)$, where N_1 is a finite (possibly empty) set, and $N_2 : {}^n N_1 \to At(\mathfrak{A})$ is a partial map. We write nodes(N) for N_1 and edges(N) for the domain of N_2. Also, we may write N for any of N, N_1, N_2.

- We write $\emptyset$ for the pre-network $(\emptyset, \emptyset)$.
- For the pre-networks N and N', we write $N \subseteq N'$ iff nodes$(N) \subseteq$ nodes(N'), edges$(N) \subseteq$ edges(N'), and $N'(f) = N(f)$ for all $f \in$ edges(N).
- Let α be an ordinal. A sequence of pre-networks $\langle N_\beta : \beta \in \alpha \rangle$ is said to be a chain if $N_\gamma \subseteq N_\beta$ whenever $\gamma \in \beta$. Supposing that $\langle N_\beta : \beta \in \alpha \rangle$ is a chain of pre-networks, define the pre-network $N = \bigcup \{N_\beta : \beta \in \alpha\}$ with

$$\text{nodes}(N) = \bigcup \{\text{nodes}(N_\beta) : \beta \in \alpha\}, \ \text{edges}(N) = \bigcup \{\text{edges}(N_\beta) : \beta \in \alpha\}$$

  and, for each $f \in$ edges(N), we let $N(f) = N_\beta(f)$, where $\beta \in \alpha$ is any ordinal with $f \in$ edges(N_β).

Definition 3.2. Let N be a pre-network and let $f, g \in$ edges(N). A sequence of edges $h_0, \ldots, h_m$ is said to be a zigzag of length m from f to g in N if the following hold:

- $h_0 = f$ and $h_m = g$.
- For each $0 \leq t \leq m$, $Rng(f) \cap Rng(g) \subseteq Rng(h_t)$.
- For each $0 \leq t < m$, there is $i_{t+1} \in n$ such that

$$h_t \neq h_{t+1}, \quad h_t \equiv_{i_{t+1}} h_{t+1} \ \text{ and } \ c_{i_{t+1}} N(h_t) = c_{i_{t+1}} N(h_{t+1}).$$

Definition 3.3. A pre-network N is said to be a network if it satisfies the following conditions for each $f, g \in$ edges(N) and each $i, j \in n$:

(a) (i) $\mathrm{K} = \mathrm{DC} \implies$ edges(N) is diagonalizable.
(ii) $\mathrm{K} = \mathrm{SC} \implies$ edges(N) is diagonalizable and permutable.

(b) $N(f) \leq d_{ij} \iff f(i) = f(j)$.

(c) If $0 < |Rng(f) \cap Rng(g)| < n$ then there is a zigzag from f to g in N.

Lemma 3.4. *Let N be a network. Let $f, g \in \text{edges}(N)$, $i, j \in n$ and $\tau \in \Omega_n$. The following are true.*

(1) If $f \equiv_i g$ then $c_i N(f) = c_i N(g)$.[1]
(2) If $f \circ [i/j] \in \text{edges}(N)$ then $N(f \circ [i/j]) = t^i_j N(f)$.
(3) Suppose that $\mathrm{K} \in \{\mathrm{DC}, \mathrm{SC}\}$. Then, $N(f \circ \tau) = \tau^{\mathfrak{A}} N(f)$.

Proof. Let N, f, g, i, j, τ be as required.

(1) Suppose that $f \equiv_i g$. If $f = g$ then we are done. Assume that $f \neq g$. So, it follows that $0 < |Rng(f) \cap Rng(g)| < n$. Thus, we can assume that $h_0, \ldots, h_m \in \text{edges}(N)$ is a zigzag from f to g.

We will define $i_1, \ldots, i_m, j_1, \ldots, j_m, k_1, \ldots, k_m$ such that

$$N(g) \leq s^{i_m}_{j_m} c_{k_m} \cdots s^{i_1}_{j_1} c_{k_1} N(f) \cdot \prod_{l \in J} d_{l\tau(l)},$$

where $i_1, \ldots, i, \tau, J$ satisfy the conditions of axiom (Ax7). Hence, by axiom (Ax7) again, $N(g) \leq c_i N(f)$ and we will be done. We define the i's as follows. For each $1 \leq t \leq m$, let $i_t \in n$ be such that $h_{t-1} \equiv_{i_t} h_t$. Let $J = \{i_1, \ldots, i_m\} \setminus \{i\}$ and $J^+ = J \cup \{i\}$. Note that $|J^+| > |J|$. We will define j_t and k_t for $1 \leq t \leq m$ by induction on t such that by letting

$$\tau_t = [i_t/j_t] \circ \cdots \circ [i_1/j_1]$$

we will have for all $t < m$ that

$$\begin{aligned} h_0(l) &= h_{t+1}(\tau_{t+1}(l)) \text{ for all } l \in J, \\ N(h_{t+1}) &\leq s^{i_{t+1}}_{j_{t+1}} c_{k_{t+1}} N(h_t) \text{ and} \\ k_{t+1} &\in J^+ \setminus \tau^*_t J. \end{aligned}$$

Let $t < m$ and assume that $j_{t'}, k_{t'}$ have been define for all $1 \leq t' \leq t$ with the above properties. Now, we have two cases.

Case 1: Suppose that $h_t(i_{t+1}) \in Rng(h_{t+1})$, say $h_t(i_{t+1}) = h_{t+1}(j)$, for some $j \in n$. Note that $h_{t+1} \neq h_t$ and $h_t(j) = h_{t+1}(j) = h_t(i_{t+1})$, then $j \neq i_{t+1}$. Hence, $N(h_{t+1}) \leq s^{i_{t+1}}_j N(h_t)$. We let $j_{t+1} \stackrel{\text{def}}{=} j$ and $k_{t+1} \in (J^+ \setminus \tau^*_t J)$ be arbitrary, note that $(J^+ \setminus \tau^*_t J) \neq \emptyset$.

[1]The proof of this item is distilled from Andréka's proof in [8].

Case 2: Suppose that $h_t(i_{t+1}) \notin Rng(h_{t+1})$. Let $j_{t+1} \stackrel{\text{def}}{=} k_{t+1} \stackrel{\text{def}}{=} i_{t+1}$. Let $l \in J$ be arbitrary. Then, by $l \neq i$, we have

$$h_t(\tau_t(l)) = h_0(l) \in Rng(f) \cap Rng(g) \subseteq Rng(h_{t+1}).$$

Hence, $i_{t+1} \neq \tau_t(l)$ as desired.

It is not difficult to check that the above choices satisfy our requirements. Then $N(g) \leq s^{i_m}_{j_m} c_{k_m} \cdots s^{i_1}_{j_1} c_{k_1} N(f)$. Also, for any $l \in J$,

$$g(l) = f(l) = h_0(l) = h_m(\tau_m(l)) = g(\tau(l)).$$

Hence $N(g) \leq d_{l\tau(l)}$ for all $l \in J$ by condition (b) of networks. Therefore, $N(g) \leq s^{i_m}_{j_m} c_{k_m} \cdots s^{i_1}_{j_1} c_{k_1} N(f) \cdot \prod_{l\in J} d_{l\tau(l)} \leq c_i N(f)$.

(2) If $i = j$ then we are done. So, we may suppose that $i \neq j$ and we also assume that $f \circ [i/j] \in \text{edges}(N)$. Thus, by the above item, $c_i N(f) = c_i N(f \circ [i/j])$. Also N is a network, then $N(f \circ [i/j]) \leq d_{ij}$. Therefore, $N(f \circ [i/j]) \leq c_i N(f) \cdot d_{ij} = t^i_j N(f)$.

(3) Let $\tau \in \Omega_n$ and assume that $\tau = [i_1/j_1] \circ \cdots \circ [i_m/j_m]$. The statement follows by an induction argument that uses item (2). □

An arbitrary sequence f (of length n) is said to be a repetition free sequence if and only if $|Rng(f)| = n$.

Definition 3.5. Let $a \in At(\mathfrak{A})$ and let f be any sequence of length n such that $(\forall i, j \in n)\ f(i) = f(j) \iff a \leq d_{ij}$. The mosaic generated by f and a, in symbols $M(f, a)$, is defined to be the pre-network N, where $\text{nodes}(N) = Rng(f)$ and:

(a) If K = RC then $\text{edges}(N) = \{f\}$ and $N(f) = a$.
(b) If K = DC then $\text{edges}(N) = \{f \circ \tau : \tau \in \Omega_n\}$, $N(f) = a$ and, for each $\tau \in \Omega_n$, $N(f \circ \tau) = \tau^{\mathfrak{A}} a$.
(c) If K = SC and $|Rng(f)| < n$ then the mosaic is defined in the same way as in item (b) above.
(d) Suppose that K = SC and assume that f is a repetition free sequence. Then $\text{edges}(N) = \{f \circ \tau : \tau \in \Delta_n\}$, and the labeling is defined as follows. First, $N(f) = a$ and for each $\tau \in \Omega_n$, $N(f \circ \tau) = \tau^{\mathfrak{A}} a$. So, each non repetition free sequence is labeled. It remains to label the repetition free sequences. Let $g_0, g_1, \ldots, g_N$ ($N \geq 1$ because $n \geq 2$) be some enumeration of the repetition free sequences such that

$$\begin{aligned} g_0 &= f \\ g_i &= g_j \circ [k, l] \text{ for some } j < i \text{ and } k \neq l \\ g_i &\neq g_j \text{ if } i \neq j \end{aligned}$$

Such enumeration is possible. Assume that $N(g_j)$ is defined for all $j < i$. Let $j < i$ and $k \neq l$ be such that $g_i = g_j \circ [k, l]$ (If there are several such j, k, l then we just select one such triple). Now we choose any atom $b \leq p_{kl}N(g_j)$ (such an atom exist by Lemma 2.6) and we define $N(g_i) = b$.

We check that the mosaic $M(f, a)$ is well defined. The essential part follows from Lemma 2.4, Suppose that $\tau, \sigma \in \Omega_n$ are such that $f \circ \tau = f \circ \sigma$. Fix an enumeration $\tau_1, \ldots, \tau_m$ of the set $\{[i/j] : i, j \in n \text{ and } f(i) = f(j)\}$. Now, let $\gamma = \tau_0 \circ \cdots \circ \tau_m$. Hence, $\gamma \circ \tau = \gamma \circ \sigma$ and by Lemma 2.2 we must have $\tau^{\mathfrak{A}}(\gamma^{\mathfrak{A}}a) = \sigma^{\mathfrak{A}}(\gamma^{\mathfrak{A}}a)$. But by the condition on the sequence f and by Lemma 2.2, $\gamma^{\mathfrak{A}}a = a$. Therefore, $\tau^{\mathfrak{A}}a = \sigma^{\mathfrak{A}}a$.

Lemma 3.6. *Let $a \in At(\mathfrak{A})$ and let f be any sequence of length n such that*

$$(\forall i, j \in n)\ f(i) = f(j) \iff a \leq d_{ij}.$$

Then the mosaic $M \stackrel{\text{def}}{=} M(f, a)$ is actually a network.

Proof. Suppose that $\mathrm{K} = \mathrm{G}_n$ and suppose that f is a repetition free sequence. We prove the above statement for this case only, the other cases are similar so we omit their details. The construction of M guarantees that it satisfies conditions (a) and (b) of networks. It remains to show that M also satisfies condition (c). First, we prove the following.

$$(\forall g \in \text{edges}(M))\ (\forall i, j \in n)\ \ M(g(i/g(j))) = t^i_j M(g). \tag{3.1}$$

Let $i, j \in n$ and let $g \in \text{edges}(M)$ be such that $i \neq j$. If $f = g$ then by the definition of M we have

$$M(g(i/g(j))) = M(f(i/f(j))) = M(f \circ [i/j]) = t^i_j M(f) = t^i_j M(g).$$

Suppose that there is a transformation $\tau \in \Omega_n$ such that $g = f \circ \tau$. Then, $g(i/g(j)) = g \circ [i/j] = f \circ \tau \circ [i/j]$. So, by the construction of M, we have $M(g(i/g(j))) = (\tau \circ [i/j])^{\mathfrak{A}}a = t^i_j \tau^{\mathfrak{A}}a = t^i_j M(g)$. Recall the enumeration $g_0, g_1, \ldots, g_N$ of the repetition free sequences given in the definition of M. To finish proving (3.1), it remains to show that

$$(\forall m \in N + 1)\ \ M(g_m(i/g_m(j))) = t^i_j M(g_m). \tag{3.2}$$

We do this by induction on m. If $m = 0$ then we are done as $g_0 = f$. Let $1 \leq p \leq N$. Assume that (3.2) is true for all $m < p$. We will show that (3.2) is true for p, too. Let $g_p = g_m \circ [k, l]$ be such that $m < p$ and suppsoe that we choose $M(g_p) \leq p_{kl}M(g_m)$.

Case 1: $i \neq k, l$ and $j = k$ or $j = l$.

Assume first that $j = k$. Consider the edges

$$h_1 = g_m \circ [i/k], \quad h_2 = g_m \circ [i/k] \circ [k/l] \text{ and } h_3 = g_m \circ [i/k] \circ [k/l] \circ [l/i].$$

Thus, $h_3 \equiv_i g_p$ because $g_p = g_m \circ [k, l]$. Hence, $g_p(l) = h_3(l) = h_3(i)$, which means that $h_3 = g_p(i/g_p(j))$. By induction hypothesis, $M(h_1) = t^i_k M(g_m)$. None of the sequences h_1, h_2 and h_3 is repetition free, so we already showed that

$$M(g_p(i/g_p(j))) = M(h_3) = t^l_i M(h_2) = t^l_i t^k_l M(h_1) = t^l_i t^k_l t^i_k M(g_m).$$

By $M(g_p) \leq p_{kl} M(g_m)$ we have $M(g_p) \leq s^i_k s^k_l s^l_i c_i M(g_m)$. Since in K_n the so-called the Merry Go Round equation $s^i_k s^k_l s^l_i c_i x = s^i_l s^l_k s^k_i c_i x$ is true, then we also have that $M(g_p) \leq s^i_l s^l_k s^k_i c_i M(g_m)$. Now, Lemma 2.7 implies that $M(g_p(i/g_p(j))) \leq c_i M(g_p)$. By condition (a) of the networks, we have $M(g_p(i/g_p(j))) \leq c_i M(g_p) \cdot d_{ij} = t^i_j M(g_p)$. The case $j = l$ is completely similar, except that we don not have to use the Merry Go Round equation.

Case 2: $i \neq k, l$ and $j \neq k, l$.

Then $[i/l] \circ [i/j] = [i/j]$. Hence, $g_p(i/g_p(j)) = g_p \circ [i/j] = g_p \circ [i/l] \circ [i/j]$. We can use the previous case to conclude that $M(g_p \circ [i/l]) = t^i_l M(g_p)$. Note that $g_p \circ [i/l]$ is not repetition free, so we know that

$$M(g_p(i/g_p(j))) = M(g_p \circ [i/l] \circ [i/j]) = t^i_j t^i_l M(g_p) = t^i_j M(g_p),$$

the last equality follows from Lemma 2.4.

Case 3: $i = k$ and $j = l$.

Then $g_p(i/g_p(j)) = g_p \circ [i/j] = g_p \circ [k/l] = g_m \circ [l/k]$. Thus, by induction hypothesis, we have $M(g_p(i/g_p(j))) = t^l_k M(g_m)$. We also have $M(g_p) \leq s^i_j c_j M(g_m)$ because of the fact that $M(g_p) \leq p_{kl} M(g_m)$. Hence, by Lemma 2.7, $M(g_p(i/g_p(j))) \leq c_i M(g_p)$. Again, since $M(g_p(i/g_p(j))) \leq d_{ij}$, $M(g_p(i/g_p(j))) = c_i M(g_p) \cdot d_{ij} = t^i_j M(g_p)$. The case $i = k$ and $j \neq l$ is as above. The case $i = l$ is completely analogous.

Therefore, (3.2) is true and thus we have shown that (3.1) is also true. We need also to show that the following holds.

(3.3) $\qquad (\forall \tau \in \Omega_n)$ there is a zigzag from $f \circ \tau$ to f in M.

Let $\tau \in \Omega_n$. So, we can suppose that $\tau = [i_1/j_1] \circ \cdots \circ [i_m/j_m]$ with the smallest possible m. Consider the edges:

$$h_0 = f,\ h_1 = h_0 \circ [i_1/j_1],\ h_2 = h_1 \circ [i_2/j_2],\ \ldots,\ h_m = h_{m-1} \circ [i_m/j_m] = f \circ \tau.$$

By (3.1), it is easy to see that $h_0, \ldots, h_m$ is a zigzag from $f \circ \tau$ to f in M. Hence, (3.3) is proved. Now we are ready to show that the mosaic M

satisfies condition (c) in Definition 3.3. Let $g, h \in \text{edges}(M)$ be such that $0 < |Rng(g) \cap Rng(h)| < n$.

(I) Suppose that none of g and h is repetition free. By (3.3) we have the following.
   - There is a zigzag $h_0, \ldots, h_m$ from g to f in M.
   - There is a zigzag $w_0, \ldots, w_d$ from h to f in M.

   It is not hard to see that $h_0, \ldots, h_m = f = w_d, \ldots, w_0$ is a zigzag from g to h in M. This true because of the facts $Rng(g) \subseteq Rng(f)$ and $Rng(h) \subseteq Rng(f)$.

(II) Suppose that one of g and h is repetition free, say g. Thus, h must not be repetition free, by the assumption $0 < |Rng(g) \cap Rng(h)| < n$. Also, we can find an element $u \in Rng(g) \setminus Rng(h)$. Let $i \in n$ be such that $g(i) = u$ and let $j \in n \setminus \{i\}$ (this j exists because $n \geq 2$). Clearly, $Rng(g \circ [i/j]) \cap Rng(h) = Rng(g) \cap Rng(h)$. Hence, by (I) above, there is a zigzag $h_0, \ldots, h_m$ from $g \circ [i/j]$ to h in M. But (3.1) and Lemma 2.1 imply $c_i M(g) = c_i M(g \circ [i/j])$. Thus, $g, h_0, \ldots, h_m$ is a zigzag from g to h in M.

Therefore, the mosaic M is indeed a network □

4. Games and representability

The games we use here are games of infinite lengths between two players $\forall$ and $\exists$. These games are basically Banach-Mazur games in disguise.

Definition 4.1. Let α be an ordinal. We define a game, denoted by $G_\alpha(\mathfrak{A})$, with α rounds, in which the players $\forall$ and $\exists$ build a chain of pre-networks $\langle N_\beta : \beta \in \alpha \rangle$ as follows. In round 0, $\exists$ starts by letting $N_0 = \emptyset$. Suppose that we are in round $\beta \in \alpha$ and assume that each N_λ, $\lambda \in \beta$, is a pre-network. If β is a limit ordinal then $\exists$ defines $N_\beta = \bigcup\{N_\lambda : \lambda \in \beta\}$. If $\beta = \gamma + 1$ is a successor ordinal then the players move as follows.

(a) $\forall$ chooses an atom $b \in At(\mathfrak{A})$, $\exists$ must respond with a pre-network $N_\beta \supseteq N_\gamma$ containing an edge g with $N_\beta(g) = b$.
(b) Alternatively, $\forall$ chooses an edge $g \in \text{edges}(N_\gamma)$, an index $i \in n$ and an atom $b \in At(\mathfrak{A})$ such that $N_\gamma(g) \leq c_i b$. In this case, $\exists$ must respond with a pre-network $N_\beta \supseteq N_\gamma$ such that for some $u \in \text{nodes}(N_\beta)$ we have $g(i/u) \in \text{edges}(N_\beta)$ and $N_\beta(g(i/u)) = b$.

$\exists$ wins if each pre-network N_β, $\beta \in \alpha$, played during the game is actually a network. Otherwise, $\forall$ wins. There are no draws.

What we have defined are the rules of the game. There are many different matches of the game that satisfy the rules. The idea of the game is that the current network is refined in the current round of the game. $\forall$ can challenge $\exists$ to find a suitable refinement in any of the two ways he likes, and she must either do or lose. Since the game then continues from there, her response must itself be refinable in any way, if she is not to lose.

Proposition 4.2. *Let α be an ordinal. $\exists$ has a winning strategy in the game $G_\alpha(\mathfrak{A})$.*

Proof. Let α be an ordinal and let $\beta \in \alpha$. Clearly, $\exists$ always wins in the round β if $\beta = 0$ or β is a limit ordinal. So, we may suppose that $\beta = \gamma + 1$ is a successor ordinal. We also may assume inductively that $\exists$ has managed to guarantee that N_γ is a network. We consider the possible moves that $\forall$ can make.

(a) Suppose that $\forall$ chooses an atom $b \in At(\mathfrak{A})$. If there is an edge in N_γ with the required conditions, then $\exists$ lets $N_\beta = N_\gamma$. Otherwise, she picks brand new nodes $f_0, \ldots, f_{n-1}$ such that

$$(\forall i, j \in n) \quad f_i = f_j \iff b \leq d_{ij}.$$

Let $f = (f_0, \ldots, f_{n-1})$. She defines $N_\beta = N_\gamma \cup M(f, a)$, where $M(f, a)$ is the mosaic generated by f and a. Note that our construction guarantees that $\text{nodes}(N_\gamma) \cap \text{nodes}(M(f, a)) = \emptyset$, so N_β is well defined. Now, Lemma 3.6 guarantees that N_β is a network.

(b) Alternatively, suppose that $\forall$ chooses an edge $f \in \text{edges}(N_\gamma)$, an index $i \in n$ and an atom $b \in At(\mathfrak{A})$ such that $N_\gamma(f) \leq c_i b$. Again, if there is a node $u \in \text{nodes}(N_\gamma)$ such that $f(i/u) \in \text{edges}(N_\gamma)$ and $N_\gamma(f(i/u)) = b$, then $\exists$ lets $N_\beta = N_\gamma$. Suppose that such u does not exist. $\exists$'s strategy goes as follows. Let $a = N_\gamma(f)$, so we have $c_i a = c_i b$ by Lemma 2.1.

   Suppose that there is some $j \in n \setminus \{i\}$ such that $b \leq d_{ij}$. Then $b \leq c_i a \cdot d_{ij}$ and thus $b = t^i_j a$. If $f(i/f_j) \in \text{edges}(N_\gamma)$ then we have $N_\gamma(f(i/f_j)) = t^i_j a = b$, which contradicts the assumptions. So, we may assume that $f(i/f_j) \notin \text{edges}(N_\gamma)$ (hence, $\mathrm{K}_n = \mathrm{Crs}_n$ because N_γ is a network). $\exists$ defines $N_\beta = N_\gamma \cup M(f(i/f_j), b)$. Thus, N_β is indeed well defined and it is not hard to see that it is a network.

   Suppose that $b \not\leq d_{ij}$ for every $j \in n \setminus \{i\}$. In this case, $\exists$ picks brand new node u and then she lets $N_\beta = N_\gamma \cup M(f(i/u), b)$. We need to check that N_β is well defined. Let $g \in \text{edges}(N_\gamma) \cap \text{edges}(M(f(i/u), b))$. Remember that u was brand new node, so the existence of such g means that $\mathrm{K} \in \{\mathrm{D}, \mathrm{G}\}$ and that there are $j \in n \setminus \{i\}$ and $\tau \in \Omega_n$ such that $g = f(i/u) \circ [i/j] \circ \tau$. Hence, $g = f \circ [i/j] \circ \tau$. By Lemma 3.4, it follows that $N_\gamma(g) = \tau^{\mathfrak{A}} t^i_j a = \tau^{\mathfrak{A}}(c_i a \cdot d_{ij})$. Let $M \overset{\text{def}}{=} M(f(i/u), b)$, by the construction of the mosaic M, we also have $M(g) = \tau^{\mathfrak{A}} t^i_j b = \tau^{\mathfrak{A}}(c_i b \cdot d_{ij})$. It remains to see that $N_\gamma(g) = M(g)$, which is true because $c_i a = c_i b$. Thus, $\exists$ could manage to guarantee that N_β is well defined. Now, we show that N_β is actually a network.

Lemma 3.6 and the induction hypothesis guarantee that N_β satisfies conditions (a) and (b) of networks. For the same reasons, it is apparent that N_β satisfies condition (c) for any two edges both lie in N_γ or in the mosaic M. Let $g \in \mathrm{edges}(N_\gamma)$ and let $h \in \mathrm{edges}(M) \setminus \mathrm{edges}(N_\gamma)$. Suppose that $0 < |Rng(g) \cap Rng(h)| < n$. We need to find a zigzag from g to h in the pre-network N_β.

Note that $u \in Rng(h)$. Let $j \in n$ be such that $h(j) = u$ and choose an index $k \in n \setminus \{j\}$. Let $h' = h \circ [j/k]$. Thus, $h' \in \mathrm{edges}(N_t)$ and $Rng(h) \cap Rng(g) = Rng(h') \cap Rng(g)$. By induction hypothesis, there is a zigzag $w_0, \ldots, w_m$ from g to h' in N_β. Remember the fact that $N_\beta(h') = M(h') = t^j_k M(h) = t^j_k N_\beta(h)$. Hence, by Lemma 2.1, we have $c_j N_\beta(h') = c_j N_\beta(h)$. Therefore, $w_0, \ldots, w_m, h$ is a zigzag from g to h in N_β as required.

Therefore, if $\exists$ plays according to the strategy above she can win any play of the game $G_\alpha(\mathfrak{A})$, regardless of what moves $\forall$ makes. □

It is true that Andréka's methods in [8] and [5] could also suggest a strategy for $\exists$ to win the game $G_\alpha(\mathfrak{A})$. Despite the technical similarities, we note that this suggested strategy is quite different than our strategy. Andréka's method would suggest adding a sequence in each step, not a mosaic, then at the end of the construction the axioms can be used to make sure that the mosaics generated by these sequences are eventually added.

In the strategy described in the above proof, we add a mosaic in each step. The axioms now are used to label the elements of the mosaics by relevant atoms. Now we show that the algebra $\mathfrak{A}$ can be represented into a full algebra whose unit is a union of mosaics.

Proposition 4.3. *$\mathfrak{A}$ is representable, i.e. there is $\mathfrak{B} \in \mathrm{Ks}_n$ such that $\mathfrak{A} \cong \mathfrak{B}$.*

Proof. Let α be an ordinal (large enough) and consider a play $\langle N_\beta : \beta \in \alpha \rangle$ of $G_\alpha(\mathfrak{A})$ in which $\exists$ plays as in Proposition 4.2, and $\forall$ plays every possible move at some stage of play. That means, (G 1) each atom $a \in At(\mathfrak{A})$ is played by $\forall$ in some round, and

(G 2) for every $b \in At(\mathfrak{A})$, every $\beta \in \alpha$, and each $f \in \text{edges}(N_\beta)$ and every $i \in n$ with $N_\beta(f) \leq c_i b$, $\forall$ plays b, i, f in some round.

Let $U = \bigcup\{\text{nodes}(N_\beta) : \beta \in \alpha\}$ and let $V = \bigcup\{\text{edges}(N_\beta) : \beta \in \alpha\} \subseteq {}^nU$. By the condition (a) of Definition 3.3, we can guarantee that $\mathfrak{P}(V) \in \text{Ks}_n$. Thus, it remains to show that $\mathfrak{A}$ is embeddable into $\mathfrak{P}(V)$. For this, we define the following function. For each $x \in \mathfrak{A}$, let

$$h(x) = \{f \in V : \exists\beta \in \alpha\ (f \in \text{edges}(N_\beta) \text{ and } N_\beta(f) \leq x)\}.$$

It is not hard to see that h is a Boolean homomorphism. Also, (G 1) above implies that h is one-to-one. We check cylindrifications and diagonals. Let $x \in \mathfrak{A}$. Then, for each $f \in V$, we have

$$\begin{aligned} f \in h(c_i x) &\iff \exists\beta \in \alpha\,(f \in N_\beta \text{ and } N_\beta(f) \leq c_i x) \\ &\iff \exists g \in V\ \exists\beta \in \alpha\ (f, g \in N_\beta, f \equiv_i g, N_\beta(g) \leq x) \\ &\iff \exists g \in V\ (f \equiv_i g \text{ and } g \in h(x)) \\ &\iff f \in C_i^{[V]} h(x). \end{aligned}$$

The second $\iff$ follows by (G 2) and Lemma 3.4 (1). For the diagonals, let $i, j \in n$, then by the second condition of networks

$$\begin{aligned} h(d_{ij}) &= \{f \in V : \exists\beta \in \alpha\ (f \in \text{edges}(N_\beta) \text{ and } N_\beta(f) \leq d_{ij})\} \\ &= \{f \in V : f(i) = f(j)\} \\ &= D_{ij}^{[V]}. \end{aligned}$$

Therefore, $\mathfrak{A}$ is isomorphic to a subalgebra $\mathfrak{B} \subseteq \mathfrak{P}(V)$ as desired. □

Proof of Main Theorem 1. Let $\text{K} \in \{\text{RC}, \text{DC}, \text{SC}\}$. It is easy to see that every concrete algebra in Ks_n satisfies all the axioms defining K_n. Conversely, let $\mathfrak{A} \in \text{K}_n$, we need to show that $\mathfrak{A} \in \mathbf{I}\text{Ks}_n$. By [1, Theorem 2.18], $\mathfrak{A}$ can be embedded into a complete and atomic $\mathfrak{A}^+ \in \text{K}_n$. Hence, Proposition 4.3 implies that $\mathfrak{A}^+$ is representable. Therefore, $\mathfrak{A}$ is representable as well. □

Main Theorem 2. *Let $n \geq 2$ be a finite ordinal and let $\text{K} \in \{\text{RC}, \text{DC}, \text{SC}\}$. Then, every $\mathfrak{A} \in \text{K}_n$ is completely representable into $\mathfrak{B} \in \mathbb{I}\text{Ks}_n$. That means, $\mathfrak{A}$ is embeddable into $\mathfrak{B}$ via an embedding that preserves infinite sums and infinite products.*

Proof. See the proofs of Proposition 4.3 and Main Theorem 1. □

5. Other networks

If $n = 2$ then one can easily see that (Ax7) follows from (Ax0)-(Ax6). Suppose that $n \geq 3$. H. Andréka and I. Németi showed that the class RCs_n can not be characterized by finitely many equations, c.f. [3, Theorem 5.5.13] and [8, Theorem 9.3], thus in this case (Ax7) can not be replaced with a finite set of axioms. However, in [5] and [9], axiom (Ax7) was omitted from the characterizations of the classes DCs_n and SCs_n, and hence it was shown that these classes are finitely axiomatizable.

Although, axiom (Ax7) is essential in the proof presented in the previous sections, this axiom can be omitted from the definitions of DC_n and SC_n. It could be possible to give a syntactical proof for this fact, but we prefer to introduced some modifications of the networks that would allow us to proceed without the need of (Ax7). Let DC_n^- and SC_n^- be the resulting classes after deleting axiom (Ax7) from the definition of DC_n and SC_n, respectively. Suppose that $\mathrm{K} \in \{\mathrm{DC}^-, \mathrm{SC}^-\}$ and let $\mathfrak{A} \in \mathrm{K}_n$ be complete and atomic. We show that $\mathfrak{A}$ is representable.

Definition 5.1. A 'modified' network N is a pre-network that satisfies the following conditions for each $f \in \mathrm{edges}(N)$ and each $i, j \in n$.

(a) $\mathrm{edges}(N)$ is diagonalizable, and $\mathrm{edges}(N)$ is permutable iff $\mathrm{K} = \mathrm{SC}^-$.
(b) $N(f) \leq d_{ij} \iff f(i) = f(j)$.
(c) $N(f \circ [i/j]) = t^i_j N(f)$.

Let N be a modified network. Let $i \in n$ and let $f, g \in \mathrm{edges}(N)$ be such that $f \equiv_i g$. We prove that $c_i N(f) = c_i N(g)$. Let $j \in n$ be such that $j \neq i$. Note that $g \circ [i/j] = f \circ [i/j] \in \mathrm{edges}(N)$. By condition (c) of the above definition and Lemma 2.1, it follows that

$$c_i N(f) = c_i N(f \circ [i/j]) = c_i N(g \circ [i/j]) = c_i N(g).$$

Hence, we have shown that N satisfies all the items of Lemma 3.4. Note that we did not use axiom (Ax7). Let α be an ordinal. We also define the modified game $G'_\alpha(\mathfrak{A})$ to be as same as $G_\alpha(\mathfrak{A})$ except that the players now have to build modified networks. It is not hard to see that if $\exists$ plays with the same strategy given in Proposition 4.2 then she will win any game regardless of what moves $\forall$ can make. It is quite straightforward to see that the mosaics in this case are also modified networks (see the proof of (3.1)). Now the proof of Proposition 4.3 works verbatim to show that $\mathfrak{A}$ is representable.

We note that the modified networks and the modified games can be used also to prove the representability of some variants of Pinter's algebras and the quasi polyadic algebras. For more details, one can see [12].

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