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What happened since ’Algebraic Logic, where does it stand today?’

Part 2: Algebraic logic; Counting varieties and models, Vaught’s conjecture and new physics

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Dedicated to my mentors András and Németi

No Institute Given

Summary. * In the article ’Algebraic logic, where does it stand today?’ published in the Bulletin of Symbolic logic in 2005 logic, we gave a fairly comprehensive review of Tarskian algebraic logic reaching the boundaries of current research back then. In this paper we take it from there, giving the reader more than a glimpse of the recent developments till the present day. In the process, we take a magical tour in algebraic logic, starting from classical results on neat embeddings due to Henkin, Monk and Tarski, all the way to recent results in algebraic logic using so-called rainbow constructions. Highlighting the connections with graph theory, model theory, and finite combinatorics, this article aspires to present topics of broad interest in a way that is hopefully accessible to a large audience. Other topics dealt with include the interaction of algebraic and modal logic, the so-called (central still active) finitizability problem, Gödel’s incompleteness Theorem in guarded fragments, counting the number of subvarieties of $\text{RCA}_\omega$ which is reminiscent of Shelah’s stability theory and the interaction of algebraic logic and descriptive set theory as means to approach Vaught’s conjecture in model theory. Rainbow constructions conjuncted with variations on András methods of splitting (atoms) are used to solve problems addressing classes of cylindric-like algebras consisting of algebras having a neat embedding property. The hitherto obtained results generalize seminal results existing in the literature on non-atom canonicity, non-first order definability and non-finite axiomatizability, proved for classes of representable cylindric algebras of finite dimension $> 2$. Recent results due to András and Németi in mathematical logic, hypercomputation and new physics are discussed in connection to the opposing philosophies of platonism and formalism. This is the second Part of the paper as in the title.

1.1 Recent results

1.1.1 Counting varieties:

There is a branch in set theory which deals with infinitary combinatorics, a classical example in this branch is the so-called Erdős-Rado Theorem. In mathematics, infinitary combinatorics,
or combinatorial set theory, is an extension of ideas in (finite) combinatorics to infinite sets. ‘Counting in the transfinite’ can be tricky. In fact, most reasonable statements in combinatorial set theory can be really challenging and many turn out independent. In set theory independent statements (like the continuum hypothesis) are the norm. Proving results in ZFC involving infinitary combinatorics, more often than not, needs a lot of ingenuity. Let $\alpha$ be an infinite ordinal. In [4] Andréka and Németi, among so many other things, count the number of varieties of $\text{RCA}_\alpha$ obtaining their infinitary combinatorial results in ZFC. This paper that appeared in the Transaction of the American Mathematical Society in August 2017, solves a long standing open problem in algebraic logic [9, Problem 4.2]. As a byproduct it solves another longstanding open problem [9, Problem 2.13], and provides yet another solution to the central problem in algebraic logic, namely, [9, Problem 4.1]. The first problem asks for the number of sub-varieties of $\text{RCA}_\alpha$, the second asks whether the class described in item (iii) of [9, Theorem 2.6.50], exhausts the variety $\text{RCA}_\alpha$. Algebras in this class are given a new name in this paper, namely, *Endo dimension-complemented* algebras, and the class consisting of all such algebras of dimension $\alpha$ is denoted by $\text{Edc}_\alpha$. There are at least continuum many sub-varieties of $\text{RCA}_\alpha$ and because there are $|\alpha|$ many equations in its signature, there are at most $2^{\alpha}$ many varieties. In [4] it is proved that this maximum is attained. Using the construction to prove this result, other equally significant results, as indicated above, are obtained. There is another even more intriguing part of the results, and that is the number of varieties of some distinguished classes of cylindric algebras such as $L_{\omega\alpha}$ (the class of locally finite $\text{CA}_{\alpha}$s) is the continuum, even if $|\alpha| > \omega$. This means that the theories allowing infinitary predicates are ‘much more’ than the ordinary ones (in $L_{\omega_1^\omega}$). In contrast, for the class of quasi-polyadic algebras $\text{QEA}_\alpha$, $\alpha$ any infinite ordinal, the number of varieties is the lowest bound, namely, $2^{\alpha}$. So if we add substitution operations corresponding to transpositions, then for any infinite ordinal $\alpha$, the number of (ordinary) first order theories expressed algebraically by $\alpha$ many variables is the same as those in first order logic with infinitary predicates using the same number of variables! Though the number of first order theories expressed algebraically by (locally finite) $\text{QEA}_\alpha$ and $\text{CA}_\alpha$ for $|\alpha| > \omega$ are the same; this result informs us that the number of theories of so-called infinitary restricted formulas in which the $p_{ij}$ ($i < j < \alpha$), are introduced as unary connectives whose semantics coincide with that of swapping the variables $x_i$ and $x_j$, is far less than the number of theories of restricted infinitary formulas without these connectives. The independence of the last substitution operators from the $\text{CA}$ operations is proved in [3]. In fact, it can be proved that $\text{RQECA}_\alpha$ cannot be axiomatized by a finite schema over $\text{RCA}_\alpha$ [18]. In [4], splitting atoms is yet again adopted to construct the so-called *witness non-symmetric representable* $\text{CA}_\alpha$, denoted by $\mathfrak{A}$, detecting together with a *master equation* denoted by $\varepsilon$, the number of sub-varieties in question. This $\mathfrak{A}$ is proved not to be outside $\text{Edc}_\alpha$, hence the strictness of the inclusion $\text{Edc}_\alpha \subset \text{RCA}_\alpha$, solving [9, Problem 2.13]. This is done by proving that every $\mathfrak{A} \in \text{Edc}_\alpha$ is symmetric (while $\mathfrak{A}$ is not). The variety generated by $\text{Edc}_\alpha$ turns out to be $\text{RCA}_\alpha$; in fact this is the case with any class containing $L_{\omega_\alpha}$, but for $\text{Edc}_\alpha$ this variety is obtained by closing it only under $\mathcal{S}$ (without the help of $\mathbf{P}$ and $\mathbf{H}$). Another class introduced and proved to be strictly between $\text{Edc}_\alpha$ and $\text{RCA}_\alpha$ is the class of $\alpha$-dimensional inductive algebras. It is also proved that an algebra $\mathfrak{A}$ is inductive $\iff$ $\mathfrak{A}$ has the same equational theory as a locally finite algebra. A variation on the construction is used to show that there are $2^{\omega\alpha}$ sub-varieties of $\text{CA}_\alpha$ containing $\text{RCA}_\alpha$. One can obtain from any non-representable algebra a symmetric non-representable one (by making it so using permutations on the dimension $\alpha$) hence there are non-representable algebras that are symmetric. Unlike symmetric algebras, all inductive algebras are representable and for that matter, symmetric. The solution of [9, Problem 4.1] in [4] paper is quite original. The class of *inductive algebras* is proved to be equationally indistinguishable from $L_{\omega_\alpha}$. From the last result a recursive enumeration of the equational theory of $\text{RCA}_\alpha$ is extracted. The reper-
1.1.2 Counting models; Vaught’s conjecture topologically and algebraically

The kind of investigation in [4] is blatantly reminiscent of Shelah’s investigation in model theory of counting models of a first order theory (better known as stability theory). The idea of stability theory is to find dichotomy between theories. There are two cases; in the first class we can find a classification theorem of the number of models of the theory, and in the second we convince ourselves that are the theories with no reasonable characterization. The first case includes $\kappa$-stable, stable, superstable and so-called totally transcendental theories.\footnote{Totally transcendental theories are those such that every formula has Morley rank less than infinity. This is equivalent to being $\omega$-stable.} Peano arithmetic, Set Theory are unstable; they belong to the second category. Other unstable theories are the theory of linear orders with dense points, theory of addition of natural numbers, any infinite Boolean algebra, any monoid with cancellation that is not a group. A branch that is wide open, is the interaction of stability theory and algebraic logic. In what follows we present briefly recent results in this direction [6]: An instance of this kind of investigations was recently applied to Vaught’ conjecture [28, 29]. In 1961, Robert Vaught asked the following question: Given a complete theory in a countable language, is it the case that it either has countably many or $2^{\omega}$ non-isomorphic countable models? A positive answer to the question is more commonly known as Vaught’s Conjecture which is widely considered as one of the most important problems still open in model theory and indeed in mathematical logic.\footnote{However, some logicians do not agree to this sweeping statement. Quoting Shelah on this: \textit{People say that settling Vaught’s conjecture is the most important problem in Model theory, because it makes us understand countable models of countable theories, which are the most important models. We disagree with all three statements.}} Morley proved that the number of countable models is either less than or equal to the first uncountable cardinal ($\leq \omega_1$) or else it has the power of the continuum. This is the best known (general) answer to Vaught’s question. Later, other logicians confirmed Vaught’s conjecture in some special cases of theories, for example for $\omega$-stable and superstable theories of finite $U$-rank, theories of linear orders with unary predicates, and theories of trees. The curious reader can consult [28] to learn about this fascinating rich topic. Recent work in algebraic logic has found contact with Vaught’s conjecture [20, 5]; we give only a tiny sample. A new proof of Morley’s Theorem (on Vaught’s conjecture) using cylindric algebras and a classical result of Burgess in descriptive set theory is given, cf. Theorem 6.

In what follows we formulate a condition of distinguishing between models, and show that Vaught’s conjecture holds in this case. If two models are isomorphic then they cannot be distinguished using the new criteria, but the converse is not true. Two non-isomorphic models may not be distinguishable using the new criterion. Vaught’s conjecture is proved to hold when considering so-called weak models for countable theories whose signature comes from certain extensions of first order logic that allow infinitary relation symbols. Also, we count the number of countable models of a countable theory $T$ that omits a given countable family of non principal types; it turns out that their number is (like in Morley’s Theorem) either $\leq \omega, \omega_1$ or $2^{\omega}$. We exhibit an example of a countable atomic theory having only one countable model (the unique prime model) omitting a countable family of non principal type $\Gamma = \langle \Gamma_i : i \in \omega \rangle$, where for each such $i \in \omega$, $\Gamma_i = \{ \neg x : x \in \text{AtNr}_{\text{Fm}} T \}$ whereas the same $T$ has continuum many non-isomorphic countable models.
We assume familiarity with basic descriptive set theory, notions like Polish spaces, \(G_{\delta}\) sets, Borel sets, analytic sets will be used without warning. Let \(\mathfrak{A}\) be any Boolean algebra, we denote its Stone space by \(\mathfrak{A}^+\). It is easy to see that if \(A\) is countable, then \(\mathfrak{A}^+\) (its Stone space) is Polish, (i.e., separable and completely metrizable). Now, suppose \(\mathfrak{A} \in \text{Lf}_{\omega_0}\) is countable. Let \(\mathcal{H}(\mathfrak{A}) = \bigcap_{i \in \omega} \mathcal{H}_i(\mathfrak{A})\) and let \(\mathcal{H}'(\mathfrak{A}) = \mathcal{H}(\mathfrak{A}) \cup \bigcup_{i \in \omega} N_{\mathfrak{A}}\). Recall that \(N_i\) is the clopen set \(\{ F \in \text{Uf}(\mathfrak{A}) : x \in F \}\) in the Stone topology. Observe that \(\mathcal{H}'(\mathfrak{A})\) are a \(G_{\delta}\) dense subset of \(\mathfrak{A}^+\) and is therefore a Polish space, too. For \(\mathfrak{B} \in \text{Lf}_{\omega_0}, x \in \mathfrak{B}\) and \(\tau : \omega \to \omega\), the substitution operator \(s_\tau x\) is defined as in [9, Definition 1.11.13]; so that particular \(s_\tau x\) is a Boolean endomorphism. Assume \(F \in \mathfrak{A}^+\). For any \(x \in A\), define the function \(\text{rep}_F\) to be \(\text{rep}_F(x) = \{ \tau \in \omega^\omega : s_\tau x \in F \}\). For a theory \(T\), \(\mathfrak{S}_{\mathfrak{T}} \in \text{Lf}_{\omega_0}\) denotes the Lindenbaum-Tarski quotient algebra corresponding to \(T\). We have the following results due to G. Sági and D. Sziráki [20]:

**Theorem 1.**

1. If \(F \in \mathfrak{A}^+\) then \(\text{rep}_F\) is a homomorphism from \(\mathfrak{A}\) onto an element of \(\text{Lf}_{\omega_0} \cap \text{Cs}_{\omega_0}^{\text{reg}}\), with base \(\omega_0\). Conversely, if \(h\) is a homomorphism from \(\mathfrak{A}\) onto an element of \(\text{Lf}_{\omega_0} \cap \text{Cs}_{\omega_0}^{\text{reg}}\) with base \(\omega_0\), then there is a unique \(F \in \mathfrak{A}^+\) such that \(h = \text{rep}_F\).

2. Let \(T\) be a consistent first order theory in a countable language. Let \(M_0\) and \(M_1\) be two models of \(T\) whose universe is \(\omega\). Suppose \(F_0, F_1 \in (\mathfrak{S}_{\mathfrak{T}})^\omega\) are such that \(\text{rep}_{F_i}\) are homomorphisms from \(\mathfrak{S}_{\mathfrak{T}}\) onto \(\text{Cs}_{\omega_0}^{\text{reg}}\), for \(i = 0, 1\). If \(\rho : \omega \to \omega\) is a bijection, then the following are equivalent:
   
   (i) \(\rho : M_0 \to M_1\) is an isomorphism.
   
   (ii) \(F_1 = s_\rho F_0 = \{ s_\rho x : x \in F_0 \}\).

These last two theorems allow us to study models and count them via corresponding ultrafilters. Consider the following action of \(S_\omega\) on the space \(H(\mathfrak{A})\), \((\tau, F) \mapsto s_\tau F\). The topological version of Vaught’s conjecture [11] states that the previous action has either countably many orbits or else continuum many. The conjecture has been confirmed when we restrict the action on a certain subgroup \(G\) of \(S_\omega\). In this case there might be isomorphic models that the group \(G\) does not ‘see’ (the isomorphism witnessing this can be outside \(G\)) so the equivalence relation is drastically different. Given an equivalence relation there are theorems that assert that either the quotient space is ‘small’ or else it contains a copy of a specific ‘large’ set. Two dichotomies showing this tendency are known. The Silver Vaught Dichotomy asserts that there are either countably many equivalence classes or there is a perfect set of pairwise inequivalent elements. For any continuous action by a Polish group \(G\) on a Polish space \(X\), the orbit equivalence relation is conjectured to satisfy the Silver Vaught Dichotomy. This conjecture both implies and is motivated by Vaught’s conjecture. In Vaught’s conjecture is the particular case, when the group is the symmetric group of permutations on \(X\), and the set \(X\), is the set of non isomorphic models of a theory with domain \(\omega\). The relation \(E\) is just the equivalence relation of isomorphism. In our case \(X\) was a \(G_{\delta}\) subset of the Stone space of a countable cylindric algebra. Another Dichotomy, called the Glimm Effros dichotomy for an equivalence relation \(E\) asserts that \(E\) contains a copy of the Vitali equivalence relation \(E_0\) (equivalently there exists a non atomic ergodic measure for \(E\)) or else there is a countable Borel separating family for \(E\). This dichotomy originates with Glimm and Effros and is motivated by questions about operator algebra. \(^3\) In all cases we consider, the Glimm Effros dichotomy implies the Silver Vaught

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\(^3\)Glimm proved that the orbit space of a Polish group \(G\) action satisfies the Glimm Effros Dichotomy if \(G\) is locally compact. Effros proved it for any Polish group \(G\), provided that the equivalence relation is \(E_G\). There exists \(S_\omega\) spaces which violate the Glimm Effros dichotomy, but for the Silver Vaught Dichotomy this is still an open question.
Theorem 2. Let $G \subseteq S_\omega$ be a cli-group, and let $E_G$ denote the corresponding orbit equivalence relation. Then $|\mathcal{H}(\mathfrak{A})/E_G| \leq \omega$ or $|\mathcal{H}(\mathfrak{A})/E_G| = 2^\omega$.

Proof. The number of orbits of $E_G$ satisfies the so-called Glimm Effros Dichotomy. By known results in the literature on the topological version of Vaught’s conjecture, we have $|\mathcal{H}(\mathfrak{A})/E_G|$ is either at most countable or $|\mathcal{H}(\mathfrak{A})/E_G|$ contains continuum many non equivalent elements (i.e non-isomorphic models).

It is well known that the number of orbits of $E_\omega$, does not satisfy the Glimm Effros Dichotomy. Cli groups cover all natural extensions of abelian groups, like nilpotent and solvable groups. Now we give a topological condition that implies Vaught’s conjecture. Let everything be as above with $G$ denoting a Polish subgroup of $S_\omega$. Give $\mathcal{H}(\mathfrak{A})/E_G$ the quotient topology and let $\pi : \mathcal{H}(\mathfrak{A}) \to \mathcal{H}(\mathfrak{A})/E_G$ be the projection map. The next Theorem is new:

Theorem 3. $\pi$ is open. If $\pi$ is clopen, i.e closed as well, then Vaught’s conjecture holds.

Proof. To show that $\pi$ is open it is enough to show for arbitrary $a \in \mathfrak{A}$ that $\pi^{-1}(\pi(N_a))$ is open (here $N_a$ means $N_a \cap \mathcal{H}(\mathfrak{A})$).

For, $\pi^{-1}(\pi(N_a)) = \{F \in \mathfrak{A}^* : (\exists F' \in N_a)(F,F') \in E \} = \{F \in \mathfrak{A}^* : (\exists F' \in N_a)(\exists p \in G)s_p^+F' = F \} = \{F \in \mathfrak{A}^* : (\exists p \in G)s_p^+F = F \} = \{F \in \mathfrak{A}^* : (\exists p \in G)a \in s_p^+F \} = \{F \in \mathfrak{A}^* : (\exists a \in G)s_p^+a \in F \} = \bigcup_{p \in G} N_{s_p^+a}$

We have $\mathcal{H}(\mathfrak{A})$ is Borel subset of $\mathfrak{A}^*$, the Stone space of $\mathfrak{A}$, and $\mathcal{H}(\mathfrak{A})/E_G$ is a continuous image of $\mathcal{H}(\mathfrak{A})$. Now assume $\pi$ is clopen. Because $\pi$ is open, $\mathcal{H}(\mathfrak{A})/E$ is second countable. Now, since $\mathcal{H}(\mathfrak{A})$ is metrizable, it is normal. Since $\pi$ is closed, open, continuous, and surjective, so $\mathcal{H}(\mathfrak{A})/E$ is also normal, hence regular. Thus $\mathcal{H}(\mathfrak{A})/E_G$ can be embedded in $\mathbb{R}^{\omega}$ (like in the proof of Urysohn’s metrization Theorem). If $\mathcal{H}(\mathfrak{A})/E_G$ is uncountable, then being the continuous image under a map between two Polish spaces of a Borel set, it is analytic. Then it has the power of the continuum.

Let $\mathfrak{A} \in \mathcal{L}_{\omega}$ be countable. For an ultrafilter $F$ of $\mathfrak{A}$ and $a \in A$, define $Sat_F(a) = \{t|\Delta a : t \in a, \omega : s^+_a \in F \}$. Let $\mathcal{E}$ be the following equivalence relation on $H(\mathfrak{A})$:

$$\mathcal{E} = \{(F_0,F_1) : (\forall a \in A)(|Sat_{F_0}(a)| = |Sat_{F_1}(a)|)\}.$$ 

Then it is proved in [5] that $\mathcal{E}$ is Borel in the product space $\mathcal{H}(\mathfrak{A}) \times \mathcal{H}(\mathfrak{A})$. We say that $F_0,F_1 \in H(\mathfrak{A})$ are distinguishable if $(F_0,F_1) \notin \mathcal{E}$. We also say that two models of a theory $T$ are distinguishable if their corresponding ultrafilters in $\mathfrak{A}$ are distinguishable. That is, two models are distinguishable if they disagree in the number of realizations they have for some formula.
Then suppose we want to count the number of distinguishable models omitting a family \( \Gamma = \{ \Gamma_i : i < \lambda \} \) (\( \lambda < \wp \)) with \( \wp \) the cardinal defined before Theorem 2 of non-isolated types of \( T \).

It is known that any Borel set satisfies the Glimm-Effros Dichotomy [11]. Thus:

**Theorem 4 (Harrington-Kechris-Louveau [11]).** Let \( X \) be a Polish space and \( E \) a Borel equivalence relation on \( X \). Then \( E \) satisfies the Glimm-Effros Dichotomy. In particular, \( E \) has either countably many equivalence classes or else perfectly many.

Now suppose we want to count the number of distinguishable models omitting a family \( \Gamma = \{ \Gamma_i : i < \lambda \} \) (\( \lambda < \wp \)) with \( \wp \) the cardinal defined before Theorem 2 of non-isolated types of \( T \).

Then

\[
\mathcal{H} = \mathcal{H}(\hat{\wp}_T) \cap \sim \bigcup_{i \in \lambda, \tau \in W} \bigcap_{\varphi \in \Gamma_i} \mathcal{N}_{\wp}(\varphi/\equiv_T)
\]

(where \( W = \{ \tau \in \wp : | i : \tau(i) \neq i < \omega \} \) is the space of ultrafilters corresponding to models of \( T \) omitting \( \Gamma \). We then have \( \mathcal{H} \) a \( G_\delta \) subset of a Polish space, so \( \mathcal{H} \) is Polish and moreover, \( \mathcal{H}' = \mathcal{H} \cap (\mathcal{H} \times \mathcal{H})' \) is a Borel equivalence relation on \( \mathcal{H} \). It follows then that the number of distinguishable models omitting \( \Gamma \) is either countable or else \( 2^{\wp} \). The same proofs goes through for locally finite quasi-polyadic algebras (first order logic without equality). We readily obtain:

**Corollary 1.** Let \( T \) be a first order theory in a countable language (with or without equality). If \( T \) has an uncountable set of countable models that are pairwise distinguishable, then actually it has such a set of size \( 2^{\wp} \).

Here is another simpler argument for Theorem 1 (but it does not work if we do not have equality). Suppose we have a language \( L \) with equality. First note that if \( L^* = L_0 \cup L_1 \) where \( L_0 \) and \( L_1 \) are disjoint copies of \( L \), then \( X_{L_0} \cong X_{L_0,1} \times X_{L_1} \), where the spaces \( X_i \)'s are defined as in [7] p. 22. For each formula \( \varphi \), let \( \varphi^* \) be the sentence \( \bigwedge_{\alpha \in \omega} (\exists \bar{x}) \varphi_0(\bar{x}) \leftrightarrow (\exists \bar{x}) \varphi_1(\bar{x}) \) where \( \varphi_0, \varphi_1 \) are the copies of \( \varphi \) in \( L_0, L_1 \), respectively, and \( \exists \) is a shorthand for “there exists at least \( n \) tuples such that ...” It is then immediate that two models \( M_0, M_1 \) of \( L \) are not distinguishable \( \iff \) the model \( M \) of \( L^* \) such that \( M|_{L_0} = M_0 \) and \( M|_{L_1} = M_1 \) satisfies \( \bigwedge_{\varphi \in \mathcal{L}^*} \varphi^* \). This means that our equivalence relation between models corresponds to the subset \( X_{L^*} \) of models of the formula \( \bigwedge_{\varphi \in \mathcal{L}^*} \varphi^* \). Such a subset is Borel by [12, Theorem 16.8].

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**Corollary 2.** Let \( T \) be a first order theory in a countable language (with or without equality). If \( T \) has an uncountable set of countable models that omit \( \wp \) many non-principal types that are pairwise distinguishable, then actually it has such a set of size \( 2^{\wp} \).

The notions of a rich language which is an extension of a first order language and weak models are defined in [?, p 212 and Definition 3.2.2(ii)]. If \( T \) is a theory in a rich language having \( \alpha \) many variables (\( \alpha \) an infinite ordinal), then \( \hat{\wp}_T \in \mathcal{D}_{\wp} \). The next theorem substantially generalizes [?, Theorem 3.2.4] when languages considered have finitely many relation symbols. It also says that theories in such languages satisfies Vaught’s conjecture with respect to weak models.

**Theorem 5.** Let \( T \) be a countable complete theory in a rich language with finitely many relation symbols. Then \( T \) has \( 2^{\wp} \) weak models that omit \( \omega \) many non-principal types.

**Proof.** We first show that if \( \mathfrak{A} \in \mathcal{D}_{\wp} \) is countable, simple and finitely generated, then the number of non-base isomorphic representations of \( \mathfrak{A} \) is \( 2^{\wp} \). Let \( V = \wp \alpha^{([\mathfrak{A}])} \). Then \( \mathfrak{A} \) cannot be atomic [9, Corollary 2.3.33], so it has \( 2^{\wp} \) ultrafilters. For an ultrafilter \( F \) that preserves the sets \( \{s_j^a : j < \omega \} \) and contains \( -d_j \) for all \( i \neq j \), that is to say \( F \in H(\mathfrak{A}) \), let \( h_F(\alpha) = \{ \tau \in V : s_\alpha \in F \} \), \( a \in \mathfrak{A} \). Then \( h_F \) is a nonzero-homomorphism from \( \mathfrak{A} \) to \( \wp(V) = \wp(V), \cap, \sim \), \( C, D_{ij}, i, j < \omega \). Now, by simplicity of \( \mathfrak{A} \), \( h_F : \mathfrak{A} \to \wp(V) \) is an isomorphism; observe that all the
\(h_F\)'s have the same target algebra. Also for ultrafilters \(G, M\) of \(\mathfrak{A}, h_G(M)\) is base-isomorphic in the sense of [9, Definition 3.1.37(ii)] to \(h_M(M)\) (which means that the corresponding models are isomorphic) \(\iff\) there exists a finite bijection \(\sigma \in V\) such that \(s_\sigma G = M\). Define the equivalence relation \(\sim\) on the set of ultrafilters by \(F \sim G\), if there exists a finite permutation \(\sigma\) such that \(F = s_\sigma G\). Thus, any equivalence class is countable, and so we have \(2^{\omega_1}\) many classes, which correspond to the non base-isomorphic representations of \(\mathfrak{A}\). Using the same reasoning as in the paragraph before Theorem 2, we get the required number of models omitting \(\leq \text{covK}\) many non-principal types.

We next give a new proof of Morley’s theorem; we also count the number of models omitting a given family of types. Let \(\text{covK}\) be the cardinal used in [?, Theorem 3.3.4]. We deal also with the uncountable cardinal \(p \leq 2^{\omega_1}\) satisfying: If \(\lambda < p\), and \((A_i : i < \lambda)\) is a family of meager subsets of a Polish space \(X\) (of which Stone spaces of countable Boolean algebras are examples) then \(\bigcup_{i \in \lambda} A_i\) is meager. It is consistent that \(\omega < p < \text{covK} \leq 2^{\omega_1}\) [?], so that the two cardinals are generally different, but it is also consistent that they are equal. Martin’s axiom implies that both cardinals coincide with \(2^{\omega_1}\).

**Theorem 6.** Suppose \(T\) is a first order complete theory in a countable language with equality.

1. (Morley) If \(T\) has more than \(\omega_1\) countable models, then it has \(2^{\omega_1}\) countable models. The same statement holds for theories not necessarily complete, in countable languages with or without equality.

2. If \((\Gamma_i : i < \omega)\) be a family of non-isolated types, then the number of non isomorphic countable models, omitting this family, is either \(\omega\), \(\omega_1\) or \(2^{\omega_1}\).

**Proof.** Let \(T\) be a first order theory in a countable language with equality, and let \(\mathfrak{A} = \mathfrak{M}_T\). Then \(S_{\omega_1}\) is a Polish group with respect to composition of functions and the topology it inherits from the Baire space \(\omega_1\). Consider the map \(J : S_{\omega_1} \times \mathcal{H}(\mathfrak{A}) \rightarrow \mathcal{H}(\mathfrak{A})\) defined by \(J(\rho, F) = s_\rho^+ F\) for all \(\rho \in S_{\omega_1}, F \in \mathcal{H}(\mathfrak{A})\). Then \(J\) is a well defined action of \(S_{\omega_1}\) on \(\mathcal{H}(\mathfrak{A})\). Also \(J\) is a continuous map from \(S_{\omega_1} \times \mathcal{H}(\mathfrak{A})\) (with the product topology) to \(\mathcal{H}(\mathfrak{A})\) because for an arbitrary \(a \in A, J^{-1}(N_a \cap \mathcal{H}(\mathfrak{A})) = \bigcup_{\tau \in S_{\omega_1}} \left(\{\mu^{-1} : \mu \in S_{\omega_1}, \mu|A_\tau = \tau|A_\tau\} \times \{N_{s_\rho^+ a} \cap \mathcal{H}(\mathfrak{A})\}\right)\).

To see why, let \(f : S_{\omega_1} \rightarrow S_{\omega_1}\) be the map given by \(f(\tau) = \tau^{-1}\). Observe that \(f\) is continuous and open. Hence, \(\{\mu^{-1} : \mu \in S_{\omega_1}, \mu|A_\tau = \tau|A_\tau\} = J^*(\{\mu \in S_{\omega_1} : \mu|A_\tau = \tau|A_\tau\})\) is the image of an open set via an open map, and is therefore open. Now, let \((\rho, F)\) be an arbitrary element in \(S_{\omega_1} \times \mathcal{H}(\mathfrak{A})\). We have the following:

\[
(\rho, F) \in J(\rho, F) = s_\rho^+ F \in N_a \cap \mathcal{H}(\mathfrak{A}) \quad \iff \quad (\rho, F) \in J^{-1}(N_a \cap \mathcal{H}(\mathfrak{A})).
\]

It follows that the the orbit equivalence relation is analytic [7, Theorem 3.2]. By Burgess’ Theorem [?] if there are more than \(\omega_1\) orbits, then there are \(2^{\omega_1}\) orbits. But the number of orbits here
is exactly the number of non-isomorphic countably infinite models of $T$. This completes the
proof. For the part on omitting types, set $\mathcal{H}_{\text{omit}} = \mathcal{H}(\emptyset) \cap \bigcap_{n \in \omega} \bigcup_{\phi \in \Gamma} N_{\emptyset}(\emptyset)$. Where $W = \{ \tau \in \emptyset : |i : \tau(i) \neq i < \omega \}$. Clearly, $\mathcal{H}_{\text{omit}}$ is $G_\emptyset$, so it is Polish. For the remaining part one uses locally finite QA$_{\omega_1}$s instead of Lf$_{\omega_1}$s.

Example 1. (i) Let $\Gamma$ be a countable theory. Then the number of non isomorphic models is
equal to the number of models omitting a given set of $< \lambda$ many types are the same $\iff$
$|\mathcal{H}(\emptyset)| = |\bigcup_{\phi \in \Gamma} N_{\emptyset}(\emptyset)|$. The next example shows that this may fail to happen:
Consider non standard models of arithmetic. $\mathbb{N}$ is an atomic model, which means that
the neat $n$-reduct of the locally finite cylindric algebra $\mathfrak{m}_T$ based on $T = \text{Th}(\mathbb{N})$ is atomic
for each $n$. For each $n \in \omega$, let $\Gamma_n$ be the set of co-atoms in the neat $n$-reduct. These are non-
principal types and a model $M$ omits them $\iff$ it is atomic, hence it is isomorphic to $\mathbb{N}$
because atomic models are unique. But Peano arithmetic is unstable, so it has $\omega^2$ many non-
isomorphic countable models (non-standard models of arithmetic). Another example exhibiting
the same phenomena: Let $T$ be the theory of algebraically closed fields of characteristic zero.
Then $T$ is stable and it has countably many non-isomorphic models; for each $\alpha \leq \omega$,
there is a model of transcendence degree $\alpha$ over the rationals. Take the types as above. In this
case the all subalgebras of the $n$-neat reducts are atomic. Then the the field of algebraic number
is the only countable model omitting this family of types. This is an atomic model. This
theory has also another countable $\omega$-saturated model, which is that of transcendence degree $\omega$.

(ii) There is a somewhat amusing Theorem of Vaught’s that says that a countable theory
cannot have exactly two models. We show that this is not the case when we require that the
constructed odes omit a given family of non-principal types. Take the language $L = \{ c_n : n \in \omega \}$. Then a model $M$ of $T$ is determined by how many extra elements it has, i.e. by $|\{ b \in M : b \neq c_n^M \}|$. So $T$ is $\aleph_0$ categorical and since $T$ has only infinite models it is complete. Also $T$ has
countably many non isomorphic models, $M_\alpha$ with $\alpha$ extra elements for $\alpha \leq \omega$. Consider the
$m$ type $\Gamma_m = \{ \emptyset \neq c_0 : v_0 \neq c_0^M : n \in \omega \}$. Then $\Gamma_m$ is non-
principal and it is omitted by exactly $m$ models namely $M_0, M_1, \ldots, M_{m-1}$. This can be generalized for complete strongly minimal theories which have countable models of
dimension $\alpha$, $\alpha \leq \omega$.

(iii) We show that there is a theory having exactly $\omega_1$ models omitting continuum many types.
Take the first order countable theory in the language $\{ <, c_0, c_1, \ldots \}$ where $<$ is a binary
relation symbols and the $c_i$‘s ($i \leq \omega$) are constants. Let $T$ be the $L$ theory which states that
$<$ is a linear order and that $c_i \neq c_j$ for $i \neq j$. Take $\Gamma_1 = \{ v_1 \neq c_i : i \in \omega \}$ and for every
injective $f \in \emptyset$, let $\Gamma_f = \{ c_{f(i)} > c_{f(i+1)} : i \in \omega \}$. Consider the set of non-principal types $\mathcal{G} = \{ \Gamma_1, \Gamma_f : f \in \emptyset \}$. Then a model $M$ omits $\mathcal{G} \iff$ it is a countable well order. The family $\mathcal{G}$ is uncountable. Making this family countable would violate Vaught’s conjecture in $L_{\emptyset, \omega}$. Indeed let $T$ be a countable theory and $\{ \Gamma_i : i \in \omega \}$ be a family of non-principal types
to each $\omega$ models. Then the $L_{\emptyset, \omega}$ sentence $\bigwedge T \land \forall \alpha \in \omega (\exists \alpha) (\emptyset \in \Gamma \land \emptyset \in F(i))$ violates
Vaught’s conjecture; for it has $\omega_1$ countable models.

1.2 The long and winding road from Hilbert’s problems in 1900
to axiomatizing general relativity in first order logic

In August 1900 the world’s best mathematicians gathered in Paris for the Second International
Congress of mathematicians. Among them was the 38 year old professor David Hilbert from the
University of Gottingen. As one of the leading mathematicians of the time, Hilbert was
due to give one of the keynote addresses to the meeting. The meeting was being held in the very first year of the twentieth century. Consequentially Hilbert chose to use his lecture not to look back over some recent work, but rather to point the way towards the future. He gave a dramatic address - an adress which is to have lasting fame and significance. He has already made fundamental contributions to algebra, number theory geometry and analysis. After 1900- he would expand his research further in analysis, then move on to mathematical physics and finally turn to mathematical logic. To emphasize this call he presented to the meeting not one but a list of twenty three major unsolved problems, problems whose solutions, if found, would each mark a significant advance in mathematics. These practically ran the gamut of the fields of mathematics of his day, from the pure to the applied, and from the most general to the more specific. The choice of these problems was to some extent subjective, restricted by Hilbert’s own knowledge and interests as broad as those were. But the work on some of them led to an extraordinary amount of important mathematics in the 20 century and beyond. In fact, certain mathematicians would become famous for solving one or another of Hilbert’s problems. Most of these problems were (or become) known by specific names as the continuum problem, or the Reimann problem (Reimann’s hypothesis). Fermat’s last Theorem also appeared on this famous list only to be solved almost a century after by the British mathematician Andrew Wiles. The problems 1,2, and 10 in his list adressed mathematical logic, better known back then as meta-mathematics, and Problem 6, on the list was simply ‘Axiomatize physics.’

Hilbert was initially a deductivist, but he considered certain metamathematical methods to yield intrinsically meaningful results and was a realist with respect to finitary arithmetic. Hilbert’s first was concerned with the continuum hypothesis $\text{CH}$ and the well ordering of $\mathbb{R}$. Hilbert’s second problem, called for a proof of consistency of arithmetical axioms, to secure the consistency of the whole of mathematics. In 1900 Hilbert was a bit vague in stating just which axioms he had in mind. But when he turned to logic in full scale in 1920s he made quite specific what axioms were to be considered. Not to beg the question he placed a strong restriction on the proof methods ‘finitary’ being one of them. In response to these questions: Gődel came up with Gődel’s incompleteness in 1931 (proved in $\text{ZF}$ without the axiom of infinity) so his proof conformed to Hilbert’s restrictions, Zermelo explicitly introduced the Axiom of Choice ($\text{AC}$) in 1918 giving the first axiomatization of set theory to show that any set $a \text{ fortiori} \mathbb{R}$ can be well ordered, answering part of the first question. Cohen proved the independence of the $\text{CH}$ and the $\text{AC}$ in 1963 answering the rest of the first question. Cohen’s proofs were the inaugural examples of a new technique, forcing, which was to become a remarkably general and flexible method for extending models of set theory. Gődel’s constructible universe had launched set theory as a distinctive field of mathematics, then Cohen’s forcing began its transformation into a modern sophisticated one. Hilbert’s tenth problem was solved in the 1970s by Yuri Matiyasevich preceded by substantial contributions from Martin Davis, Putnam, Julia Robinson. This was done by showing that Diophantine sets and recursively enumerable sets coincide on $\mathbb{Z}$ hence Hilbert’s tenth problem is algorithmically undecidable. Hilbert’s first was concerned with the continuum hypothesis $\text{CH}$ and the well ordering of $\mathbb{R}$. Hilbert’s second problem, called for a proof of consistensy of arithmetical axioms, to secure the consistency of the whole of mathematics. In 1900 Hilbert was a bit vague in stating just which axioms he had in mind. But when he turned to logic in full scale in 1920s he made quite specific what axioms were to be considered. Not to beg the question he placed a strong restriction on the proof methods ‘finitary’ being one of them. In response to these questions: Gődel came up with Gődel’s incompleteness in 1931 (proved in $\text{ZF}$ without the axiom of infinity) so his proof conformed to Hilbert’s restrictions, Zermelo explicitly introduced the Axiom of Choice ($\text{AC}$) in 1918 giving the first axiomatization of set theory to show that any set $a \text{ fortiori} \mathbb{R}$ can be well ordered, answering part of the first question. Cohen proved the independence of the $\text{CH}$
and the AC in 1963 answering the rest of the first question. Cohen’s proofs were the inaugural examples of a new technique, forcing, which was to become a remarkably general and flexible method for extending models of set theory. Hilbert’s tenth was finally solved in the 1970s by Yuri Matiyasevich preceded by substantial contributions from Martin Davis, Putnam, Julia Robinson. This was done by showing that Diophantine sets and recursively enumerable sets coincide on \( \mathbb{Z} \) hence Hilbert’s tenth problem is algorithmically undecidable [27]. It is worthy of note that the analogous problem for Diophantine equations on \( \mathbb{Q} \) is open [17].

Hilbert gave a second order many sorted axiomatization of Euclidean geometry. Tarski went further. In his 1926-27 lectures at the university of Warsaw, Tarski gave an axiomatic development of elementary Euclidean geometry in one sorted first order logic. He proved that his system of geometry admits elimination of quantifiers, the theory is complete and decidable. Later, he gave a first order axiomatization to different geometries whether Euclidean or not [25]. Suppes, and before him Reinenbach suggested that theories like the special theory of relativity whose mathematical model is Minkowski’s geometry can be, and indeed, should be formalized in first order logic. Andréka, Madárasz, and Németi applied this logistic viewpoint to axiomatizing Einstein’s general theory of relativity in first order logic. The investigations triggered off by Andréka and Németi to deal with Problem 6 in Hilbert’s list, are flourishing till the present day. Márásasz [16] gives an excellent lucid account of the Andréka and Németi axiomatic approach to the special theory of relativity, dealing also with so-called accelerating observers as an intermediate phase between special and general relativity. In her strictly first order axiomatization of special relativity, Márásasz deduces bizarre consequences demonstrated by Einstein such as masses increase, clocks run slower, measuring rods shrink when physical systems move at speeds close to that of light and simultaneity is frame dependent from simple and transparent axioms formulated in first order logic; these effects are not postulated \textit{a priori}.

**Challenging the Church-Turing Thesis using general relativity:** Gödel’s incompleteness Theorem implies that the concept of mathematical truth cannot be encapsulated in any formalistic scheme. Mathematical truth is something that goes beyond mere formalism. Gödel’s incompleteness theorems thus seemed to be a blow to the logistic view envisaged by Gottlob Frege. Gottlob Frege is regarded as the greatest philosopher of logic since Aristotle for developing his 1879 \textit{Begriffsschrift} establishing a logical foundation for arithmetic, and generally stimulating the analytic tradition in philosophy. The architect of that tradition is Bertrand Russell who in his early years influenced by Frege and Peano, wanted to found all of mathematics, on the certainty of logic. The vauling expression of that ambition was the 1910 three volumes \textit{Principia Mathematica} by Whitehead and Russell. But Russell was exercised by his well known paradox, one which led to the tottering of Frege’s mature formal system. It remained for Hilbert to shift the ground and establish mathematical logic as a field of mathematics. Russell’s philosophical disposition precluded his axiomatizing logic. The pioneer representative of logicism is Frege. Logicism, basically asserts that mathematics is reducible to logic and hence is nothing but a chapter of logic. Logicists hold that mathematics can be known a priori, but suggest that our knowledge of mathematics is just a part of our knowledge in logic, and thus is analytic, not requiring any special faculty of mathematical intuition. In this view logic is a proper foundation of mathematics, and all mathematical statements are necessary logical truths. Hilbert brought mathematical logic under scrutiny as he did Euclidean geometry by establishing the axiomatic context and raising the crucial questions of consistency and completeness. This largely syntactical approach was soon given a superstructure, when Hilbert developed proof theory and proposed his program establishing the consistency of classical mathematics with his metamathematics. However, Godel’s incompleteness theorems, knocked down Hilbert’s Program. But it remained to be seen whether Hilbert’s Consistency
Program is still viable in any way, either by restricting its scope or by somehow enlarging the methods of proof admitted. Taking into account the laws of the general theory of relativity, more specifically, the (infinite) time dilation in strong gravitational fields of (rotating) black holes, then Andréka and Németi show that one can imagine thought experiments in which Church thesis is no longer valid!

**Theorem 7.** *(Andréka and Németi).* It is consistent with Einstein’s equations that by certain kinds of relativistic experiments, future generations can find the answers to non-computable questions like the Halting problem, Hilbert’s tenth and the consistency of ZFC.

**Idea:** A slowly rotating black hole has two event horizons. The gravitational pull of such black holes grows without limit as one approaches the (outside) event horizon. Assume that two observers $H$ and $L$ are hovering over the outside event horizon, with $H$ being higher up. Then $L$’s clock runs slower than $H$’s clock since he is experiencing a stronger gravitational field (this is a prediction of general relativity). Moreover, as $L$ moves towards the horizon, this discrepancy between the ticking of both clocks gets larger and larger. In fact, by lowering $L$ appropriately, this time lag can be controlled. Now, if a programmer $P$ gets very close to the outside event horizon while leaving his computer $C$ higher up, then in a few days time relative to the programmer, the computer does a few million’s year’s job. Accordingly, we can reach an infinite speed up by lowering $P$ to the right position; hence breaking the Turing barrier. The rotation of the black hole induces a repelling effect (a centrifugal force in the language of Newtonian mechanics) that counter-balances the strong gravitational pull of the black hole. In this way, $L$ can slow down as desired without being crushed. It is possible for an observer $L$ to stay at a fixed distance from the center of the rotating black hole. In this way, the infinite time dilation speed-up of the computer $C$ safely outside the outer event horizon with respect to the programmer $P$, is accomplished. The creation of a computer that can compute tasks beyond the Turing limit can be achieved as follows. The programmer $P$ leaves earth in a spaceship towards a huge slowly rotating black hole. As $P$ is heading towards his target, $C$ checks one by one the theorems of set theory. If $C$ finds a contradiction, he sends a signal to $P$. Otherwise, he does nothing. From $C$’s point of view, as the programmer $P$ approaches the event horizon, his clock will be ticking slower and slower relative to $C$’s clock. At the limit, that is, when $P$ reaches the inner horizon, his clock freezes coming to a halt relative to $C$. From the point of view of $P$, however, the $C$’s clock appears to be running faster and faster. Moreover, assuming that the black hole is huge, $P$ will safely cross the inner event horizon. If $P$ receives a light signal from $C$, $P$ will know that $C$ has found an inconsistency in ZFC. Otherwise, $P$ concludes that ZFC is consistent. Because the black hole is huge, the center of the black hole is relatively far from the event horizon. More importantly, The matter content, that is, the singularity is not a point (as the case of a static black hole), but is actually a ring (one fascinating property of rotating black holes). This makes $P$ comfortably pass through the middle of the ring without being torn apart! We emphasize that the last Theorem is not Science fiction. It is a scientific thought experiment that becomes more plausible especially after the first black hole was actually seen!.

The two major paradigms of computing arising from new physics are quantum computing and general relativistic computing. Quantum computing challenges complexity barriers in computability, while general relativistic computing challenges the Physical Church Thesis (PhCT), namely, that recursiveness is the mathematical equivalence of computability. This was formulated and accepted in the 1930’s as being well supported by rigorous mathematics and common sense. But ‘common sense today’ means ‘physics a century ago’. Due to the major paradigm shift in our physical world view this thesis can now be challenged by thought experiments involving huge rotating black holes. Andréka and Németi’s (exotic) re-
sults, depending essentially on the non-absoluteness of time in general relativity, in the last theorem is a cry away from Newtonian philosophy and a counterblow to Gödel’s theorems on incompleteness and impossibility of proving ZFC consistent!

**Gödel’s incompleteness property in guarded decidable fragments:** For a class K of algebras, and a cardinal $\beta > 0$, $\exists \tau_\beta K$ stands for the $\beta$-generated free $K$ algebra. In particular, $\exists \tau_\beta \mathcal{C}A_n$ denotes the $\beta$-generated free cylindric algebra of dimension $n$. The following is known: If $\beta \geq \omega$, Pigozzi proved that $\exists \tau_\beta \mathcal{C}A_n$ is atomless (has no atoms) [9, Theorem 2.5.13]. Assume that $0 < \beta < \omega$. If $n < 2$ then $\exists \tau_\beta \mathcal{C}A_\omega$ is finite, hence atomic, [9, Theorem 2.5.3 (i)], $\exists \tau_\beta \mathcal{C}A_\omega$ is infinite but still atomic, a result of Henkin’s [9, Theorems 2.5.3 (ii), 4.7 (ii)]. If $3 \leq n < \omega$, Tarski proved that $\exists \tau_\beta \mathcal{C}A_n$ has infinitely many atoms [9, Theorem 2.5.9] and it was posed as an open question, cf [9, Problem 2.14] whether it is atomic or not. Recent research in algebraic logic has revealed however that some guarded fragments of first order logic, surviving undecidability and other undesirable properties, made contact with a deep and subtle algebraic reformulation of Gödel’s incompleteness theorem. This was proved first by Németi for the ‘unguarded’ $L_n (n \geq 3)$ with square Tarskian semantics by translating a form of Gödel’s incompleteness property to non-atomicity of the free algebras and then proving for any finite $m$, $\exists \tau_m \mathcal{C}A_n$ is not atomic. The key idea is that if $T$ is a finite consistent complete $L_n$ theory, then the equivalence class of $\bigwedge T$ will be an atom in the formula algebra of pure logic, built up of the symbols occurring in formulas in $T$. Now if one finds a formula that cannot be extended to such a theory then there will be no atoms below the equivalence class of this formula, and here is where Gödel’s Theorem intervenes. The idea in Német’s proof uses a translation function of $L_{\omega_1 \omega}$ into $L_3$ together with the pairing technique of Tarski’s suitably redefined to adapt the case of three variables rather than four; the latter being the natural habitat of relation algebras. Using similar but more sophisticated methods, Andréka and Németi later proved the analogous result for the $\exists \tau_\omega \mathcal{D}f_n$ free algebras, solving a long standing open problem in algebraic logic, posed by Tarski, Maddux, Németi and others. This was deduced from the fact that the whole of ZFC can be coded in $\mathcal{D}f_3$: in other words, $\mathcal{D}f_3$, which is substantially weaker than $\mathcal{C}A_3$ a fortiori strictly weaker than the calculus of relations is an adequate vehicle for the whole of mathematics. Tarski and Givant had formalized set theory in the calculus of relations establishing an intriguing ‘variable free’ approach to meta-mathematics. Their joint work in this fascinating topic is published in the monograph A formalization of Set Theory without variables. In op.cit it is shown that in principle mathematics can be developed in the very simple framework of equations and substitution of equals for equals rather than the customary basis using set theory formalized in first order logic, which is, to say the least, an impressive tour de force with profound metamathematical and philosophical repercussions. The first chapter of Andréka et al [1] gives an excellent account of these results. Mohamed Khaled [13], another student of Andréka and Németi, proved that the free $\mathcal{C}r_n$s with finitely many generators are not atomic and developed his proof to show that many guarded logics has a form of Gödel’s incompleteness property: There is a formula $\phi$ (in the signature of the guarded logic under investigation) that cannot be extended to a finite complete recursive theory. Such ‘incompletable formulas are called inseparable formulas by Németi. This solves another long standing open problem in algebraic logic posed by Németi in the early eighties of the last century. This, in turn, shows that after all ‘the taken for granted’ implication ‘Gödel’s incompleteness $\implies$ undecidability’ does not always hold. Results of this kind are open to huge philosophical considerations, revisions and repercussions, and are far from being fully understood. Indeed such results tend to raise more questions than answers. The question that bears a lot of discussion and reflection in this connection is how faithfull the algebraic translation to non-atomicity of the free algebras, vis a vis the in built dependence of the proof of Gödel’s incompleteness Theorem in Peano arithmetic (without the axiom of infinity) using
Gödel numbering. Is Gödel’s celebrated incompleteness theorem intrinsic to arithmetic (and richer formal systems like set theory), or can it lend itself to different, possibly more general frameworks? Is the idea of Gödel numbering—mirroring statements about numbers to statements about other statements of numbers, possibly themselves-applicable only to the entities known as natural numbers? The two ingenious components in Gödel’s proof are diagonalization and self-reference. Such methods and antimonies were known before Gödel. Diagonalization is implemented in Cantor’s proof of the uncountability of \( \mathbb{R} \) and self reference appeared (philosophically) with the liar paradox, later getting a more mathematical manifestation in the famous hugely influential Russell’s paradox with several scattered re-incarnations in interdisciplinary literature between mathematics, logic and philosophy [17]. But combining the two is certainly a master stroke proving one of the most important Theorems in mathematics in the 20th century and beyond. These two ingredients of Gödel’s proof simply vanish in the algebraic proofs for \( Cr_n \) proving the non-atomicity of their finitely generated free algebras. The new proofs use an ingenious purely algebraic method [14]. Non-atomicity of finitely generated free algebras are also proved for \( D_n \) and \( G_n \) [15]. The consistency of a form of Gödel’s incompleteness theorem and decidability (for guarded fragments of \( L_0 \)) is certainly an exciting and a telling co-existence.

**From the Vienna circle to The Budapest Group** A great influence on the philosophy of Science in the early 20th century that ultimately led to the formalist view is the school of Logic positivism in philosophy. Logical positivism also known as logical empiricism or logical neo-positivism was a philosophical movement risen in Austria and Germany in the 1920’s, primarily concerned with the logical analysis of scientific knowledge, which affirmed that statements about metaphysics, religion and ethics are void of cognitive reasoning; only statements in mathematics, logic and natural science have a definitive meaning. The chief influences on the early logical positivists were Rudolf Carnap. The advocates of logical positivism including at one point of time Bertrand Russell and Albert Einstein became known as the Vienna circle. Ernst Mach and Ludwig Wittgenstein. Wittgenstein’s *Tractatus Logico Philopolicus* introduced the conception of philosophy as a critique of language. Leibniz taught us through his theory of so-called *monads* to reject any reference to *a-priori* and immutable structure, such as Newton’s absolute space and time. But he did not tell us what to replace them with. Mach did, for he showed us that every use of such an absolute entity hides an implicit reference to something real and tangible that has so far been left out of the picture. What we feel pushing against us when we accelerate cannot be absolute space, for there is no such thing. It must somehow be the whole of the matter of the universe. Einstein took a third step in the transformation from an absolute to a relational conception of space and time. In this step, the absolute elements, identified by Mach as the distance galaxies, are tied into an interwoven, dynamical cosmos. The final result is that the geometry of space and time - which was for Newton absolute and eternal - became dynamical, contingent and lawful. Mach’s philosophy well prepared for Einstein’s (non-Newtonian) relativistic philosophy. In his special theory of relativity, Einstein was saying that the *mathematical structure* of a physical law must *not change* as we go from one observer to the other. Laws of nature have the same form relative to all inertial observers. Space is different for different (inertial) observers, time is different for different (inertial) observers, but *spacetime is the same for all (inertial) observers*. In general relativity, Einstein’s field equations say that *matter curves spacetime and spacetime tells matter how to move*. One must only imagine the experience of falling and recall that those who fall have no sensation of weight. In the hands of Einstein, this everyday fact became the opening to a profound shift in our way of understanding the world: while you can abolish the effects of gravity locally, by freely falling, this can never be done over a large region of spacetime. Therefore, while curved space(time) can be approximated by a patchwork of small
flat regions, these regions will always have discontinuities where we try to join them at their edges. This could be taken to mean that the overall space is curved. The very fact of this failure to join smoothly is the curvature of space. Einstein has specified the mechanism by which gravity is transmitted: the wrapping of spacetime. Einstein tells us that the gravitational pull holding the earth in orbit is not, as Newton claimed, a mysterious instantaneous action of the sun; rather, it is the wrapping of the spatial fabric induced by the sun. This dynamic relation between matter and geometry; this feedback loop, is coded in the following very elegant (and simple!) equation $G_{ab} = 8\pi T_{ab}$. On the right, stands the source of curvature, namely, the energy-momentum tensor. On the left, stands the receptacle of curvature in the form of what one wants to know, the metric coefficients twice differentiated (the Einstein tensor). The matter distribution $T_{ab}$ determines the geometry $G_{ab}$, and hence is a source of inertial effects. When Einstein had created his general theory of relativity, he is supposed to have said that while the left hand side had been curved in marble, the right hand side was built out of straw. The left hand side of Einstein’s equations (refering to the actual geometry of spacetime) is surely one of the great insights of science. The right hand side (describing how the mass and energy produces this curvature) did not follow with such elegance as the geometric part of the field equations. The field equations show how the stress energy of matter generates an average curvature in its neighbourhood. It governs the external spacetime curvature of a static and dynamic source, the generation of gravitational waves (ripples in the curvature of spacetime) by stress energy in motion, the external spacetime geometry of a (static and rotating) black hole and, last but not least, the expansion and the contraction of the universe. It is fair to say that all modern theories in physics nowadays, namely, string theory, Penrose’s twistor geometry are attempts to fully understand the right hand side of Einstein’s field equations [19]!

Einstein’s theory of relativity exerted a great influence over the origin of positivism. A scientific theory, according to positivism, is an axiomatic system which acquires an empirical interpretation from suitable statements, called co-ordinate definitions, which establish a correlation between real objects or processes and the abstract concepts of the theory. Pragmatic aspects of scientific research were usually dismissed by positivists, who were not interested in the real process of discovering, in so much as they were concerned with rational reconstruction of scientific knowlege, that is the study of the logical relationships between statements, hypothesis and empirical evidence. Positivists were interested in clarifying the philosophical significance of the theory of relativity. Schlick wrote in 1915 and 1917 two essays on relativity. Reinbakh (who attended Einstein’s lectures on theory of relativity in Berlin University in 1917), wrote several books on relativity. These writings largely influenced Suppes, Tarski and much later Andrêka and Németi, in their axiomatic (formalist) approach to new physics. Axiomatizations offered by the Budapest group are based on quite novel primitive concepts such as the world view relation connecting different observers bodies and coordinates, which they have shown to be capable of encapsulating the entire structure and behaviour of spacetime arena, and subjecting it to intricate mathematical analysis reaching Einstein’s field’s equations in an absolutely breathtaking achievement. The insisting of using first order logic here is very useful and innovative, and allows the basic assumptions of relativity to be reduced of simple and transparent principles clarifying what the predictions of physical reality depend on. This is a very valuable contribution to knowledge in a subject whose concepts have been notoriously counterproductive and difficult to grasp. This programme is proving to be a very productive and unique research programme with great potential and highly realistic achievable valuable goals: the main one is to prove deep theorems of the special and the general theory of relativity from a small number of simple, easily understandable and convincing axioms. The underlying idea here is that general relativity can be defined (in the very broad sense of the word), or obtained, as patching together its local parts, which reflect the laws of special relativity, in this
way obtaining ultimately a manifold that represents or is a mathematical model for general relativity. Category theory intervenes here; one can view such a spacetime manifold as a colimit of its local ‘special relativistic’ parts [22, 23]. Locally spacetime is apparently flat but globally it is curved. The patching here involves subtle very deep elaborations on definability issues in logic. Indeed, in her dissertation [16] Judit Madarasz presents special relativity in light of a definability theory. Having established this methodology and demonstrated its effectiveness the research team now are extending it to new directions, such as the study of space time with rotating black holes and wormholes.

**Platonism versus formalism:** András and Németi are closer to being formalists in the sense that they are tolerant and inviting to new approaches in logic as long as the rule of the games are sound and the games are exciting and challenging. However, their motivations are drawn from existing mathematical or philosophical concerns, so the games are not completely arbitrary manipulation games. Gödel was a realist platonist who believed in an objective mathematical reality that could be perceived in a manner analogous to sense perceptions. A platonist insists upon the absolute, immutable nature of mathematics, it still has an a priori aspect. Many mathematicians have been mathematical realists: discovering naturally occurring objects like the Hungarian hugely prolific mathematician Paul Erdős. Paul Erdős envisaged that there is book written by God, in which beautiful proofs in mathematics are present, and he referred to beautiful proofs as proofs from the book. For example triangles and circles from the viewpoint of a platonist are real entities, not creations of the mind. Platonism is the form of realism that suggests that mathematical entities are abstract, have no spatiotemporal or causal properties, and are eternal and unchanging. This is often claimed to be the view most people have of natural numbers. According to extreme platonism, mathematical objects are real, real as any thing in the world we live in. For example infinite sets exist, not just as a mental construct, but in a real sense, perhaps in a hyperworld. Similarly uncountable sets exist, real numbers, choice functions, Lebesgue measure and the category of all categories. Since all of the mathematical objects are real, the job of a mathematician is empirical as that of a geologist or physicist; the mathematician looks at a special aspect of nature and tries to discover (not invent) some of the facts. For the platonist independence results are not about mathematics but rather about the formalism of mathematics. A Formalist on the other hand, does not believe that any mathematical objects have a real existence independent of himself. For him, mathematics is just a collection of axioms, theorems and formal proofs. Of course the activity of mathematics is not just randomly writing down formal proofs for random theorems. The choice of axioms, problems, of research directions are influenced by a variety of considerations, practical, artistic, mystical but really not mathematical. A formalist is not concerned with ‘what is’ but rather with ‘what would be if’. According to this view mathematics is a purely logical discipline and, like logic, is carried on entirely within the confines of language; it has nothing whatever to do with reality, or with pure intuition; on the contrary it deals exclusively with the use of signs or symbols. These signs or symbols can be used as we like, in conformity with rules that we have set. The only restriction on our freedom is that we may under no circumstances contradict the self established rules. The final criterion of mathematical existence thus becomes freedom from contradiction; that is to say, mathematical existence can be ascribed to every concept whose use does not enmesh us in contradiction. According to Formalism, mathematical truths are not about numbers and sets and the like- in fact they aren’t about anything at all. A formalist would be content with Cohen’s ‘independence solution’ of CH and AC; all resulting set theories are on equal footing from his viewpoint. For a formalist, there is no preference of ZF + ¬AC to ZFC + ¬CH, say. The formalist is indifferent to such inquiries concerning which set theory is ‘better’ and if so on what grounds. But a platonist would stipulate that AC is ‘true’, but CH is ‘false’ objectively; neither can be both in two different contradictory worlds.
existing among many, for there is only one ‘platonic immutable objective’ world where choice functions exist for any relation and $\omega_1 < 2^{\omega}$. A platonist would view such results not about the mathematics but rather about the formalism mathematics. Some leading mathematicians-who it seems are platonists at heart- such as Woodin and Shelah argue that Cohen’s solution is not final, trying to settle the CH one way or another by modifying the axioms of Set Theory introducing so-called determinacy axioms (like projective determinacy) that turn out equiconsistent with the existence of certain large large cardinals like the so-called super compact ones. There is no obvious and compelling unique path of axioms that supplemented ZFC and settle important independent problems. But grossemode in current set theoretic landscape two opposing theoretic approaches dominate to solve the continuum problem and its likes. The most complete and concise of these two stems from Gødel’s constructible universe $L$. Gødel showed that CH and AC hold in $L$, and in fact $V = L$ answers most (if not all) outstanding questions concerning sets. The second theory begins from a style of axioms for which Gødel had high hopes namely large cardinal axioms. The first of those large cardinals are the inaccessible cardinals. The idea is that the two main operations for generating new sets from old postulated by ZFC replacement and power set are not enough to exhaust all ordinals. This is ripe for generalization. The heuristic of generalization from $\omega$ also came to be used to motivate various large cardinals. Recalling Cantor’s unitary view of the finite and the transfinite large cardinal properties satisfied by $\omega$ would be too accidental if they were not also ascribable to higher cardinals in an eternal recurrence. A cardinal $\kappa$ is inaccessible if it is uncountable, it is not a sum of fewer than $\kappa$ cardinals that are less than $\kappa$, and $\alpha < \kappa \Rightarrow 2^{\alpha} < \kappa$. A large cardinal is necessarily inaccessible. Large cardinal axioms cannot be proved from ZFC, because if they can, and $\kappa$ is such, then the universe of sets $V_\kappa$ will be a model of ZFC contradicting Gødel’s incompleteness Theorems. in fact the following cannot be proved: If $\kappa$ is a large cardinal, then “consistency of ZFC implies consistency of ZFC + the existence of this large cardinal” is consistent. Solovay showed that the axiom “there exists an inaccessible cardinal” leads to that every set of reals is measurable, and subsequently Shelah proved the converse. There are other even larger types of cardinals: Mahlo, weakly compact, ineffable, measurable, Ramsey supercompact, huge, $n$ huge and Woodin cardinals. The existence of a measurable cardinal (which is a large large cardinal) contradicts $V = L$, a result due to Dana Scott. The assumption of the existence of certain large large cardinals, imply certain determinacy axioms. The so-called full Axiom of determinacy AD refers to certain two-person topological games of length $\omega$. AD states that every game of a certain type is determined. The axiom of determinacy is inconsistent with AC, and it implies implies that all subsets of the real numbers are Lebesgue measurable, have the property of Baire, and the perfect set property. The last implies a weak form of the CH, namely, that every uncountable set of real numbers has the same cardinality as $\mathbb{R}$. The axiom of projective determinacy (PD) states that for any two-player infinite game of perfect information of length in which the players play natural numbers, if the victory set (for either player, since the projective sets are closed under complementation) is projective, then one player or the other has a winning strategy, so that such game are not fully deterministic. The axiom is not a theorem of ZFC (assuming ZFC is consistent), but unlike the so-called full axiom of determinacy which is fully deterministic as the name might suggest and contradicts the axiom of choice, it is not known to be inconsistent with ZFC. PD follows from certain large cardinal axioms, such as the existence of infinitely many Woodin cardinals or the existence of a super compact cardinal. The proofs of determinacy from large cardinal assumptions are not related by implication but rather by relative consistency strengths. There appears to be a central axis of axioms to which all independent propositions are comparable in consistency strength. This axis is delineated by large cardinal axioms. There are no known counterexamples to this behaviour. Such results seem to support the formalist view.
It frequently happens that controversy arises in a particular physical problem, due to undefined basic concepts. In Roger Penrose’s *The Emperor’s new mind* the central thesis is that nature produces essentially harnessable non-computable processes, but at the quantum level only; and Penrose goes on to speculate that human intelligence may be such a process. Like Roger Penrose, András and Németi are also concerned with questions concerning artificial intelligence; can we devise a machine that ”thinks”? Godel’s incompleteness theorem seems to tell us that Turing machines cannot. However, the recent work of András and Németi in Theorem 7 says that (certain relativistic) machines can compute non-computable functions if we are willing to replace the underlying Newtonian physics by the general theory of relativity; surely not an unreasonable; in fact a highly plausible, thing to do. To understand the human mind, conciousness and intelligence, we need new insights in both physics and mathematics. The huge project initiated by András and Németi in the last two decades provides just that! If Hilbert was to re-appear in a third meeting of the International Congress of Mathematicians in the year 2020 say, he would most likely have to take cognizance of the unfolding momentous work of András and Németi in mathematical logic, computer science, artificial intelligence, hypercomputation and general relativity.

References

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