Splitting methods in algebraic logic in connection to non–finite axiomatizability, non-atom–canonicity and non-first order definability

Tarek Sayed Ahmed
Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt.

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Abstract . We deal with various splitting methods in algebraic logic. The word ‘splitting’ refers to splitting some of the atoms in a given relation or cylindric algebra each into one or more subatoms obtaining a bigger algebra, where the number of subatoms obtained after splitting is adjusted for a certain combinatorial purpose. This number (of subatoms) can be an infinite cardinal. The idea originates with Leon Henkin. Splitting methods existing in a scattered form in the literature, possibly under different names, proved useful in obtaining (negative) results on non–finite axiomatizability, non-atom canonicity, and non-first order definability for various classes of relation and cylindric algebras. In a unified framework, we give several known and new examples of each. Our framework covers Monk’s splitting, Andréka’s splitting, and, also, so-called blow up and blur constructions involving splitting (atoms) in finite Monk–like algebras and rainbow algebras.

1 Introduction
Fix $2 < n < \omega$. We assume familiarity with the basic notions of the (duality) theory of Boolean algebras with operators BAOs, like atom structures and complex algebras. A good reference is [17, §2.5, §2.6, §2.7]. Classes of algebras considered throughout this paper, like relation algebras (RA) and cylindric algebras of dimension $n$ ($\text{CA}_n$), $n$ any ordinal, are BAOs endowed with a semantical notion of representability. For any such class $L$ we write $RL$ for the class of representable algebras in $L$. For example RRA is the class of representable RAs, and $\text{RCA}_n$ is the class of representable $\text{CA}_n$s.

The idea of splitting one or more atoms in an algebra to get a (bigger) superalgebra tailored to a certain purpose seems to originate with Henkin [13, p.378, footnote 1]. Let $2 < n < \omega$. Monk proved his seminal result that $\text{RCA}_n$ is not finitely axiomatizable by constructing finite non–representable algebras (referred to together with variations thereof in the literature as Monk–like algebras), whose ultraproduct is representable. The idea involved in the construction of a non–representable finite Monk (–like) $\mathfrak{A} \in \text{CA}_n$s is not so hard. Such $\mathfrak{A}$ is finite, hence atomic, more precisely its Boolean reduct is atomic. The algebra $\mathfrak{A}$ is obtained by splitting some atoms in a finite $\text{CA}_n$ each into one or more subatoms. The new atoms are given colours, and cylindrifications and diagonals are re-defined by stating that monochromatic triangles are inconsistent. If the atoms resulting after splitting are ‘enough’, that is, a Monk’s algebra has many more atoms than colours, it follows by using a fairly standard form of Ramsey’s Theorem that any

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representation of $\mathfrak{A}$ will contain a monochromatic triangle, so $\mathfrak{A}$, by definition, cannot be representable.

Maddux modified Monk’s algebras using $\mathbb{C}A_m$s based on atomic relation algebras generated by a single element. This improvement is highly rewarding for it transfers incompleteness theorems for $n$–variable first order logic from languages having countably many $n$–ary relation symbols to incompleteness theorems for languages in a signature having only one binary relation symbol. Furthermore, the modification is far from being trivial for one generation makes the automorphism group of the algebra rigid.

In the cylindric paradigm, Andréka modified such splitting methods re-inventing (Andrěka’s) splitting. In this new setting, Andrěka proved a plethora of relative non-finite axiomatizability results in the following sense.

Let $\mathbb{K}$ be a variety having signature $t$, and let $\mathbb{V}$ be a variety having signature $t' \subseteq t$, such that if $\mathfrak{A} \in \mathbb{K}$ then the reduct of $\mathfrak{A}$ obtained by discarding the operations in $t \sim t'$, $\mathfrak{A}_{t'} \mathfrak{A}$ for short, is in $\mathbb{V}$. We say that a set of first order formulas $\Sigma$ in the signature $t$ axiomatizes $\mathbb{K}$ over $\mathbb{V}$, if for any algebra $\mathfrak{A}$ in the signature $t$ whenever $\mathfrak{A} \models \Sigma$ and $\mathfrak{A}_{t'} \mathfrak{A} \in \mathbb{V}$, then $\mathfrak{A} \in \mathbb{K}$. This means that $\Sigma$ ‘captures’ the properties of the operations in $t \sim t'$. A relative non–finite axiomatizability result is of the form: There is no set ‘of a special form’ of first order formulas satisfying a ‘finitary condition’ that axiomatizes $\mathbb{K}$ over $\mathbb{V}$. Such special forms may be equations, or universal formulas. By finitary, we exclusively mean that $\Sigma$ is finite (this makes no sense if the signature at hand is infinite), or $\Sigma$ is a finite schema in the sense of Monk’s schema [13, Definition 5.6.11-5.6.12], or $\Sigma$ contains only finitely many variables. The last two cases apply equally well to varieties having infinite signature like $\mathbb{RCA}_\omega$. In the last case a finite schema is understood in the sense of [13, Definition 4.1.4] namely, in a two-sorted sense, one sort for ordinals $< \omega$, the other sort for the usual first order situation. finitely many variables. For example Andrěka proved that $\mathbb{RQE}A_n$ is not finitely axiomatizable over $\mathbb{RQA}_n$ nor $\mathbb{RCA}_n$. In the former case Andrěka went further excluding universal axiomatizations containing only finitely many variables, a result that we lift to the transfinite below.

In Andrěka’s splitting one typically splits an atom in a given representable (set) algebra to finitely many subatoms $m$ say, obtaining a bigger non–representable superalgebra $\mathfrak{A}_m$ whose subalgebras generated by $< m$ elements are representable, and various proper reducts of $\mathfrak{A}_m$ remain representable. Iterating such a splitting argument for $m$, with $m$ getting arbitrarily large, excludes universal finite variable axiomatizations of the variety of representable algebras at hand, over such strict subreducts.

Here, besides dealing with a variant of Andrěka’ splitting, we present other constructions such as the so–called blow up and blur constructions which are splitting arguments at heart involving splitting (atoms) in finite Monk–like algebras and rainbow algebras. This methods proves useful on obtaining results on non–atom canonicity of varieties of relation and cylindric algebras. We recall that a class $L$ of $\mathbb{BAO}$s is atom canonical if whenever $\mathfrak{A} \in L$ is completey additive, then its Dedekind-MacNeille completion, namely, the complex algebra of its atom structure (in symbols $\mathbb{CmAt}\mathfrak{A}$) is also in $L$.

Now some (not necessarily all) of the atoms in a finite algebra $\mathfrak{D}$ outside a given class $\mathbb{K} \supseteq \mathbb{RL}$, $\mathbb{L} \in \{\mathbb{RA, CA}_n\}$, where $\mathbb{K}$ is required to be closed under subalgebras, are split each to infinitely many subatoms obtaining an atom structure $\mathfrak{At}$ for which there is an atomic $\mathfrak{A} \in \mathbb{RL}$ such that the atom structure of $\mathfrak{A}$ coincides with $\mathfrak{At}$. In this case $\mathfrak{At}$ is called weakly representable. So here by splitting the atoms we pass from a given finite non–representable algebra to an infinite atomic (weakly) representable one. Showing that $\mathfrak{D}$ embeds into $\mathbb{CmAt}$, it follows that although $\mathfrak{A}$ is representable its Dedekind-MacNeille completion, namely, $\mathbb{CmAt}$ is outside $\mathbb{K}$ (because $\mathbb{K}$ is closed under forming subalgebras), least representable, hence $\mathbb{K}$ is not atom–canonical. In
particular, At is not strongly representable; there is a non−representable atomic algebra whose atom structure coincides with At, namely, CmAt. We conclude that there is an atomic algebra outside K having a dense representable subalgebra. This method will be applied to K, when K is any of the varieties SRAi, l ≥ 5 and SNrCA_n+k and k ≥ 3 showing that they are not atom−canonical.

As another application to splitting methods for both cylindric and relation algebras, together with a rainbow construction, we also prove the following new results. Let CRCA_n denote the class of completely representable CA_n s, CRRA denote the class of completely representable RA s, Sc denote the operation of forming complete subalgebras, and Sd denote the operation of forming dense subalgebras. We show that any class K such that Nr_nCA_n ∩ CRCA_n ⊆ K ⊆ Sc Nr_nCA_n+3, K is not elementary, and that for any class L such that Sd RaCA_5 ∩ CRRA ⊆ L ⊆ Sc RaCA_5, L is not elementary.

Furthermore, the construction used in all constructions in this paper involves a variation on the following single theme typical of Andrêka’s splitting:

Split some (possibly all) atoms in an algebra (that need not be finite nor even atomic) each into one or more subatoms forming a bigger superalgebra that constitutes the starting point for a construction serving the purpose at hand. For cylindric−like algebras of dimension α, α an ordinal, if the atom x ∈ A is split to the subatoms X = (x_i : i ∈ I) in B ⊇ A, then X is a partition of x in the sense that x_l x_m = 0 for l ≠ m ∈ I, and \(\sum_{i∈ I} x_i = x\). Furthermore, for each i ∈ I, x_i is cylindrically equivalent to x, in the sense that for all j < α, c^B_j x_i = c^B_j x.

This roughly means that cylindric algebras of dimension α, α an ordinal, if the atom x ∈ A is split to the subatoms X = (x_i : i ∈ I) in B ⊇ A, then X is a partition of x in the sense that x_l x_m = 0 for l ≠ m ∈ I, and \(\sum_{i∈ I} x_i = x\). Furthermore, for each i ∈ I, x_i is cylindrically equivalent to x, in the sense that for all j < α, c^B_j x_i = c^B_j x.

Henceforth, we follow the notation of [3] which is in conformity with the notation of [13]. We deal with the following cylindric−like algebras Sc (Pinter’s substitution algebras), CA, QA(QEA) quasi−polyadic (equality) algebras. For K any of these classes and α any ordinal, we write K_α for the variety of α−dimensional K algebras, and (C)RK_α for the class of (completely) representable K_α s. By the same token, while RA denotes the class of relation algebras, (C)RRA will denote the class of (completely) representable RAs.

Layout

Fix 2 < n < ω.

• In §2 we modify Andrêka’s splitting to show that RQEA_n is not axiomatizable over its diagonal free reduct by universal formulas containing finitely many variables. Several related relative non−finite axiomatizability results are obtained, theorem 2.1, corollary 2.2.

• In §3 several blow up and blur constructions for relation and cylindric algebras are presented to show non-atom−canonicity of the varieties mentioned in the in-
troduction, namely, $\text{SRaCA}_m$ and $\text{SNr}_n\text{CA}_{n+k}$, $m \geq 5$ and $k \geq 3$, theorems 3.1, 3.2, and 3.4.

- In §4 a splitting argument conjuncted with rainbow constructions are used to show that several classes related to $\text{CRCA}_n$ and $\text{CRRA}$ defined via the operators $\text{Nr}_n$ and $\text{Ra}$ (applied to $\text{CA}$s of higher dimension as given in the introduction) are not elementary, theorem 4.1. The result in [16] for relation algebras is strengthened by showing that the class $\text{CRRA}$ is not closed under $\equiv_{\infty, \omega}$, theorem 4.10.

- In §5, the class $\text{CRCA}_n$ is characterized syntactically using the operator $\text{Nr}_n$, theorem 5.1. Results in §2 and §3 are reformulated in semantical form in terms of $m$-relativized (complete) representations, for $2 < n < m \leq \omega$. An $m$-relativized representation is a locally well behaved representation where the ‘locality’ is measured by the parameter $m$; we deal (following terminology introduced by Hirsch and Hodkinson for relation algebras) with so-called $m$-square and $m$-flat representations. Complete $m$-relativized representations are characterized via existence of atomic $m$-dilations and atomic games by restricting the number of nodes used in play to $m$ nodes. Several (counter) examples are given marking the boundaries of ‘characterization results’ obtained in theorem 5.1, theorem 5.8. Various notions of (ordinary) representability formulated for atom structures, like ‘satisfying so-called Lyndon conditions’, ‘strongly representable atom structures’, are lifted from atom structures to atomic algebras, and reformulated for relativized $m$-representations, theorems, 5.14, 5.15 and 5.16.

- In §6 metalogical applications to the clique guarded fragments of $n$–variable first order logic, with respect to clique guarded semantics provided by $m$-square and $m$-flat models ($2 < n < m \leq \omega$) are given, theorems ??, 6.1.

- In §7 constructions related to the construction providing the solution of the neat embedding problem [13, Problem 2.12], presented in [20], as well as in [21], are reviewed in the context of splitting atoms in finite relation algebras, theorems 7.1, 7.2, 7.4. Furthermore a conditional omitting types analogue of [13, Problem 2.12] is worked out, theorem 7.6.

2 Non–finite axiomatizability: A variation on Andréka’s (most famous) splitting:

We split an atom in a representable algebra to finitely many subatoms getting a non–representable algebra whose ‘small subalgebras’ are representable and several proper reducts of which are representable. For a start fix the dimension to be $\omega$. For $\text{CA}_\omega$, this method typically shows, that for any positive positive $k$, there is an a non–representable algebra having $\text{CA}_\omega$ signature, such that all its $k$–generated subalgebras are representable, where by a $k$–generated subalgebra we mean a subalgebra generated by at most $k$ elements. Such an algebra is obtained by splitting an atom in a set algebra into finitely many subatom. From such constructions relative non–finite axiomatizability concerning number of variables used in universal axiomatizations can be easily proved. We let $\mathcal{R}_\omega$ denote ‘cylindric reduct’ and $\mathcal{R}_\omega$ denote ‘quasi–polyadic reduct.’

Theorem 2.1. The variety $\text{RQEA}_\omega$ cannot be axiomatized by a set of universal formulas containing finitely many variables over $\text{RQA}_\omega$. 

Proof. The proof is an instance of Andréka’s splitting [2]. For each positive integer \( k \) we construct \( \mathfrak{A} \notin \mathcal{RQE}_\omega \) such that its \( k \)-subalgebras (subalgebras generated by at most \( k \)-elements) are in \( \mathcal{RQE}_\omega \), \( \mathcal{Rd}_\omega \mathfrak{A} \notin \mathcal{RCA}_\omega \) and \( \mathcal{Rd}_\omega \mathfrak{A} \notin \mathcal{RQA}_\omega \). But for fixed \( k \) not only one splitting is done, but infinitely many each (to an atom) in a different set algebra; the resulting algebras (obtained after splitting) form a chain; their directed union will be the \( \mathfrak{A} \) we want. This can (and will) be done for each positive \( k \). Accordingly, throughout the proof fix a positive \( k \).

(1) Splitting a single atom to finitely many subatoms getting non–representable algebras from representable ones: For fixed \( 2 < n < \omega \), take a finite \( m \geq 2^{k \times n! + 1} \).

Suppose that the signature consists of \( \omega \)-many cylindric function symbols, \( c_i : i < \omega \), diagonal constants \( d_{ij}, i < j < \omega \), and \( n^2 - n \) substitutions \( s_{[ij]} : i < j < n \). One forms, for each such \( n \), an algebra \( \operatorname{split}(\mathfrak{A}_n, R, m) \) in this specified signature, by splitting the \( \omega \)-ary relation \( R = \prod_{i < \omega} U_i \) with \( U_0 = m - 1 \) and \( |U_i| = m \) for \( 0 < i < \omega \) in the algebra \( \mathfrak{A}_n = \Sigma \{ \varphi^{(\omega \omega)}(U) \} \{ R \} \), where \( U = \bigcup_{i \in I} U_i \) into \( m \) abstract copies. Observe that here \( R \) depends on \( n \), because \( m \) depends on \( n \) and \( R \) depends on \( U_0 = m - 1 \). The resulting algebra \( \operatorname{split}(\mathfrak{A}_n, R, m) \) therefore has the signature expanding \( \mathcal{CA}_{\omega} \) by the finitely many substitution operators \( s_{[ij]}, i < j < n \). Here the set algebra \( \varphi^{(\omega \omega)}(U) \) is taken in the specified signature with operations interpreted the usual way as in set algebras, e.g. \( \Sigma_{S, \{ R \}} \{ R \} \) = \{ \{ s \in U : \{ \{ s, s \} \} \in R \} \}. It can be easily checked that for all \( i < j < n \), \( s_{[ij]} \) is an atom in \( \mathfrak{A}_n \). In particular, \( R \) is partitioned into a family \( \{ R_i : i < m \} \) of atoms in the bigger algebra \( \operatorname{split}(\mathfrak{A}_n, R, m) \{ \geq \mathfrak{A}_n \} \), so that \( \mathfrak{A}_n \subseteq \operatorname{split}(\mathfrak{A}_n, R, m) \). Setting \( R = \sum_{i < m} R_i \), where \( m = |U_0| + 1 \). Furthermore in \( \operatorname{split}(\mathfrak{A}_n, R, m) \), we have \( s_{[ij]} R = \sum_{i < m} s_{[ij]} R_i \) and \( c_{s_{[ij]} R} = c_{s_{[ij]} R} \) for all \( i, j < n, i < m \) and \( t < \omega \); so that, in particular, \( R \) is cylindrically equivalent to its abstract copies. The algebra \( \operatorname{split}(\mathfrak{A}_n, R, m) \) is determined uniquely (up to isomorphism) by \( \mathfrak{A}_n, R, m \), hence the notation, and it will not be representable. Even more, the algebra \( \mathfrak{Rd}_\omega \operatorname{split}(\mathfrak{A}_n, R, m) \) having \( \mathcal{CA}_{\omega} \) signature will not be representable for the following reasoning: One defines the term \( \tau(x) = (\bigwedge_{i < m} s_{[ij]} x) \cdot \bigwedge_{i < j < n} d_{ij} \) as in [2, Top of p.157]. Then \( \mathfrak{A}_n \models \tau(R) = 0 \) hence \( \operatorname{split}(\mathfrak{A}_n, R, m) \models \tau(R) = 0 \) because \( \mathfrak{A}_n \subseteq \operatorname{split}(\mathfrak{A}_n, R, m) \). Identifying set algebras with their domain, for an algebra \( \mathfrak{A} \) and a non–zero \( a \in \mathfrak{A} \), we say that a representation \( h : \mathfrak{A} \rightarrow \varphi^{(\omega \omega)}(U) \) respects the non–zero element \( a \) if \( h(a) \neq \emptyset \). If \( \operatorname{split}(\mathfrak{A}_n, R, m) \) are representable, then it will have a representation that respects \( R \). But any such representation \( h \) will satisfy that \( \tau(h(R)) \neq 0 \) which is impossible.

(2) Representability of \( k \)-generated subalgebras: Now we show that the \( k \)-generated subalgebras are representable. Let \( G \subseteq \operatorname{split}(\mathfrak{A}_n, R, m) \), \|G\| \leq k \). Let \( \mathfrak{P} = \{ R_l : l < m \} \) be the abstract partition of \( R \) in the bigger algebra \( \operatorname{split}(\mathfrak{A}_n, R, m) \) obtained by splitting \( R \) in \( \mathfrak{A}_n \) into \( m \) (abstract) subalgebras \( \{ R_l : l < m \} \). One defines the following relation on \( \mathfrak{P} \): For \( l, t < m, R_l \sim R_t \iff (\forall g \in G)(\forall i, j < n)(s_{[ij]} R_l \leq g \iff s_{[ij]} R_t \leq g) \). Then it is straightforward to check that \( \sim \) is an equivalence relation on \( \mathfrak{P} \) having \( p < m \) many equivalence classes, because \( |G| \leq k, n^2 - n < n! \) and (recall that) \( m \geq 2^{k \times n! + 1} \). One next takes \( B = \{ a \in B_k, n : (\forall l, t < m)(\forall i, j < n)(R_l \sim R_t, s_{[ij]} R_l \leq a \iff s_{[ij]} R_t \leq a) \} \), then \( G \subseteq B, R \in B, \) and \( B \) is closed under the operations, so that \( \mathfrak{A}_n \subseteq \mathfrak{B} \subseteq \operatorname{split}(\mathfrak{A}_n, R, m) \), where \( \mathfrak{B} \) is the algebra with universe \( B \). Furthermore, \( \mathfrak{B} \) is the smallest such subalgebra of \( \operatorname{split}(\mathfrak{A}_n, R, m) \), where for each \( i, j < n, s_{[ij]} R \) is partitioned into \( p < m \) many parts cylindrically equivalent to \( s_{[ij]} R \). The non–representability of the algebra \( \operatorname{split}(\mathfrak{A}_n, R, m) \) can be pinned down to the existence of ‘one more extra atom’ leading to the incomputability condition \( |U_0| < m \) (= number of subalgebras) witnessed by the term \( \tau \) using diagonal elements. Using that \( |\mathfrak{P}| = m \), we showed that a representation \( h \) of \( \operatorname{split}(\mathfrak{A}_n, R, m) \) that respects \( R \), has to respect the atoms below it, and this forces that \( |U_0| \geq m \), which contradicts
the construction of \( \mathfrak{A}_n \). But this cannot happen with \( \mathfrak{B} \), because \( p < m \) (by the condition \( |G| \leq k \)), so that this ‘one more extra atom and possibly more’ vanish in \( \mathfrak{B} \). In representing \( \mathfrak{B} \), we use the following optimal compatibility condition between the cardinality of \( |U_0| \) and the number of concrete copies of \( R \) represented genuinely in \( \mathfrak{B} \): 

(*) If \( m \) is any cardinal, \( \alpha \) is an ordinal \( \geq \omega \), \( (U_i : i \in \alpha) \) is a system of sets each having cardinality \( \geq m \), and \( U \supseteq \bigcup \{ U_i : i \in \alpha \} \), then there is a partition \( (R_j : j < m) \) of \( R = \prod_{i \in \mathbb{I}} U_i \) such that \( c_i^j R_j = c_i^j \) for all \( i < \alpha \) and \( j < m \) \cite{2, Lemma 3}. Representing \( \mathfrak{B} \) is done by embedding it into a representable algebra \( \mathfrak{C} \) having the same top element as \( \mathfrak{A}_n \), namely, \( \{ u \in \mathfrak{C} \mid R \in \mathfrak{C} \text{ is partitioned concretely into } m - 1 \text{ real atoms, that is, there exists } R_1 \subseteq \mathfrak{C} \text{, l < m - 1 real atoms in } \mathfrak{C} \text{ such that for all } i < j < n, S_{ij} R_i = S_{ij} \cup i < m - 1 R_i = \bigcup_{i < m - 1} S_{ij} R_i \} \) and \( C, R_i = c_i R \text{ for all } l < m - 1 \text{ and } i < \omega \). This concrete partition exists by (*) because \( |U_0| = m - 1 \) and by the condition \( |G| \leq k \), the value of \( p \), which is the new number of subatoms of \( R \) in \( \mathfrak{B} \) (depending on \( G \)) cannot exceed \( m - 1 \).

(3) Forming the directed union getting the required algebra: For fixed \( k \), obtaining the algebras \( \mathfrak{B}_{k,n} = \text{split}(\mathfrak{A}_n, R, m) \) for each each \( 2 < n < \omega \) we proceed as follows. The constructed non-representable algebras form a chain in the following sense: 

For \( 2 < n_1 < n_2 \), \( \mathfrak{B}_{k,n_1} \) embeds into \( \mathfrak{B}_{k,n_2} \), where the last algebra is the reduct obtained by discarding substitution operations not in the signature of the former, that is the substitution operations \( s_{i,j} : i, j \geq n_1, i \neq j \). Take the directed union \( \mathfrak{B}_k = \bigcup_{n \in \omega} \mathfrak{B}_{k,n} \) having the signature of \( \text{QEA}_\omega \). The cylindric reduct of \( \mathfrak{B}_k \) is not representable because the cylindric reduct of every \( \mathfrak{B}_{k,n} \) is not representable. Using that the \( k \) generated subalgebras of \( \mathfrak{B}_{k,n} \) for each \( 2 < n < \omega \) are representable, it follows without difficulty that the \( k \)-generated subalgebras of \( \mathfrak{B}_k \) remain representable. One constructs such an algebra \( \mathfrak{B}_k \) having the signature of \( \text{QEA}_\omega \) for each positive \( k \). But one can even go further, by showing that the diagonal free reduct of \( \mathfrak{B}_k \), so constructed is in \( \text{RQA}_\omega \) for each \( k \), by showing that this is the case for every \( \mathfrak{B}_{k,n} (n > 2) \). Recall that for fixed positive \( k \) and \( 2 < n < \omega \), the algebra \( \mathfrak{B}_{k,n} \) is not representable because of the incompatibility of \( |U_0| < \) number of subatoms. One now adds ‘one extra element or more’ to \( |U_0| \) forming \( W_0 \) to compensate for such an incompatibility. The diagonal free reduct of \( \mathfrak{B}_{k,n} \) can now be represented by a set algebra \( \mathfrak{C} \) obtained by splitting an \( \omega \)-ary relation \( R = W_0 \times \prod_{i \in \omega} U_i \) where \( |W_0| \geq m + 1 \) and (as before) \( |U_i| = m, i \in \omega \), a set algebra generated by \( R \), into \( m \) real atoms, as described in (*). Here, in the absence of diagonal elements, we cannot count the elements in \( |W_0| \) (like we did with \( |U_0| \) using the term \( \tau \) defined above), so adding this element to \( U_0 \) does not clash with the concrete interpretation of the other operations. In short, \( \text{RQA}_\omega \mathfrak{B}_{k,n} \) can be represented via \( \mathfrak{C} \). This gives the required relative non-finite axiomatizability result.

(4) Relative non-finite axiomatizability; the required result: For a set \( X \) we denote by \( \mathfrak{B}(X) \) the Boolean algebra \( \langle \wp(X), \cup, \cap, \neg \rangle \). Now we show that any universal axiomatization of \( \text{RQA}_\omega \), must contain a formula with more than \( k \) variables and containing at least one diagonal constant getting the required relative non-finite axiomatizability result. Fix \( n \in \omega \sim 3 \). Let \( \Sigma_n^u \) be the set of universal formulas using only \( n \) substitutions and \( k \) variables valid in \( \text{RQA}_\omega \), and let \( \Sigma_n^d \) be the set of universal formulas using only \( n \) substitutions and no diagonal elements valid in \( \text{RQA}_\omega \). By \( n \) substitutions we understand the set \( \{ s_{i,j} : i, j \in n \} \). Then \( \mathfrak{B}_{k,n} = \text{split}(\mathfrak{A}_n, R, m) \models \Sigma_n^u \cup \Sigma_n^d \). \( \mathfrak{B}_{k,n} \models \Sigma_n^u \) because the \( k \)-generated subalgebras of \( \mathfrak{B}_{k,n} \) are representable, while \( \mathfrak{B}_{k,n} \models \Sigma_n^d \) because \( \mathfrak{B}_{k,n} \) has a representation that preserves all operations except for diagonal elements. Indeed, let \( \phi \in \Sigma_n^d \), then there is a representation of \( \mathfrak{B}_{k,n} \) in which all operations are the natural ones except possibly for the diagonal elements. This means that (after discarding the diagonal elements) there is a injective homomorphism \( h : \mathfrak{A}_d \to \mathfrak{B}_d \) where
The variety \( \mathfrak{A}^d = \langle B_{k,n}, +, \cdot, c_k, s_i^d, s_{i,j}^d \rangle_{k\in \omega, i,j \in \omega} \) and \( \mathfrak{Q}^d = \langle \mathfrak{B}(\omega^W), C_k, S_i^j, S_{i,j} \rangle_{k\in \omega, i,j \in \omega} \), for some infinite set \( W \).

Now let \( \mathfrak{P} = \langle \mathfrak{B}(\omega^W), C_k, S_i^j, S_{i,j}^W, D_{k,l} \rangle_{k,l\in \omega, i,j \in \omega} \). Then we have that \( \mathfrak{P} \models \phi \) because \( \phi \) is valid and so \( \mathfrak{Q}^d \models \phi \) due to the fact that no diagonal elements occur in \( \phi \). It thus follows that \( \mathfrak{Q}^d \models \phi \), because \( \mathfrak{Q}^d \) is isomorphic to a subalgebra of \( \mathfrak{Q}^d \) and \( \phi \) is quantifier free. Therefore \( \mathfrak{B}_{k,n} \models \phi \).

Let \( \Sigma^c = \bigcup_{n\in \omega^\omega} \Sigma^c_n \) and \( \Sigma^d = \bigcup_{n\in \omega^\omega} \Sigma^d_n \). It follows that \( \mathfrak{B}_k = \bigcup_{n\in \omega^\omega} \mathfrak{B}_{k,n} \models \Sigma^c \cup \Sigma^d \). For if not, then there exists a quantifier free formula \( \phi(x_1, \ldots, x_m) \in \Sigma^c \cup \Sigma^d \), and \( b_1, \ldots, b_m \) such that \( \phi[b_1, \ldots, b_n] \) does not hold in \( \mathfrak{A}_k \). We have \( b_1, \ldots, b_m \in \mathfrak{B}_{k,i} \) for some \( 2 < i < \omega \). Take \( n \) large enough \( \geq i \) so that \( \phi \in \Sigma^c \cup \Sigma^d \). Then \( \phi \) does not hold in \( \mathfrak{B}_{k,n} \), which is a contradiction. Let \( \Sigma \) be a set of quantifier free formulas axiomatizing \( RQEA_\omega \), then \( \mathfrak{B}_k \) does not model \( \Sigma \) since \( \mathfrak{B}_k \) is not representable, so there exists a formula \( \phi \in \Sigma \) such that \( \phi \notin \Sigma^c \cup \Sigma^d \). Then \( \phi \) contains more than \( k \) variables and a diagonal constant occurs in \( \phi \).

For fixed positive \( k \), various other reducts of \( \mathfrak{B}_{k,n} = split(\mathfrak{A}_n, R, m) \) can be proved to be representable, for each \( 2 < n < \omega \). For example, there are representations that preserve all operations except for infinitely many cylindrifiers, representations that preserve all operations except the Boolean \( \cap \) and \( \cup \), and representations that preserve all operations except for finitely many substitution operations and diagonal elements having a common pre-assigned index in \( \omega \). Using several variations on this theme, it can be shown, without much difficulty, that all complexity results in [2] generalize to \( RQEA_\omega \) by combining the method of ‘infinite splitting’ given above together with the techniques used in [12]. Using transfinite induction such results lift to an arbitrary infinite ordinal \( \alpha \), one of which is the following result. If \( \Sigma \) is any universal axiomatization of \( RQEA_\alpha \), \( l, k, k' < \alpha \), then \( \Sigma \) contains infinitely formulas in which one of the Boolean join or Boolean meet occurs, a diagonal or a substitution operation with an index \( l \) occurs, more than \( k' \) cylindrifications, and more than \( k \) variables occur. Even stronger results can be obtained if \( \Sigma \) is an equational axiomatization.

Andréka’s splitting argument in constructing \( \mathfrak{B}_{k,n} \) \( (2 < n < \omega) \) from \( \mathfrak{B}_n \) by splitting one atom into \( m \)–many subatoms, where \( m \) is finite \( > 2^{k,n+1} \), is in fact an ingenious combination of Monk–like constructions with the concept of dilations in the sense of [13, Construction 3.6.69]. Andréka’s construction avoids ‘colouring’ (expressed in Monk’s construction by an application of Ramsey’s theorem.) Roughly, in dilations one adds atoms to an atomic algebra, if it is not down right ‘impossible’ to do so, witness [13, Last paragraph p. 88]. The finite number \( |U_0| \) plays the role of the number of colours used in Monk’s original construction of bad Monk algebras. Here too, in ‘Andréka’s splitting argument’ the incompatibility condition between the number of atoms \( m \) and the number of colours \( |U_0| \) \( (m > |U_0|) \), leads to an impossibility in case there is a representation of \( \mathfrak{B}_{k,n} \), for the existence of a representation concretely represents the \( m \) atoms below \( R \) forcing \( |U_0| \geq m \). This answers a question of Andréka’s posed explicitly in [2, p.193]; we quote:

“For \( n \geq \omega \), the analogous algebras are called representable quasi-polyadic equality algebras, and their class is denoted by \( RQPEA_m \). We do not know whether Theorem 6 remains true if we drop the condition \( n < \omega \) and replace \( RPEA_m \) by \( RPQEA_m \) in it.”

**Corollary 2.2.**

1. The variety \( RQEA_\omega \) cannot be axiomatized by a set of universal formulas containing finitely many variables over \( RSc_\omega \) nor over \( RDF_\omega \).

2. [2] The variety \( RCA_\omega \) cannot be axiomatized by a set of universal formulas containing finitely many variables over \( RSc_\omega \) nor over \( RDF_\omega \).
Proof. For each positive $k$, using the notation in the previous proof, $\mathfrak{A}_\omega \mathfrak{B}_k \in \text{RSc}_\omega$ and $\mathfrak{A}_\omega \mathfrak{B}_k \in \text{Rdf}_\omega$, while $\mathfrak{B}_k \not\in \text{RQEA}_\omega$. This gives the required in the first item. By observing that in fact, for each positive $k$, $\mathfrak{A}_\omega \mathfrak{B}_k \not\in \text{RCA}_\omega$, we get the required in the second item. 

In the next table, we consider non–finite axiomatizability results for $\omega$–dimensional representable algebras. The answers that do not follow from theorem 2.1 and corollary 2.2 will be provided after the table.

<table>
<thead>
<tr>
<th>Algebras</th>
<th>Finite variable universal axiomatization</th>
<th>Finite schema</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{RQEA}<em>\omega$ over $\text{RQA}</em>\omega$</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>$\text{RQEA}<em>\omega$ over $\text{RCA}</em>\omega$</td>
<td>?</td>
<td>no</td>
</tr>
<tr>
<td>$\text{RQEA}<em>\omega$ over $\text{RSc}</em>\omega$</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>$\text{RQA}<em>\omega$ over $\text{Rdf}</em>\omega$</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>$\text{RCA}<em>\omega$ over $\text{Rdf}</em>\omega$</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>$\text{SNR}<em>{\text{QEA}</em>{\omega+p}, p \geq 2}$</td>
<td>no</td>
<td>no</td>
</tr>
</tbody>
</table>

The proof of the ‘no’ in the second row (proving a weaker result) goes differently using the construction in [6]. In op.cit a non–representable $\text{QEA}_\omega$ whose CA reduct is an $\omega$–dimensional weak set algebra, briefly, a $\text{W}_3$, is constructed. The unit of the weak set algebra is sequences agreeing finitely with $\prod_{i \in \omega} Z_i$ where $Z_0 = Z_1 = 3 = \{0, 1, 2\}$ and $Z_i = \{2i - 1, 2i\}$ for $i > 1$; denote this $\text{QEA}_\omega$ with representable CA reduct by $\mathfrak{A}_3$. For $4 \leq n < \omega$, take $Z_0 = Z_1 = n = \{0, 1, 2, \ldots, n - 1\}$ and $Z_i = \{(n - 1)i - 1, (n - 1)i\}$ for $i > 1$. In exactly the same way, one can construct an algebra $\mathfrak{A}_n \in \text{QEA}_\omega \sim \text{RQEA}_\omega$ and $\mathfrak{A}_\omega \mathfrak{B}_n \in \text{W}_n$ with top element agreeing finitely with sequences in $\prod_{i \in \omega} Z_i^n$.

The proof for $\mathfrak{A}_n$ is identical to the proof for $\mathfrak{A}_3$ replacing $3$ by $n$. It can also be shown that $\prod_{i \in \omega} \mathfrak{A}_i / F \in \text{RQEA}_\omega$ for any non–principal ultrafilter on $\omega$, proving the required.

Now we prove the last ‘no’. Let $n > 1$ be finite and $m = 2^{k \times n! + 1}$. Let

$$e_n = \bigwedge_{i \leq m} c_0(x \cdot x_i) \bigwedge_{i \neq j \leq m} -x_j \leq c_0 \cdots c_m \bigwedge_{i, j \leq m, i \neq j} s_i c_1 \cdots c_m x \cdot -d_{ij},$$

as defined in [2, Remark 2]. Fix positive $k$. Let $\mathfrak{B}_{k,n} = \text{split}(\mathfrak{A}_n, R, m)$ with finite $m \geq 2^{m \times k + 1}$ and $\mathfrak{B}_k$, be as in the proof of theorem 2.1. Then $e_n$ for $t \geq n$, does not hold in $\mathfrak{B}_{k,n}$ because it does not hold in its CA reduct [2, Remark 2]. Let $\mathfrak{R}_\omega \mathfrak{B}_k$ be the reduct obtained from $\mathfrak{B}_k$ by discarding all operations $s_i$, $i \neq n$, $i \neq j$. Then $\mathfrak{B}_{k,n} \subseteq \mathfrak{R}_\omega \mathfrak{B}_k$, so $e_n$ does not hold in $\mathfrak{B}_k$. Else, $e_n$ would hold in $\mathfrak{R}_\omega \mathfrak{B}_k$ since $e_n$ is in the signature of $\text{CA}_n$, and so it holds in $\mathfrak{B}_{k,n}$ (equations are preserved in subalgebras), but we know that this is not the case. By theorem 2.1, $\text{SNR}_{\text{QEA}_{\omega+2}} \models e_n$ for all $n \in \omega \sim 2$, since $\text{SNR}_{\text{CA}_{\omega+2}} \models e_n$ (recall that $e_n$ is in the signature of $\text{CA}_n$).

Now using the same reasoning on [2, p.163], one thereby obtains that $\text{SNR}_{\text{QEA}_{\omega+p}}$ is not axiomatizable with any set of quantifier free formulas containing only finitely many variables for $p \geq 2$.

3 Non–atom canonicity: Blow up and blur constructions

We give some specific examples to the subtle ‘blow up and blur construction’ a term used in [4] that proves highly efficient in proving non–atom canonicity.

8
3.1 Blowing up and blurring a finite Maddux algebra

Here we review and elaborate on the flexible construction in [4] as our first instance of a blow up and blur construction. Let \( \mathcal{R} \) be a relation algebra, with non-identity atoms \( I \) and \( 2 < n < \omega \). Assume that \( J \subseteq \wp(I) \) and \( E \subseteq \wp^2(\omega) \). \( (J, E) \) is an \( n \)-blur for \( \mathcal{R} \), if \( J \) is a complex \( n \)-blur and the tenary relation \( E \) is an index blur defined as in item (ii) of [4, Definition 3.1].

We say that \( (J, E) \) is a strong \( n \)-blur, if it \( (J, E) \) is an \( n \)-blur, such that the complex \( n \)-blur satisfies: \((\forall V_1, \ldots, V_n, W_2, \ldots, W_n \in J)(\forall T \in J)(\forall 2 \leq i \leq n) \text{safe}(V_i, W_i, T) \) (with notation as in [4]).

The atom structure of an atomic algebra \( \mathfrak{A} \) will be denoted by \( \text{At}(\mathfrak{A}) \) or simply \( \text{At}\mathfrak{A} \). The following theorem concisely summarizes the construction in [4] and says some more easy facts. We denote the relation algebra \( \mathfrak{B}b(\mathfrak{R}, J, E) \) with atom structure \( \text{At}\mathfrak{A} \) obtained by blowing up and blurring \( \mathfrak{R} \) (with underlying set is denoted by \( \text{At} \) on [4, p.73]) by \( \text{split}(\mathfrak{R}, J, E) \). By the same token, we denote the algebra \( \mathfrak{B}b(\mathfrak{R}, J, E) \) as defined in [4, Top of p. 78] by \( \text{split}(\mathfrak{R}, J, E) \). This switch of notation is motivated by the fact that we wish to emphasize the role of splitting some (possibly all) atoms into infinitely subatoms during blowing up and blurring a finite algebra. For a relation atom structure \( \alpha \) and \( n > 2 \), \( \text{Mat}_n(\alpha) \) denotes the set of all \( n \) by \( n \) basic matrices on \( \alpha \).

**Theorem 3.1.** Let \( 2 < n \leq l < m \leq \omega \). Let \( \mathfrak{R} \) be a finite relation algebra with an \( l \)-blur \( (J, E) \) where \( J \) is the \( l \)-complex blur and \( E \) is the index blur.

1. The set of \( l \) by \( l \)-dimensional matrices \( \text{At}_{\text{cm}} = \text{Mat}_l(\text{At}) \) is an \( l \)-dimensional cylindric basis, that is a weakly representable atom structure [4, Theorem 3.2]. The algebra \( \text{split}_l(\mathfrak{R}, J, E) \) with atom structure \( \text{At}_{\text{cm}} \) is in \( \text{RCA}_l \). Furthermore, \( \mathfrak{R} \) embeds into \( \text{CmAt} \) which embeds into \( \text{RaCm}(\text{At}_{\text{cm}}) \). If \( (J, E) \) is a strong \( m \)-blur for \( \mathfrak{R} \), then \( (J, E) \) is a strong \( n \)-blur for \( \mathfrak{R} \), \( \text{split}_l(\mathfrak{R}, J, E) \cong \text{Nr}_l(\text{split}_m(\mathfrak{R}, J, E)) \) and \( \text{split}(\mathfrak{R}, J, E) \cong \text{Ra}(\text{split}_l(\mathfrak{R}, J, E)) \).

2. For every \( n < l \), there is an \( \mathfrak{R} \) having a strong \( l \)-blur \( (J, E) \) but no infinite representations (representations on an infinite base). Hence the atom structures defined in the first item for this specific \( \mathfrak{R} \) are not strongly representable.

3. Let \( m < \omega \). If \( \mathfrak{R} \) is a finite relation algebra having a strong \( l \)-blur, and no \( m \)-dimensional hyperbasis, then \( l < m \). If \( n = l < m < \omega \) and \( \mathfrak{R} \) as above has an \( n \) blur \( (J, E) \) and no infinite \( m \)-dimensional hyperbasis, then \( \text{CmAt}(\text{split}(\mathfrak{R}, J, E)) \) and \( \text{CmAt}(\text{split}_l(\mathfrak{R}, J, E)) \) are outside \( \text{SRArCA}_m \) and \( \text{SNr}_n \text{CA}_m \), respectively, and the latter two varieties are not atom-canonical.

**Proof.** [4, Lemmata 3.2, 4.2, 4.3]. We start by an outline of (1). Let \( \mathfrak{R} \) be as in the hypothesis. Let \( 3 < n \leq l \). We blow up and blur \( \mathfrak{R} \). \( \mathfrak{R} \) is blown up by splitting all of the atoms each to infinitely many. \( \mathfrak{R} \) is blurred by using a finite set of blurs (or colours) \( J \). This can be expressed by the product \( \text{At} = \omega \times \text{At}\mathfrak{R} \times J \), which will define an infinite atom structure of a new relation algebra. Then two partitions are defined on \( \text{At} \), call them \( P_1 \) and \( P_2 \). Composition is re-defined on this new infinite atom structure; it is induced by the composition in \( \mathfrak{R} \), and a ternary relation \( E \) on \( \omega \), that ‘synchronizes’ which three rectangles sitting on the \( i, j, k \) \( E \)-related rows compose like the original algebra \( \mathfrak{R} \). The first partition \( P_1 \) is used to show that \( \mathfrak{R} \) embeds in the complex algebra of this new atom structure, namely \( \text{CmAt} \). The second partition \( P_2 \) divides \( \text{At} \) into finitely many (infinite) rectangles, each with base \( W \in J \), and the term algebra (denoted in [4] by \( \mathfrak{B}b(\mathfrak{R}, J, E) \)) over \( \text{At} \), denoted here by \( \text{split}(\mathfrak{R}, J, E) \), consists of the sets that intersect co-finitely with every member of this partition. The algebra \( \text{split}(\mathfrak{R}, J, E) \) is representable using the finite number of blurs. Because \( (J, E) \) is a
complex set of $l$–blurs, this atom structure has an $l$–dimensional cylindric basis, namely, $\mathsf{At}_{ca} = \mathsf{Mat}_l(\mathsf{At})$. The resulting $l$–dimensional cylindric term algebra $\mathcal{M}\mathsf{Mat}_l(\mathsf{At})$, and an algebra $\mathcal{C}$ having atom structure $\mathsf{At}_{ca}$ (denoted in [4] by $\mathcal{B}_b((\mathcal{R}, J, E))$) and denoted now by $\mathsf{split}_l(\mathcal{R}, J, E)$ such that $\mathcal{M}\mathsf{Mat}_l(\mathsf{At}) \subseteq \mathcal{C} \subseteq \mathcal{C}\mathsf{Mat}_l(\mathsf{At})$ is shown to be representable. Assume that the $m$–blur $(J, E)$ is strong. Then by [4, item (3) pp. 80], $\mathsf{split}_l(\mathcal{R}, J, E) \cong \mathcal{M}\mathsf{r}\mathsf{split}_m(\mathcal{R}, J, E)$.

For (2): Like in [4, Lemma 5.1], one takes $l \geq 2n - 1, k \geq (2n - 1)l, k \in \omega$. The Maddux integral relation algebra $\mathcal{E}_k(2, 3)$ where $k$ is the number of non-identity atoms is the required $\mathcal{R}$. In this algebra a triple $(a, b, c)$ of non-identity atoms is consistent $\iff |\{a, b, c\}| \neq 1$, i.e only monochromatic triangles are forbidden.

We prove (3). Let $(J, E)$ be the strong $l$–blur of $\mathcal{R}$. Assume for contradiction that $m \leq l$. Then we get by [4, item (3), p.80], that $\mathcal{A} = \mathsf{split}_l(\mathcal{R}, J, E) \cong \mathcal{M}\mathsf{r}\mathsf{split}_m(\mathcal{R}, J, E)$. But the cylindric $l$–dimensional algebra $\mathsf{split}_l(\mathcal{R}, J, E)$ is atomic, having atom structure $\mathsf{Mat}_l(\mathsf{split}(\mathcal{R}, J, E))$, so $\mathcal{A}$ has an atomic $l$–dilation. So $\mathcal{A} = \mathcal{M}\mathsf{r}\mathcal{D}$ where $\mathcal{D} \in \mathcal{CA}_l$ is atomic. Thus $\mathcal{R} \subseteq \mathcal{M}\mathsf{At}\mathcal{D} \subseteq \mathcal{C}\mathsf{Mat}\mathcal{D}$. By [17, Theorem 13.45 (6) $\iff$ (9)], $\mathcal{R}$ has a complete $l$–flat representation, thus it has a complete $m$–flat representation, because $m < 1$ and $l \in \omega$. This is a contradiction. For the second part. Let $\mathcal{B} = \mathsf{split}_n(\mathcal{R}, J, E)$. Then, since $(J, E)$ is an $n$ blur, $\mathcal{B} \in \mathcal{RCA}_n$. But $\mathcal{C} = \mathcal{C}\mathsf{Mat}\mathcal{B} \notin \mathcal{SN}\mathsf{r}\mathcal{C}\mathsf{A}_m$, because $\mathcal{R} \notin \mathcal{SR}\mathcal{C}\mathsf{A}_m, \mathcal{R}$ embeds into $\mathcal{B}(\mathsf{At}\mathsf{split}(\mathcal{R}, J, E))$ which, in turn, embeds into $\mathcal{R}\mathcal{C}\mathsf{Mat}\mathcal{B}$. Similarly, $\mathsf{split}(\mathcal{R}, J, E) \in \mathcal{RCA}$ and $\mathsf{C}\mathsf{m}(\mathsf{At}\mathsf{split}(\mathcal{R}, J, E)) \notin \mathcal{SR}\mathcal{C}\mathsf{A}_m$. Hence the alleged varieties are not atom–canonical.

3.2 Blowing up and blurring finite rainbow algebras

In theorem 3.1, we used a single blow up and blur construction to prove non-atom–canonicity of $\mathsf{RRA}$ and $\mathcal{RCA}_n$ for $2 < n < \omega$. This construct is based on relation algebras that have an $n$–dimensional cylindric basis denoted above by $\mathcal{E}_k(2, 3)$. To obtain finer results, we use two blow up and blur constructions. For the RA case we blow up and blur the finite rainbow relation algebra (denoted below by) $\mathcal{R}_{4,3}$ and for the CA case we blow up and blur the finite rainbow $\mathcal{C}\mathsf{A}_n$ (denoted below by) $\mathcal{A}_{m+1,n}$.

While the Maddux algebra used in [4] and in the second item of theorem 3.1, $\mathcal{E}_k(2, 3)$ has an $n$–dimensional cylindric basis (by suitably choosing $k$), the relation rainbow algebra $\mathcal{R}_{4,3}$ does not have a 4-dimensional cylindric basis. So for CAs we start anew.

Relation algebras: We briefly review the blow up and blur construction in [17, 17.32, 17.34, 17.36] for relation algebras. Let $2 \leq n \leq \omega$ and $r \leq \omega$. Let $\mathcal{R}$ be an atomic relation algebra. Then the $r$–rounded game $G^n_r(\mathsf{At}\mathcal{R})$ [17, Definition 12.24] is the (usual) atomic game played on networks of an atomic relation algebra $\mathcal{R}$ using $n$ nodes.

Let $L$ be a relational signature. Let $G$ (the greens) and $R$ (the reds) be $L$ structures and $p, r \leq \omega$. The game $\mathsf{EF}^p(G, R)$, defined in [17, Definition 16.1.2], is an Ehrenfeucht–Fraissé forth ‘pebble game’ with $r$ rounds and $p$ pairs of pebbles. In [17, 16.2], a relation algebra rainbow atom structure is associated for relational structures $G$ and $R$. We denote by $\mathcal{R}_{A,B}$ the (full) complex algebra over this atom structure. The Rainbow Theorem [17, Theorem 16.5] states that: If $G, R$ are relational structures and $p, r \leq \omega$, then $\exists$ has a winning strategy in $G^{2+ p}_{1+r}(\mathcal{R}_G, \mathcal{R}) \iff$ she has a winning strategy in $\mathsf{EF}^p(G, R)$.

For $5 \leq l < \omega$, $\mathcal{R}_{A_l}$ is the class of relation algebras whose canonical extensions have an $l$–dimensional relational basis [17, Definition 12.30]. $\mathcal{R}_{A_l}$ is a variety containing properly the variety $\mathcal{SR}\mathcal{C}\mathsf{A}_l$. Furthermore, $\mathcal{R} \in \mathcal{R}_{A_l} \iff \exists$ has a winning strategy in $G^n_r(\mathsf{At}\mathcal{R})$. Cf. [17, Proposition 12.31] and [17, Remark 15.13]. We now show:
proof. We follow the notation in [18, lemmas 17.32, 17.34, 17.35, 17.36] with the sole exception that we denote by \(m\) (instead of \(K_m\)) the complete irreflexive graph on \(m\) defined the obvious way; that is we identify this graph with its set of vertices. Fix \(2 < n < m < \omega\). Let \(\mathfrak{R} = R_{m,n}\). Then by the rainbow theorem \(\exists\) has a winning strategy in \(G_{m+2}^\omega(\text{At}\mathfrak{R})\), since it clearly has a winning strategy in the Ehrenfeucht–Fraisse (EF) game \(\text{EF}^m_{\omega}(m,n)\) because \(m\) is `longer' than \(n\). Then \(\mathfrak{R} \notin \mathfrak{RA}_{m+2}\) by [17, Proposition 12.25, Theorem 13.46 (4) \(\iff\) (5)], so \(\mathfrak{R} \notin \mathfrak{SRaCA}_{m+2}\). Next one ‘splits’ every red atom to \(\omega\)-many copies obtaining the infinite atomic countable (term) relation algebra denoted in \(\text{op.cit}\) by \(T\), which we denote by \(\text{split}(\mathfrak{R}, r, \omega)\) (blowing up the reds by splitting each into \(\omega\)-many subatoms) with atom structure \(a\), cf. [17, item (4) top of p. 532]. Then \(\mathfrak{CR} \notin \mathfrak{SRaCA}_{m+2}\) because \(\mathfrak{R}\) embeds into \(\mathfrak{CR}\) by mapping every red to the join of its copies, and \(\mathfrak{SRaCA}_{m+2}\) is closed under \(S\). Finally, one (completely) represents (the canonical extension of) \(\text{split}(\mathfrak{R}, r, \omega)\) like in [17]. By taking \(m = 4\) and \(n = 3\) the required follows. \(\square\)

Cylindric algebras: From now on, unless otherwise indicated, \(n\) is fixed to be a finite ordinal \(> 2\). For an atomic \(\mathfrak{A} \in \mathfrak{CA}_n\), the \(\omega\)-rounded game \(G^\omega(\text{At}\mathfrak{A})\) or simply \(G^\omega\) is like the usual atomic \(\omega\)-rounded game \(G_n^\omega(\text{At}\mathfrak{A})\) using \(m\) nodes, except that \(\forall\) has the option to re-use the \(m\) nodes in play. We need the following lemma:

Lemma 3.3. Let \(K\) be any any class having signature between \(\mathfrak{SC}\) and \(\text{QEA}\), \(\mathfrak{A} \in K_n\) and \(\mathfrak{A} \in S_n R_n K_m\). Then \(\exists\) has a winning strategy in \(G^\omega(\text{At}\mathfrak{A})\). In particular, If \(\mathfrak{A}\) is finite and \(\forall\) has a winning strategy in \(G^\omega(\text{At}\mathfrak{A})\), then \(\mathfrak{A} \notin \mathfrak{SN} R_n K_m\).

Proof. We give the proof only for \(\text{CA}\). The proof lifts the ideas in [14, Lemma 29, 26, 27] formulated for relation algebras to \(\text{CA}\). This is tedious but not too hard. \(\square\)

For rainbow constructions for \(\text{CA}\), we follow [16, 18]. Fix \(2 < n < \omega\). Given relational structures \(G\) (the greens) and \(R\) (the reds) the rainbow atom structure of a \(\text{QEA}_n\) consists of equivalence classes of surjective maps \(a: n \rightarrow \Delta\), where \(\Delta\) is a coloured graph.

A coloured graph is a complete graph labelled by the rainbow colours, the greens \(g \in G\), reds \(r \in R\), and whites; and some \(n - 1\) tuples are labelled by `shades of yellow'. In coloured graphs certain triangles are not allowed for example all green triangles are forbidden. A red triple \((r_{ij}, r_{jk}, r_{ki})\) \(i, j, j', k', k'' \in R\) is not allowed, unless \(i = i', j = j'\) and \(k = k''\), in which case we say that the red indices match, cf.[16, 4.3.3]. The equivalence relation relates two such maps \(\iff\) they essentially define the same graph [16, 4.3.4]. We let \([a]\) denote the equivalence class containing \(a\). The accessibility (binary relations) corresponding to cylindric operations are like in [16]. For transpositions \([i,j], i < j < n\) they are defined as follows: \([a][b] \iff a = b[i,j]\).

Special coloured graphs typically used by \(\forall\) during implementing his winning strategy are called cones: Let \(i \in G\) and let \(M\) be a coloured graph consisting of \(n\) nodes \(x_0, \ldots, x_{n-2}, z\). We call \(M\) an \(i\) - cone if \(M(x_0, z) = g_{00}\) and for every \(1 \leq j \leq n - 2, M(x_j, z) = g_j\), and no other edge of \(M\) is coloured green. \((x_0, \ldots, x_{n-2})\) is called the base of the cone, \(z\) the apex of the cone and \(i\) the tint of the cone.

For \(2 < n < \omega\), we use the graph version of the games \(G^\omega_m(\beta)\) and \(G^\omega_\omega(\beta)\) where \(\beta\) is a \(\text{QEA}_n\) rainbow atom structure, cf. [16, 4.3.3]. The (complex) rainbow algebra based on \(G\) and \(R\) is denoted by \(\mathfrak{A}_{G,R}\). The dimension \(n\) will always be clear from context. \(\text{RDF}_{n}\) denotes the class of representable diagonal free \(\text{CA}_{n,S_i}\). \(\mathfrak{RDF}_{S}\) denotes `diagonal free reduct' and \(\mathfrak{RDF}_{SC}\) denotes `\(\mathfrak{SC}\) reduct'. We now formulate and prove:
Theorem 3.4. Let \( n \) be a finite ordinal > 2 and \( K \) is a class between \( Sc \) and QEA. Assume that \( m \geq n+3 \). Then the varieties \( SNr_nK_m \) and \( RDf_n \), are not atom–canonical. In particular, such varieties are not closed under Dedekind-MacNeille completions and are not Sahlgqvist axiomatizable.

Proof. The idea for CAs is like that for RAs by blowing up and blurring (the CA reduct of) \( \mathfrak{A}_{n+1,n} \) in place of \( R_{1,3} \). We work with \( m = n+3 \) and any \( K \) between \( Sc \) and QEA. This gives the result for any larger \( m \). Fix \( 2 < n < \omega \).

Blowing up and blurring \( \mathfrak{A}_{n+1,n} \) forming a weakly representable atom structure \( \mathfrak{A} \): Take the finite quasi–polyadic equality algebra rainbow algebra \( \mathfrak{A}_{n+1,n} \) where the reds \( R \) is the complete irre
exive graph \( n \), and the greens are \( G = \{ g_i : 1 \leq i < n−1 \} \cup \{ g_{i,j} : 1 \leq i \leq n \} \), endowed with the polyadic operations. Denote its finite atom structure by \( \mathfrak{A}f \); so that \( \mathfrak{A}f = \mathfrak{A}(\mathfrak{A}_{n+1,n}) \). One then replaces the red colours of the finite rainbow algebra of \( \mathfrak{A}_{n+1,n} \) each by infinitely many reds (getting their superscripts from \( \omega \)), obtaining this way a weakly representable atom structure \( \mathfrak{A} \). The resulting atom structure after ‘splitting the reds’, namely, \( \mathfrak{A} \), is like the weakly (but not strongly) representable atom structure of the atomic, countable and simple algebra \( \mathfrak{A} \) as defined in [22, Definition 4.1]; the sole difference is that we have \( n+1 \) greens and not \( \omega \)–many as is the case in [22]. We denote the algebra \( \mathfrak{TmAt} \) by \( \text{split}(\mathfrak{A}_{n+1,n}, r, \omega) \) short hand for blowing up \( \mathfrak{A}_{n+1,n} \) by splitting each red graphs (atoms) into \( \omega \) many. By a red graph is meant (an equivalence class of) a surjection \( a : n \rightarrow \Delta \), where \( \Delta \) is a coloured graph in the rainbow signature of \( \mathfrak{A}_{n+1,n} \) with at least one edge labelled by a red label (some \( r_{ij}, i < j < n \)). It can be shown exactly like in [22] that \( \exists \) \( \omega \) can win the rainbow \( \omega \)–rounded game and build an \( n \)–homogeneous model \( M \) by using a shade of red \( \rho \) outside the rainbow signature, when she is forced a red; [22, Proposition 2.6, Lemma 2.7]. Using this, one proves like in op.cit that \( \text{split}(\mathfrak{A}_{n+1,n}, r, \omega) \) is representable as a set algebra having top element \( ^nM \). (The term algebra in [22]; which is the subalgebra generated by the atoms of \( \mathfrak{A} \) as defined in [22, Definition 4.1] is just \( \text{split}(\mathfrak{A}_{*,n}, r, \omega) \).)

Embedding \( \mathfrak{A}_{n+1,n} \) into \( \mathfrak{mAt}(\text{split}(\mathfrak{A}_{n+1,n}, r, \omega)) \): Let \( \text{CRG}_f \) be the class of coloured graphs on \( \mathfrak{At} \) and \( \text{CRG} \) be the class of coloured graph on \( \mathfrak{At} \). We can assume that \( \text{CRG}_f \subseteq \text{CRG} \). Write \( M_a \) for the atom that is the (equivalence class of the) surjection \( a : n \rightarrow M, M \in \text{CRG} \). Here we identify \( a \) with \( [a] \); no harm will ensue. We define the (equivalence) relation \( \sim \) on \( \mathfrak{At} \) by \( M_a \sim M_b, (M, N \in \text{CRG}) \iff \) they are identical everywhere except at possibly at red edges:

\[
M_a(a(i), a(j)) = t^l \iff N_b(b(i), b(j)) = t^k, \text{ for some } l, k \in \omega.
\]

We say that \( M_a \) a copy of \( N_b \) if \( M_a \sim N_b \) (by symmetry \( N_b \) is a copy of \( M_a \).) Indeed, the relation ‘copy of’ is an equivalence relation on \( \mathfrak{At} \). An atom \( M_a \) is called a red atom, if \( M \) has at least one red edge.

Any red atom has \( \omega \) many copies that are cylindrically equivalent, in the sense that, if \( N_a = M_b \) with one (equivalently both) red, with \( a : n \rightarrow N \) and \( b : n \rightarrow M \), then we can assume that \( \text{nodes}(N) = \text{nodes}(M) \) and that for all \( i < n \), \( a \mid n \sim \{ i \} = b \mid n \sim \{ i \} \).

In \( \mathfrak{mAt} \), we write \( M_a \) for \( \{ M_a \} \) and we denote suprema taken in \( \mathfrak{mAt} \), possibly finite, by \( \sum \). Define the map \( \Theta \) from \( \mathfrak{A}_{n+1,n} \) to \( \mathfrak{mAt} \), by specifying first its values on \( \mathfrak{At}_f \), via \( M_a \rightarrow \sum_j M_a^{(j)} \) where \( M_a^{(j)} \) is a copy of \( M_a \). So each atom maps to the suprema of its copies. This map is well-defined because \( \mathfrak{mAt} \) is complete. It can be checked that \( \Theta \) is an injective a homomorphism, hence \( \Theta \) is the required embedding.

\( \forall \) has a winning strategy in \( G^{n+1}\mathfrak{At}(\mathfrak{A}_{n+1,n}) \): It is straightforward to show that \( \forall \) has winning strategy first in the Ehrenfeucht–Fraïssé forth private game played between \( \exists \) and \( \forall \) on the complete irreflexive graphs \( n+1 \) and \( n \) in \( n+1 \) rounds, namely, the game \( \mathcal{EF}^{n+1}(n+1, n) \) [18, Definition 16.2]. \( \forall \) lifts his winning strategy from the private
Ehrenfeucht–Fraïssé forth game, to the graph game on $\mathbf{At}_f = \mathbf{At}(\mathfrak{A}_{n+1}, n)$ [16, pp. 841] forcing a win using $n+3$ nodes. He bombards $\exists$ with cones having common base and distinct green tints until $\exists$ is forced to play an inconsistent red triangle (where indices of reds do not match).

By lemma 3.3, $\mathbf{Rd_{sc}} \mathfrak{A}_{n+1,n} \not\in S_n \mathbf{Sn}_{n} \mathfrak{S}_{n+3}$. Since $\mathfrak{A}_{n+1,n}$ is finite, then $\mathbf{Rd_{sc}} \mathfrak{A}_{n+1,n}$ is not in $\mathbf{Sn}_{n} \mathfrak{S}_{n+3}$.

But $\mathfrak{A}_{n+1,n}$ embeds into $\mathbf{CmAtA3}$, hence $\mathbf{Rd_{sc}} \mathbf{CmAtA3} = \mathbf{CmRd_{sc}} \mathbf{AtA3}$ is outside $\mathbf{Sn}_{n} \mathfrak{S}_{n+3}$, too. Since $\mathbf{CmAtA3}$ is generated using infinite (countable) unions by $\{ x \in \mathfrak{C} : \Delta x \neq n \}$, then easily adapting the proof of [13, Lemma 5.1.50, Theorem 5.1.51] and [13, Theorem 5.4.26], we get that $\mathbf{Rd_{sf}} \mathbf{CmAtA3} \not\in \mathbf{Rdf}_n$. This proves the non–atom canonicity of $\mathbf{Rdf}_n$, too.

If $\mathfrak{A} \subseteq \mathfrak{Rv}_n \mathfrak{B}$, $\mathfrak{B} \in \mathfrak{CA}_m$, $n < m \leq \omega$, we say that $\mathfrak{B}$ is an $m$ dilation of $\mathfrak{A}$. It is known that $\mathfrak{B} \in \mathfrak{CA}_m$ is representable $\iff$ it has an $\omega$–dilation. By adjusting the number of greens in the proof of theorem 3.4 to be $n+1$ we got a result finer than Hodkinson’s [22], where Hodkinson uses an ‘overkill’ of infinitely many greens excluding $\omega$–dilations of $\mathbf{CmAtA3}$ with $\mathfrak{A}$ as defined in [17, Definition 4.1]. As indicated above $\mathbf{CmAtA3}$ is just $\mathbf{split}(\mathfrak{A}_{n+1,n}, r, \omega)$. From Hodkinson’s construction, we know that $\mathbf{CmAtA3} \not\in \mathbf{Sn}_{n} \mathbf{CA}_m$ for some $m > n$, but the (semantical) argument used in [22] does not give any information on the value of such $m$. By truncating the greens to be $n+1$, and using a syntactical blow up and blur construction, we could pin down such a value of $m$, namely, $m = n+3 (= \text{number of greens} + 2)$ by showing that $\mathbf{CmAt(split(\mathfrak{A}_{n+1,n}, r, \omega))} \not\in \mathbf{Sn}_{n} \mathbf{CA}_m$.

In our next table, result on atom–canonicity and canonicity for various varieties of RAs and $\mathfrak{CA}_n$s are summarized. We include first order definable expansions of $L_n$ [25] and $\Sigma^\ast$ [7] algebraically as expansions of $\mathbf{RCA}_n$ and $\mathbf{RRA}$. We deviate from the the notation in the former reference by writing $\mathbf{RCA}_n^+$ for a first order definable expansion of $\mathbf{RCA}_n$. Following [7], we denote a first order definable expansion of $\mathbf{RRA}$ by $\mathbf{RRA}^+$. For $\mathbf{CA}$s and its expansions, the dimension $n$ is finite $> 2$. For the ordinals $k$ and $m$, appearing in the table, $k \geq 3$ and $m \geq 6$. In the fourth row $\mathbf{SN}_{n+2} \mathbf{CA}_{n+2}$ would not be atom–canonical if there exists finite $\mathfrak{A}$ with $n$–blur (not necessarily strong) and no infinite $n + 2$–dimensional hyperbasis, cf. theorem ??.

<table>
<thead>
<tr>
<th>Algebras</th>
<th>Atom–canonical</th>
<th>Canonical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{RCA}_n, \mathbf{RRA}$</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>$\mathbf{SN}<em>{n+1} \mathbf{CA}</em>{n+1}, \mathbf{SRaCA}_3$</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>$\mathbf{SN}<em>{n+2} \mathbf{CA}</em>{n+2}, \mathbf{SRaCA}_4, \mathbf{SRaCA}_5$</td>
<td>?</td>
<td>yes</td>
</tr>
<tr>
<td>$\mathbf{SN}<em>{n+3} \mathbf{CA}</em>{n+3}, \mathbf{SRaCA}_m$</td>
<td>no, thms 3.4, 3.2</td>
<td>yes</td>
</tr>
<tr>
<td>$\mathbf{D}_n, \mathbf{G}_n$</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>$\mathbf{RCA}_n^+, \mathbf{RRA}^+$</td>
<td>no, if completely additive</td>
<td>yes</td>
</tr>
</tbody>
</table>

The results in the second row are known [16, 22, 17, 18]. The results in the fifth row (proved in theorems 3.4, 3.2) refine the results in the second row on atom–canonicity with the second row being the limiting case $k = \omega$. Canonicity follows from the observation that if $\mathfrak{A} \subseteq \mathfrak{Rv}_n \mathfrak{B}$, then $\mathfrak{A} \subseteq \mathfrak{Rv}_n \mathfrak{B}^+$ ($\mathfrak{B} \in \mathfrak{CA}_m$, $m > n$) and an analogous result holds for RAs replacing $\mathfrak{Rv}_n$ by $\mathfrak{Ra}$. The affirmative answers in the last row follow from the fact that there are finite equational axiomatizations of both $\mathbf{G}_n$ and $\mathbf{D}_n$ that are positive in the wider sense, hence Sahliqvist, so both varieties are canonical and atom–canonical [17, Theorems 2.77, 2.80].

We prove only non–atom–canonicity of $\mathbf{RCA}_n^+$. Take $\mathfrak{A} \in \mathbf{RCA}_n \cap \mathbf{NR}_n \mathbf{CA}_{n+4}$, where $l > \text{the number of variable used in defining the extra operations}$, and $\mathfrak{A}$ is countable,
having no complete representation. Such an algebra exists by blowing up and blurring $\mathcal{C}_k(2,3)$ for large enough $k$ depending on the strong $l$-blur $(J,E)$ that $\mathfrak{R}$ posses (as indicated in the proof of the first item of theorem 3.1) getting split$_n(\mathcal{C}_k(2,3), J,E) \in \mathcal{RCA}_n \cap \mathcal{N}r_n\mathcal{C}A_{n+l}$. We can (and will) assume that we have only one extra operation $f$ definable by a first order formula $\phi$, say, using $m(< n + l)$ variables with at most $n$ free variables. Now $\phi$ defines a $\mathcal{C}A_n$ term $\tau(\phi)$ which, in turn, defines the unary operation $f$ on $\mathfrak{A}$, via $f(a) = \tau(\phi)^{\mathfrak{A}}(a)$. This is well defined, in the sense that $f(a) \in \mathfrak{A}$, because $\mathfrak{A} \in \mathcal{N}r_n\mathcal{C}A_{n+l}$ and the first order formula $\phi$ defining $f$, has at most $n$ free variables. Call the expanded structure $\mathfrak{A}'((\in \mathcal{RCA}_n^+)$. By the stipulated condition of complete additivity of the first order definable operation $f$, we get that $\mathcal{RCA}_n^+$ is a completely additive variety, so $\mathcal{C}m\mathcal{A}t\mathfrak{A}^*$ is the Dedekind-MacNeille completion of $\mathfrak{A}^*$. But we also know, from theorem 3.1, that $\mathfrak{R}d_{ec}\mathcal{C}m\mathcal{A}t\mathfrak{A}^* = \mathcal{C}m\mathcal{A}t\mathfrak{A} \notin \mathcal{RCA}_n$, a fortiori, $\mathcal{C}m\mathcal{A}t\mathfrak{A}^* \notin \mathcal{RCA}_n^+$, completing the proof.

4 Non–elementary classes

Cylindric algebras: For a class $L$, we write $Ell(L)$ for the elementary closure of $L$. We write $AtL$ for the class $\{At\mathfrak{A} : \mathfrak{A} \in K \cap At\}$ (of first order structures). We write $\mathfrak{A} \subseteq_d \mathfrak{B}$, if $\mathfrak{A}$ is dense in $\mathfrak{B}$, that is for all non–zero $b \in \mathfrak{B}$, there exists a non zero $a \in A$ such that $a \leq b$. We let $S_d$ denote the operation of forming dense subalgebras, and we let $S_n$ denote the operation of forming complete subalgebras. That is, if $K$ is a class of BAOs, then $\mathfrak{A} \in S_nK \iff$ there exists $\mathfrak{B} \in K, \mathfrak{A} \subseteq \mathfrak{B}$ and for all $X \subseteq \mathfrak{A}$, whenever $\sum^X X = 1$, then $\sum^n X = 1$. It is not hard to see that $K \subseteq S_dK \subseteq S_nK$. The inclusions are strict when $K$ is the class of Boolean algebras.

Theorem 4.1. Let $2 < n < \omega$ and let $k \geq 3$. For any class $K$, such that $\mathcal{CRCA}_n \cap \mathcal{N}r_n\mathcal{C}A_n \subseteq K \subseteq S_d\mathcal{N}r_n\mathcal{C}A_{n+3}$, $K$ is not elementary. In particular, any class between $\mathcal{At}(\mathcal{N}r_n\mathcal{C}A_n \cap \mathcal{CRCA}_n)$ and $\mathcal{At}S_d\mathcal{N}r_n\mathcal{C}A_{n+3}$ is not elementary.

Idea of proof: The proof uses a rainbow construction followed by a splitting argument applied to another construction. Let $2 < n < \omega$. We define a rainbow–like algebra $\mathcal{C} \in \mathcal{C}A_n$ based on $\mathbb{Z}$ (the greens) and $\mathbb{N}$ (the reds) having countably many atoms. We show that $\forall$ has winning strategy in $G_{\omega}^{n+3}(At\mathcal{C})$ so that using lemma 3.3, $\mathcal{C} \notin S_d\mathcal{N}r_n\mathcal{C}A_{n+3}$. We define a $k$–rounded atomic game $H_k (k \leq \omega)$ stronger than $G_k$, and show that $\exists$ has a winning strategy in a game $H_k(At\mathcal{C})$ for all $k < \omega$, from which we infer that $\exists$ has a winning strategy in $H_\omega(\alpha)$ for some countable atomic structure $\alpha$ such that $\Gamma \alpha \equiv \mathcal{C}$. The game $H$ is designed, so that a winning strategy in $H_\omega(\alpha)$ implies that $\exists \alpha \in S_d\mathcal{N}r_n\mathcal{C}A_{\omega} \cap \mathcal{CRCA}_n$. This leads to (*): Any class $K$ between $\mathcal{CRCA}_n \cap S_d\mathcal{N}r_n\mathcal{C}A_n$ and $S_d\mathcal{N}r_n\mathcal{C}A_{n+3}$, $K$ is not elementary. To exclude elementary classes between $\mathcal{N}r_n\mathcal{C}A_n \cap \mathcal{CRCA}_n$ and $S_d\mathcal{N}r_n\mathcal{C}A_n$ (which is strictly bigger by the forthcoming example 4.2) we use a variation on the construction in [27, Theorem 5.1.4], presented in the framework of a splitting argument.

Before giving details, we need some preparation. The next example will be used several times below.

Example 4.2. Assume that $1 < n < \omega$. Let $V = ^n\mathbb{Q}$ and let $\mathfrak{A} \in \mathcal{C}S_n\mathfrak{A}$ has universe $\varphi(V)$. Then clearly $\mathfrak{A} \in \mathcal{N}r_n\mathcal{C}A_{\omega}$. To see why, let $W = \omega\mathbb{Q}$ and let $\mathfrak{D} \in \mathcal{C}S_{\omega}$ have universe $\varphi(W)$. Then the map $\theta : \mathfrak{A} \rightarrow \varphi(\mathfrak{D})$ defined via $a \rightarrow \{s \in W : (s \mid \alpha) \in a\}$, is an injective homomorphism from $\mathfrak{A}$ into $\mathfrak{D}_{\omega},\mathcal{D}$ that is onto $\mathcal{N}r_n\mathcal{D}$.

Let $y$ denote the following $n$–ary relation: $y = \{s \in V : s_0 + 1 = \sum_{i>0}s_i\}$. Let $y_s$ be the singleton containing $s$, i.e. $y_s = \{s\}$ and $\mathfrak{B} = \mathcal{E}\varphi(\mathfrak{A}, y, y_s : s \in y)$. It is shown in [28] that $\{s\} \in \mathfrak{B}$, for all $s \in V$. 

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Now \( \mathcal{B} \) and \( \mathfrak{A} \) having same top element \( V \), share the same atom structure, namely, the singletons, so \( \mathcal{B} \subseteq_d \mathfrak{A} \) and \( \mathfrak{CmAt}\mathcal{B} = \mathfrak{A} \). Furthermore, plainly \( \mathfrak{A}, \mathcal{B} \in \text{CRCA}_n \); the identity maps establishes a complete representation for both, since \( \bigcup_{x \in V} \{ s \} = V \). As proved in [28], \( \mathcal{B} \notin \text{ElNr}_n \text{CA}_{n+1} \), hence for any \( \beta > n \), \( \mathcal{B} \in \text{S}_d \text{Nr}_n \text{CA}_\beta \sim \text{Nr}_n \text{CA}_\beta \).

If \( K \) is class of BAOs, and \( \mathfrak{A} \in K \) is atomic, then plainly \( \text{At}\mathfrak{A} \in \text{At}K \). But the converse might not be true. Let \( 2 < n < \omega \). In theorem 3.4, it was shown that there is an atom structure \( \text{At} \) such that \( \mathfrak{CmAt} \in \text{RCA}_n \), but \( \mathfrak{CmAt} \notin \text{SNr}_n \text{CA}_{n+3} \), thus there is an algebra outside \( \text{SNr}_n \text{CA}_{n+k} \) for all \( k \geq 3 \), with a representable dense subalgebra.

From example 4.2 taken from 4.2, it can be easily deduced that for any infinite cardinal \( \kappa \), the atom structure \( \text{At} = (\kappa, \{ \kappa, x \}, \{ D_{ij} \}_{i,j \leq \kappa}) \) is the atom structure of an algebra in \( \text{Nr}_n \text{CA}_\omega \), namely, the complex algebra \( \mathfrak{CmAt} \). Furthermore, if we assume that \( \kappa \) is the base of a field of characteristic zero, then one can construct an algebra \( \mathcal{B} \), with \( |\mathcal{B}| = \kappa \), \( \text{At}\mathcal{B} = \text{At} \) and \( \mathcal{B} \notin \text{ElNr}_n \text{CA}_{n+1} \). But, conversely, if \( \mathfrak{A} \in \text{CRCA}_n \) has atom structure \( \text{At} \), then if \( \mathfrak{A} \in \text{CA}_n \) and \( \text{At}\mathfrak{A} = \text{At} \), then \( \mathfrak{A} \in \text{CRCA}_n \). This motivates:

**Definition 4.3.**

1. The class \( K \) is **gripped by its atom structures**, or simply **gripped**, if for all \( \mathfrak{A} \in \text{CA}_n \), whenever \( \text{At}\mathfrak{A} \in \text{At}K \), then \( \mathfrak{A} \in K \).

2. An -rounded game \( H \) **grips** \( K \), if whenever \( \mathfrak{A} \in \text{CA}_n \) is atomic with countably many atoms and \( \exists \) has a winning strategy in \( H(\mathfrak{A}) \), then \( \mathfrak{A} \in K \). The game \( H \) **weakly grips** \( K \), if whenever \( \mathfrak{A} \in \text{CA}_n \) is atomic with countably many atoms and \( \exists \) has a winning strategy in \( H(\text{At}\mathfrak{A}) \), then \( \text{At}\mathfrak{A} \in \text{At}K \). The game \( H \) **densely grips** \( K \), if whenever \( \mathfrak{A} \in \text{CA}_n \) is atomic with countably many atoms and \( \exists \) has a winning strategy in \( H(\text{At}\mathfrak{A}) \), then \( \text{At}\mathfrak{A} \in \text{At}K \) and \( \mathfrak{CmAt}\mathfrak{A} \in K \).

**Example 4.4.** Let \( 2 < n < m \leq \omega \). Then the classes \( \text{RCA}_n \) and \( \text{Nr}_n \text{CA}_m \) are not gripped, by theorem 3.4 and example 4.2. For any \( n < \omega \), the class \( \text{CRCA}_n \), and its elementary closure \( \text{LCA}_n \) (the class of algebras satisfying the Lyndon conditions [18]) is gripped. Fix \( 2 < n < \omega \). The usual atomic game \( G \) weakly grips, densely grips and grips \( \text{CRCA}_n \).

Now we define the game \( H \), mentioned above, that densely grips, hence weakly grips, \( \text{Nr}_n \text{CA}_\omega \), but we show that \( H \) does not grip \( \text{Nr}_n \text{CA}_\omega \).

The game \( H \) is similar to the game devised by Hirsch for relation algebras, denoted by \( H \) in [14, Definition 28].

A hypernetwork is roughly a network endowed with hyperedges of length \( \neq n \) allowed to get arbitrarily long but are of finite length, and such hyperedges get their labels from a non-empty set of labels \( \Lambda \). The board of the game consists of so-called \( \lambda \)-neat hypernetworks which are hypernetworks with the so-called short hyperedges constantly labelled by (a label) \( \lambda \in \Lambda \). In more detail:

**Definition 4.5.** For an \( n \)-dimensional atomic network \( N \) on an atomic \( \text{CA}_n \) and for \( x, y \in \text{nodes}(N) \), set \( x \sim y \) if there exists \( \bar{z} \) such that \( N(x, y, \bar{z}) \leq \text{d}_0 \). Define the equivalence relation \( \sim \) over the set of all finite sequences over \( \text{nodes}(N) \) by \( \bar{x} \sim \bar{y} \) iff \( |\bar{x}| = |\bar{y}| \) and \( x_i \sim y_i \) for all \( i < |\bar{x}| \). (It can be easily checked that this indeed an equivalence relation).

A **hypernetwork** \( N = (N^a, N^h) \) over an atomic \( \text{CA}_n \) consists of an \( n \)-dimensional network \( N^a \) together with a labelling function for hyperlabels \( N^h : \text{nodes}(N) \to \Lambda \) (some arbitrary set of hyperlabels \( \Lambda \)) such that for \( \bar{x}, \bar{y} \in \text{nodes}(N) \) if \( \bar{x} \sim \bar{y} \Rightarrow N^h(\bar{x}) = N^h(\bar{y}) \). If \( |\bar{x}| = k \in \mathbb{N} \) and \( N^h(\bar{x}) = \lambda \), then we say that \( \lambda \) is a \( k \)-ary hyperlabel. \( \bar{x} \) is referred to as a \( k \)-ary hyperedge, or simply a hyperedge.
We may remove the superscripts $a$ and $h$ if no confusion is likely to ensue. A hyperedge $\bar{x} \in <\omega \text{nodes}(N)$ is short, if there are $y_0, \ldots , y_{n-1}$ that are nodes in $N$, such that $N(x_i, y_0, \bar{z}) \leq d_{01}$ or $\ldots N(x_i, y_{n-1}, \bar{z}) \leq d_{01}$ for all $i < |x|$, for some (equivalently for all) $\bar{z}$. Otherwise, it is called long.

This game involves, besides the standard cylindrifier move, two new amalgamation moves. Concerning his moves, this game with $m$ rounds, call it $H_m$, $\forall$ can play a cylindrifier move, like before but now played on $\lambda$-neat hypernetworks ($\lambda$ a constant label). Also $\forall$ can play a transformation move by picking a previously played hypernetwork $N$ and a partial, finite surjection $\theta : \omega \rightarrow \text{nodes}(N)$, this move is denoted $(N, \theta)$. $\exists$’s response is mandatory. She must respond with $N\theta$. Finally, $\forall$ can play an amalgamation move by picking previously played hypernetworks $M, N$ such that $M \cap \text{nodes}(M) \cap \text{nodes}(N)$, and $\text{nodes}(M) \cap \text{nodes}(N) \neq \emptyset$. This move is denoted $(M, N)$. To make a legal response, $\exists$ must play a $\lambda_0$-neat hypernetwork $L$ extending $M$ and $N$, where $\text{nodes}(L) = \text{nodes}(M) \cup \text{nodes}(N)$.

The next Lemma is the CA analogue of [14, Theorem 39], but it proves more:

**Theorem 4.6.** Let $\alpha$ be a countable atom structure. If $\exists$ has a winning strategy in $H_\omega \alpha$, then any algebra $\mathfrak{A}$ having atom structure $\alpha$ is completely representable, $\mathfrak{C} \alpha \in \mathfrak{N}_n \mathfrak{C} \omega$ and $\alpha \in \mathfrak{A} \mathfrak{N}_n \mathfrak{D} \omega$. In fact, there exists a complete $\mathfrak{D} \in \mathfrak{R} \mathfrak{C} \omega$ such that $\mathfrak{C} \alpha \cong \mathfrak{N}_n \mathfrak{D}$ and $\alpha \cong \mathfrak{A} \mathfrak{N}_n \mathfrak{D}$.

**Proof.** Fix some $\alpha \in \alpha$. The game $H_\omega$ is designed so that using $\exists$’s winning strategy in the game $H_\omega(\alpha)$ one can define a nested sequence $M_0 \subseteq M_1, \ldots$ of $\lambda$-neat hypernetworks where $M_0$ is $\exists$’s response to the initial $\forall$-move $\alpha$, such that: If $M_r$ is in the sequence and $M_r(\bar{x}) \leq c_\alpha a$ for an atom $a$ and some $i < n$, then there is $s \geq r$ and $d \in \text{nodes}(M_s)$ such that $M_s(\bar{y}) = a$, $\bar{y} = d$ and $\bar{y} \equiv_i \bar{x}$. In addition, if $M_r$ is in the sequence and $\theta$ is any partial isomorphism of $M_r$, then there is $s \geq r$ and a partial isomorphism $\theta^+$ of $M_s$ extending $\theta$ such that $\text{rng}(\theta^+) \supseteq \text{nodes}(M_r)$ (This can be done using $\exists$’s responses to amalgamation moves).

Now let $\mathfrak{M}_\alpha$ be the limit of this sequence, that is $\mathfrak{M}_\alpha = \bigcup M_i$, the labelling of $n - 1$ tuples of nodes by atoms, and hyperedges by hyperlabels done in the obvious way using the fact that the $M_i$s are nested. Let $L$ be the signature with one $n$-ary relation for each $b \in \alpha$, and one $k$-ary predicate symbol for each $k$-ary hyperlabel $\lambda$. Now we work in $L_{\infty, \omega}$. For fixed $f_a \in <\omega \text{nodes}(\mathfrak{M}_\alpha)$, let $\Lambda_a = \{f \in <\omega \text{nodes}(\mathfrak{M}_\alpha) : \{i < \omega : g(i) \neq f_a(i)\} \text{ finite}\}$.

We make $\Lambda_a$ into the base of an $L$ relativized structure $\mathcal{M}_\alpha$ like in [14, Theorem 29] except that we allow a clause for infinitary disjunctions. In more detail, for $b \in \alpha, l_0, \ldots , l_{n-1}, i_0 \ldots , i_{k-1} < \omega$, $k$-ary hyperlabels $\lambda$, and all $L$-formulas $\phi, \phi_i, \psi$, and $f \in U_a$:

\[
\begin{align*}
\mathcal{M}_\alpha, f \models b(x_0, \ldots , x_{l-1}) & \iff \mathcal{M}_\alpha(f(l_0), \ldots , f(l_{n-1})) = b, \\
\mathcal{M}_\alpha, f \models \lambda(x_0, \ldots , x_{i_k-1}) & \iff \mathcal{M}_\alpha(f(i_0), \ldots , f(i_{k-1})) = \lambda, \\
\mathcal{M}_\alpha, f \models \neg \phi & \iff \mathcal{M}_\alpha, f \not\models \phi, \\
\mathcal{M}_\alpha, f \models \bigvee_{i \in I} \phi_i & \iff (\exists i \in I)(\mathcal{M}_\alpha, f \models \phi_i), \\
\mathcal{M}_\alpha, f \models \exists x_i \phi & \iff \mathcal{M}_\alpha, f[i/m] \models \phi, \text{ some } m \in \text{nodes}(\mathcal{M}_\alpha).
\end{align*}
\]

We are now working with (weak) set algebras whose semantics is induced by $L_{\infty, \omega}$ formulas in the signature $L$, instead of first order ones. For any such $L$-formula $\phi$, write $\phi^{\mathcal{M}_\alpha}$ for $\{f \in \Lambda_a : \mathcal{M}_\alpha, f \models \phi\}$. Let $D_a = \{\phi^{\mathcal{M}_\alpha} : \phi \text{ is an } L\text{-formula}\}$ and $\mathcal{D}_a$.
be the weak set algebra with universe $D_\alpha$. Let $\mathcal{D} = \mathcal{P}_{a \in \alpha} D_a$. Then $\mathcal{D}$ is a generalized complete weak set algebra [13, Definition 3.1.2 (iv)]. By complete we mean that infinite suprema exist. This is true because we chose to work with $L_{\infty, \omega}$ while forming the dilations $D_a (a \in \alpha)$. Each $D_a$ is complete, hence so is their product $\mathcal{D}$.

Now we show that $\alpha \cong \mathfrak{AtNr}_\mathcal{D} \mathfrak{D}$ and $\mathfrak{Cma} \cong \mathfrak{At}_{\mathcal{D}} \mathcal{D}$. The argument used is like the argument used in [14, Theorem 39] adapted to CASs. Let $x \in \mathcal{D}$. Then $x = (x_a : a \in \alpha)$, where $x_a \in D_a$. For $b \in \alpha$ let $\pi_b : \mathcal{D} \to \mathcal{D}_b$ be the projection map defined by $\pi_b(x_a : a \in \alpha) = x_b$. Conversely, let $\iota_a : D_a \to \mathcal{D}$ be the embedding defined by $\iota_a(y) = (x_b : b \in \alpha)$, where $x_a = y$ and $x_b = 0$ for $b \neq a$.

Suppose $x \in \mathfrak{At}_{\mathcal{D}} \mathcal{D} \setminus \{0\}$. Since $x \neq 0$, then it has a non-zero component $\pi_a(x) \in D_a$, for some $a \in \alpha$. Assume that $\emptyset \neq \phi(x_{i_0}, \ldots, x_{i_{k-1}}) D_a = \pi_a(x)$, for some $L$-formula $\phi(x_{i_0}, \ldots, x_{i_{k-1}})$. We have $\phi(x_{i_0}, \ldots, x_{i_{k-1}}) D_a \in \mathfrak{At}_{\mathcal{D}} \mathcal{D}_a$. Pick $f \in \phi(x_{i_0}, \ldots, x_{i_{k-1}}) D_a$ and assume that $M_{i_0}, f \models b(x_0, \ldots, x_{n-1})$ for some $b \in \alpha$. We show that $b(x_0, x_1, \ldots, x_{n-1}) D_a \subseteq \phi(x_{i_0}, \ldots, x_{i_{k-1}}) D_a$. Take any $g \in b(x_0, x_1, \ldots, x_{n-1})$, so that $M_{i_0}, g \models b(x_0, \ldots, x_{n-1})$. The map $\{(f(i), g(i)) : i < n\}$ is a partial isomorphism of $M_a$. Here that short hyperedges are constantly labelled by $\lambda$ is used. This map extends to a finite partial isomorphism $\theta$ of $M_a$ whose domain includes $f(i_0), \ldots, f(i_k-1)$. Let $g' \in M_a$ be defined by

$$g'(i) = \begin{cases} \theta(i) & \text{if } i \in \text{dom}(\theta) \\ g(i) & \text{otherwise} \end{cases}$$

We have $M_{i_0}, g' \models \phi(x_{i_0}, \ldots, x_{i_{k-1}})$. But $g'(0) = \theta(0) = g(0)$ and similarly $g'(n-1) = g(n-1)$, so $g$ is identical to $g'$ over $n$ and it differs from $g'$ on only a finite set. Since $\phi(x_{i_0}, \ldots, x_{i_{k-1}}) D_a \in \mathfrak{At}_{\mathcal{D}} \mathcal{D}_a$, we get that $M_{i_0}, g \models \phi(x_{i_0}, \ldots, x_{i_k})$, so $g \in \phi(x_{i_0}, \ldots, x_{i_{k-1}}) D_a$ (this can be proved by induction on quantifier depth of formulas). This proves that

$$b(x_0, x_1 \ldots x_{n-1}) D_a \subseteq \phi(x_{i_0}, \ldots, x_{i_k}) D_a = \pi_a(x),$$

and so

$$\iota_a(b(x_0, x_1 \ldots x_{n-1}) D_a) \leq \iota_a(\phi(x_{i_0}, \ldots, x_{i_{k-1}}) D_a) \leq x \in D_a \setminus \{0\}.$$

Now every non-zero element $x$ of $\mathfrak{At}_{\mathcal{D}} \mathcal{D}_a$ is above a non-zero element of the following form $\iota_a(b(x_0, x_1, \ldots, x_{n-1}) D_a)$ (some $a, b \in \alpha$) and these are the atoms of $\mathfrak{At}_{\mathcal{D}} \mathcal{D}_a$.

The map defined via $b \mapsto (b(x_0, x_1, \ldots, x_{n-1}) D_a : a \in \alpha)$ is an isomorphism of atom structures, so that $\alpha \cong \mathfrak{At}_{\mathcal{D}} \mathcal{D}_\omega$. proving the first required. Since $\mathcal{D}$ is complete, then $\mathfrak{At}_{\mathcal{D}} \mathcal{D}$ is complete. To see why, let $X \subseteq \mathfrak{At}_{\mathcal{D}} \mathcal{D}$. Then by completeness of $\mathcal{D}$, we get that $d = \sum \mathcal{D} X$ exists. Assume that $i \notin n$, then $c_i d = c_i \sum X = \sum_{x \in X} c_i x = \sum X = d$, because the $c_i$s are completely additive and $c_i x = x$, for all $i \notin n$, since $x \in \mathfrak{At}_{\mathcal{D}} \mathcal{D}$. We conclude that $d \in \mathfrak{At}_{\mathcal{D}} \mathcal{D}$, hence $d$ is an upper bound of $X$ in $\mathfrak{At}_{\mathcal{D}} \mathcal{D}$. Since $d = \sum \mathcal{D} X$ there can be no $b \in \mathfrak{At}_{\mathcal{D}} \mathcal{D}$ with $b < d$ such that $b$ is an upper bound of $X$ for else it will be an upper bound of $X$ in $\mathcal{D}$. Thus $\sum_{x \in X} X = d$ and so, as claimed, $\mathfrak{At}_{\mathcal{D}} \mathcal{D}$ is complete. Making the legitimate identification $\mathfrak{At}_{\mathcal{D}} \mathcal{D} \subseteq d \mathfrak{Cma}$ by density, we get that $\mathfrak{At}_{\mathcal{D}} \mathcal{D} = \mathfrak{Cma}$ (since $\mathfrak{At}_{\mathcal{D}} \mathcal{D}$ is complete), hence $\mathfrak{Cma} \in \mathfrak{At}_{\mathcal{D}} \mathcal{D}_\omega$. \hfill \qed

4.1 Proof of theorem 4.1

Take the rainbow–like $\mathcal{C}_\alpha$, call it $\mathcal{C}$, based on the ordered structure $Z$ and $\mathbb{N}$. The reds $R$ is the set $\{r_{ij} : i < j < \omega(= \mathbb{N})\}$ and the green colours used constitute the set $\{g_i : 1 \leq i < n - 1\} \cup \{g_0^i : i \in \mathbb{Z}\}$. In complete coloured graphs the forbidden triples are like the usual rainbow constructions based on $Z$ and $\mathbb{N}$ specified above, but now
the triple \((g_i, g_j, r_{kl})\) is also forbidden if \(\{(i, k), (j, l)\}\) is not an order preserving partial function from \(\mathbb{Z} \to \mathbb{N}\).

Then \(\mathcal{C} \not\in S_c \mathcal{N}_n \mathcal{C} A_{n+3}\). This can be proved using lemma 3.3 by showing that \(\forall\) has a winning strategy in \(G^{n+3}(\mathcal{A}\mathcal{C})\). His winning strategy is to bombard \(\exists\) with cones having common base and distinct green tints. \(\exists\) has to label edges between apexes of cones created during the game by reds. The newly added consistency condition of ‘order preserving’ restricts \(\exists\)’s ‘red choices’. To conform to the rules of the play, \(\exists\) is forced to play reds \(r_{ij}\) with the first index forming a decreasing sequence in \(\mathbb{N}\). Having the option to re–use the nodes in play, \(\forall\) needs to use and reuse exactly \(n + 3\) nodes to force a win in \(\omega\) rounds (but not before).

He plays as follows: In the initial round \(\forall\) plays a graph \(M\) with nodes \(0, 1, \ldots, n - 1\) such that \(M(i, j) = w_0\) for \(i < j < n - 1\) and \(M(i, n - 1) = g_i\) \((i = 1, \ldots, n - 2)\), \(M(0, n - 1) = g_0^0\) and \(M(0, 1, \ldots, n - 2) = y_2\). This is a 0 cone. In the following move \(\forall\) chooses the base of the cone \((0, \ldots, n - 2)\) and demands a node \(n\) with \(M_2(i, n) = g_i\) \((i = 1, \ldots, n - 2)\), and \(M_2(0, n) = g_0^0\). \(\exists\) must choose a label for the edge \((n + 1, n)\) of \(M_2\). It must be a red atom \(r_{mk}\), \(m, k \in \mathbb{N}\). Since \(-1 < 0\), then by the ‘order preserving’ condition we have \(m < k\). In the next move \(\forall\) plays the face \((0, \ldots, n - 2)\) and demands a node \(n + 1\), with \(M_3(i, n) = g_i\) \((i = 1, \ldots, n - 2)\), such that \(M_3(0, n + 2) = g_0^0\). Then \(M_3(n + 1, n)\) and \(M_3(n + 1, n - 1)\) both being red, the indices must match. \(M_3(n + 1, n) = r_{1k}\) and \(M_3(n + 1, r - 1) = r_{klm}\) with \(l < m \in \mathbb{N}\). In the next round \(\forall\) plays \((0, 1, \ldots, n - 2)\) and re-uses the node 2 such that \(M_4(0, 2) = g_0^0\). This time we have \(M_4(n, n - 1) = r_{jl}\) for some \(j < l < m \in \mathbb{N}\). Continuing in this manner leads to a decreasing sequence in \(\mathbb{N}\), so \(\forall\) can implement a winning strategy in \(\omega\) many rounds. By lemma 3.3, \(\mathcal{C} \not\in S_c \mathcal{N}_n \mathcal{C} A_{n+3}\).

In [10], it is shown that for \(k < \omega\), \(\exists\) has a winning strategy \(\rho_k\), in \(G_k(\mathcal{A}\mathcal{C})\). Let \(\mathcal{D}\) be a non–principal ultrapower of \(\mathcal{C}\). Then \(\exists\) has a winning strategy \(\sigma\) in \(G_\omega(\mathcal{A}\mathcal{D})\) — essentially she uses \(\sigma_k\) in the \(k\)'th component of the ultraproduct so that at each round of \(G_\omega(\mathcal{A}\mathcal{D})\), \(\exists\) is still winning in co–finally many components, this suffices to show she has still not lost. Now one can use an elementary chain argument to construct countable elementary subalgebras \(\mathcal{C} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots \subseteq \mathcal{D}\) in the following way. One defines \(\mathcal{A}_{i+1}\) to be a countable elementary subalgebra of \(\mathcal{D}\) containing \(\mathcal{A}_i\) and all elements of \(\mathcal{D}\) that \(\sigma\) selects in a play of \(G_\omega(\mathcal{A}\mathcal{D})\) in which \(\forall\) only chooses elements from \(\mathcal{A}_i\). Now let \(\mathcal{B} = \bigcup_{i<\omega} \mathcal{A}_i\). This is a countable elementary subalgebra of \(\mathcal{D}\), hence necessarily atomic, and \(\exists\) has a winning strategy in \(G_\omega(\mathcal{A}\mathcal{B})\), so \(\mathcal{B}\) is completely representable. Thus \(\mathcal{A} \equiv \mathcal{C} \equiv \mathcal{B}\), hence \(\mathcal{A} \in EI\mathcal{C} \mathcal{R} \mathcal{C} A_n\). But we can go further.

It can be shown that \(\exists\) has a winning strategy in \(H_k(\mathcal{A}\mathcal{C})\) for all \(k < \omega\), hence \(\exists\) has a winning strategy in \(H_\omega(\mathcal{A}\mathcal{C})\) for a countable atom structure \(\alpha\), such that \(\mathcal{A}\mathcal{C} \equiv \alpha\). Let \(\mathcal{B}' = T\mathcal{A}\alpha\). Then \(\mathcal{B}' \subseteq_{\alpha} \mathcal{C}\mathcal{M}\mathcal{A} \in \mathcal{N}_n \mathcal{C} A_n \cap \mathcal{C} \mathcal{R} \mathcal{C} A_n \subseteq \mathcal{K}\), \(\mathcal{C} \not\in S_c \mathcal{N}_n \mathcal{C} A_{n+3} \supseteq \mathcal{K}\), and \(\mathcal{C} \equiv \mathcal{B}'\) proving (*). Observe that this already proves the second part, because the atom structure of an algebra is first order definable in the algebra, so that \(\alpha \equiv \mathcal{A}\mathcal{C}\). Also \(\alpha \in \mathcal{A}\mathcal{N}_n \mathcal{C} A_\omega\). Finally, \(\mathcal{A}\mathcal{C} \not\in S_c \mathcal{N}_n \mathcal{C} A_{n+3}\) because for any \(m > n\), and any atomic \(\mathcal{D} \in \mathcal{C} A_n\), \(\mathcal{D} \in S_c \mathcal{N}_n \mathcal{C} A_n \iff \mathcal{A}\mathcal{D} \in \mathcal{A}\mathcal{S}_c \mathcal{N}_n \mathcal{C} A_m\). This does not hold if we omit \(S_c\), as indicated in example 4.2.

To complete the proof, we first need to slightly modify the construction in [27, Lemma 5.1.3, Theorem 5.1.4] reformulating it as a ‘splitting argument’. The algebras \(\mathcal{A}\) and \(\mathcal{B}\) constructed in \(op\text{.}cit\) satisfy that \(\mathcal{A} \in \mathcal{N}_n \mathcal{C} A_\omega\), \(\mathcal{B} \notin \mathcal{N}_n \mathcal{C} A_{n+1}\) and \(\mathcal{A} \equiv \mathcal{B}\). As they stand, \(\mathcal{A}\) and \(\mathcal{B}\) are not atomic, but they it can be fixed that they are to be so giving the same result, by interpreting the uncountably many tenary relations in the signature of \(\mathcal{M}\) defined in [27, Lemma 5.1.3], which is the base of \(\mathcal{A}\) and \(\mathcal{B}\) to be disjoint in \(\mathcal{M}\), not just distinct.
We work with $2 < n < \omega$ instead of only $n = 3$. The proof presented in op.cit lift verbatim to any such $n$. Let $u \in \mathbb{N}$. Write $\mathfrak{u}_n$ for $\chi^M_u$ (denoted by $\mathfrak{u}_n$ for $n = 3$) in [27, Theorem 5.1.4].) We denote by $\mathfrak{A}_n$ the Boolean algebra $R\mathfrak{I}_{1n} \mathfrak{A} = \{x \in \mathfrak{A} : x \leq \mathfrak{u}_n\}$ and similarly for $\mathfrak{B}_n$, writing $\mathfrak{B}_u$ short hand for the Boolean algebra $R\mathfrak{I}_{1u} \mathfrak{B} = \{x \in \mathfrak{B} : x \leq \mathfrak{u}_u\}$.

We show that $\exists$ has a winning strategy in an Ehrenfeucht–Fraissé-game over ($\mathfrak{A}, \mathfrak{B}$) concluding that $\mathfrak{A} \equiv \mathfrak{B}$. At any stage of the game, if $\forall$ places a pebble on one of $\mathfrak{A}$ or $\mathfrak{B}$, $\exists$ must place a matching pebble, on the other algebra. Let $\bar{a} = \langle a_0, a_1, \ldots, a_{n-1} \rangle$ be the position of the pebbles played so far (by either player) on $\mathfrak{A}$ and let $\bar{b} = \langle b_0, \ldots, b_{n-1} \rangle$ be the the position of the pebbles played on $\mathfrak{B}$. $\exists$ maintains the following properties throughout the game: For any atom $x$ (of either algebra) with $x \cdot \mathfrak{u}_u = 0$ then $x \in a_i \iff x \in b_i$ and $\bar{a}$ induces a finite partition of $\mathfrak{1}_u$ in $\mathfrak{A}$ of $2^n$ (possibly empty) parts $p_i : i < 2^n$ and $\bar{b}$ induces a partition of $\mathfrak{1}_u$ in $\mathfrak{B}$ of parts $q_i : i < 2^n$. Furthermore, $p_i$ is finite $\iff q_i$ is finite and, in this case, $|p_i| = |q_i|$. That such properties can be maintained is fairly easy to show.

Using that $M$ has quantifier elimination we get, using the same argument in op.cit that $\mathfrak{A} \in \mathfrak{N}_n \mathfrak{C} \mathfrak{A}_n \cap \mathfrak{R} \mathfrak{C} \mathfrak{A}_n$ and $\mathfrak{S} \mathfrak{N}_n \mathfrak{C} \mathfrak{A}_n$. So hoping for a contradiction, we can only assume that there is a class $M$ between $\mathfrak{N}_n \mathfrak{C} \mathfrak{A}_n \cap \mathfrak{R} \mathfrak{C} \mathfrak{A}_n$ and $\mathfrak{S} \mathfrak{N}_n \mathfrak{C} \mathfrak{A}_n \cap \mathfrak{R} \mathfrak{C} \mathfrak{A}_n$ that is first order definable. Then $\mathfrak{E}(\mathfrak{N}_n \mathfrak{C} \mathfrak{A}_n \cap \mathfrak{R} \mathfrak{C} \mathfrak{A}_n) \subseteq M \subseteq \mathfrak{S} \mathfrak{N}_n \mathfrak{C} \mathfrak{A}_n \cap \mathfrak{R} \mathfrak{C} \mathfrak{A}_n$. We have $\mathfrak{B} \equiv \mathfrak{A}$, and $\mathfrak{A} \in \mathfrak{N}_n \mathfrak{C} \mathfrak{A}_n \cap \mathfrak{R} \mathfrak{C} \mathfrak{A}_n$, hence $\mathfrak{B} \in \mathfrak{E}(\mathfrak{N}_n \mathfrak{C} \mathfrak{A}_n \cap \mathfrak{R} \mathfrak{C} \mathfrak{A}_n) \subseteq \mathfrak{S} \mathfrak{N}_n \mathfrak{C} \mathfrak{A}_n \cap \mathfrak{R} \mathfrak{C} \mathfrak{A}_n$.

We show that $\mathfrak{B}$ is in fact outside $\mathfrak{S} \mathfrak{N}_n \mathfrak{C} \mathfrak{A}_n \cap \mathfrak{A} \mathfrak{T} \supseteq \mathfrak{S} \mathfrak{N}_n \mathfrak{C} \mathfrak{A}_n \cap \mathfrak{R} \mathfrak{C} \mathfrak{A}_n$ getting the hoped for contradiction, and consequently the required. Take $\kappa$ the signature of $M$ to be $2^{2^n}$ and assume for contradiction that $\mathfrak{B} \in \mathfrak{S} \mathfrak{N}_n \mathfrak{C} \mathfrak{A}_n \cap \mathfrak{A} \mathfrak{T}$. Then $\mathfrak{B} \subseteq \mathfrak{N}_n \mathfrak{D}$, for some $\mathfrak{D} \in \mathfrak{C} \mathfrak{A}_n$ and $\mathfrak{N}_n \mathfrak{D}$ is atomic. For brevity, let $\mathfrak{C} = \mathfrak{N}_n \mathfrak{D}$. Then $R\mathfrak{I}_d \mathfrak{B} \subseteq R\mathfrak{I}_d \mathfrak{C}$. Since $\mathfrak{C}$ is atomic, then $R\mathfrak{I}_d \mathfrak{C}$ is also atomic. Using the same reasoning as above, we get that $|R\mathfrak{I}_d \mathfrak{C}| > 2^n$ (since $\mathfrak{C} = \mathfrak{N}_n \mathfrak{D}$). By the choice of $\kappa$, we get that $|\mathfrak{A} R\mathfrak{I}_d \mathfrak{C}| > \omega$.

By density, $\mathfrak{A} R\mathfrak{I}_d \mathfrak{C} \subseteq R\mathfrak{I}_d \mathfrak{B}$, so $|\mathfrak{A} R\mathfrak{I}_d \mathfrak{B}| \geq |\mathfrak{A} R\mathfrak{I}_d \mathfrak{C}| > \omega$. But by the construction of $\mathfrak{B}$, we have $|\mathfrak{A} R\mathfrak{I}_d \mathfrak{B}| = |\mathfrak{A} R\mathfrak{I}_d \mathfrak{C}| = \omega$, which is a contradiction and we are done.

4.2 Marking the boundaries

**Example 4.7.** Let $1 < n < \omega$. Here we show that for $\mathfrak{B}(\in \mathfrak{R} \mathfrak{C} \mathfrak{A}_n)$ given in example 4.2, $\exists$ has a winning strategy in $H_\omega(\mathfrak{A} \mathfrak{T} \mathfrak{B})$. Recall that $\mathfrak{B} \notin \mathfrak{N}_n \mathfrak{C} \mathfrak{A}_n$ but $\mathfrak{A} \mathfrak{B} \in \mathfrak{A} \mathfrak{T} \mathfrak{N}_n \mathfrak{C} \mathfrak{A}_n$. We describe the winning strategy of $\exists$ in $H_\omega(\mathfrak{A} \mathfrak{T} \mathfrak{B})$. We start by describing $\exists$’s strategy dealing with $\lambda$–neat hypernetworks, where $\lambda$ is a constant label kept on short hyperedges. In a play, $\exists$ is required to play $\lambda$–neat hypernetworks, so she has no choice about the the short edges, these are labelled by $\lambda$. In response to a cylindermove by $\forall$ extending the current hypernetwork providing a new node $k$, and a previously played coloured hypernetwork $M$ all long hyperedges not incident with $k$ necessarily keep the hyperlabel they had in $M$.  

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All long hyperedges incident with \( k \) in \( M \) are given unique hyperlabels not occurring as the hyperlabel of any other hyperedge in \( M \). In response to an amalgamation move, which involves two hypernetworks required to be amalgamated, say \((M, N)\) all long hyperedges whose range is contained in \( \text{nodes}(M) \) have hyperlabel determined by \( M \), and those whose range is contained in \( \text{nodes}(N) \) have hyperlabels determined by \( N \). If \( \bar{x} \) is a long hyperedge of \( \exists \)'s response \( L \) where \( \text{rng}(\bar{x}) \not\subseteq \text{nodes}(M), \text{nodes}(N) \) then \( \bar{x} \) is given a new hyperlabel, not used in any previously played hypernetwork and not used within \( L \) as the label of any hyperedge other than \( \bar{x} \). This completes her strategy for labelling hyperedges.

The winning strategy for \( \exists \) is to play \( \lambda \)-neat hypernetworks \((N^a, N^h)\) with \( \text{nodes}(N_a) \subseteq \omega \) such that \( (N^a)^+ \neq 0 \) (recall that \( (N^a)^+ \) is as defined in the proof of lemma 3.3). In the initial round, let \( \forall \) play \( a \in A_t \). \( \exists \) plays a network \( N \) with \( N^a \in 0, \ldots, n - 1 \) = \( a \).

Then \( (N^a)^+ = a \neq 0 \). The response to the cylindrifier move is exactly like in the first part of lemma 3.3 because \( B \) is completely representable so \( B \subseteq S, NR_tCA_\omega \) \cite{27, Theorem 5.3.6}. For transformation moves: if \( \forall \) plays \((M, \theta)\), then it is easy to see that we have \((M^\theta)^+ \neq 0 \), so this response is maintained in the next round.

For the amalgamation (new) move, as far as the proof of lemma 3.3 is concerned, we need some preparing to do. We use the argument in \cite{14, Lemma 34}. For each \( J \subseteq \omega, |J| = n \) say, let \( NR_tD = \{ x \in D : c_jx = x, \forall l \in \omega \sim J \} \). Then it can be shown, using that \( AtB \in AtNR_tCA_\omega \), that for all \( y \in NR_tD \), where \( J = \{ i_0, i_1, \ldots, i_{n-1} \} \), the following holds for \( a \in \alpha : s_{i_0j_1n_{i_1}} \cdots y \not= 0 \implies s_{i_0j_1n_{i_1}} \cdots a y \not= 0 \).

Now we are ready to describe \( \exists \)'s strategy in response to amalgamation moves. For better readability, we write \( \bar{i} \) for \( \{ i_0, i_1, \ldots, i_{n-1} \} \), if it occurs as a set, and we write \( s_i \) short for \( s_{i_0i_1} \cdots i_{n-1} \). Also we only deal with the network part of the game.

Now suppose that \( \forall \) plays the amalgamation move \((M, N)\) where \( \text{nodes}(M) \cap \text{nodes}(N) = \{ \bar{i} \} \), then \( M(\bar{i}) = N(\bar{i}) \). Let \( \mu = \text{nodes}(M) \sim \bar{i} \) and \( v = \text{nodes}(N) \sim \bar{i} \). Then \( c_vM^+ = M^+ \) and \( c_vN^+ = M^+ \). Hence using \((*)\), we have: \( c_vM^+ = s_\mu \bar{M} = s_\mu N(\bar{i}) = c_vN^+ \) so \( c_vM^+ = M^+ \leq c_vN^+ = c_vN^+ \) and \( M^+ \cdot N^+ \neq 0 \). So there is \( L \) with \( \text{nodes}(L) = \text{nodes}(M) \cup \text{nodes}(N) \neq 0 \), and \( L^+ \cdot x \not= 0 \), where \( M^+ \cdot N^+ = x \), thus \( L^+ \cdot M^+ \neq 0 \) and consequently \( L^+ \neq 0 \).

4.3 Relation algebras

**Theorem 4.8.** Any class between \( S_tRaCA_\omega \cap \text{CRRA} \) and \( S_tRaCA_5 \) and any class between \( AtRaCA_\omega \) and \( AtS_tRaCA_5 \) are not elementary.

**Proof.** We prove the second part. One uses the arguments in \cite{14, Theorem 39, 45}, but resorting to the game \( H_k \) \( (k < \omega) \), as defined for relation algebras \cite{14, Definition 28}. Now we have the countable relation algebra atom structure \( \beta \) based on \( \mathbb{N} \) and \( \mathbb{Z} \) as defined in \cite{14}, for which \( \exists \) has a winning strategy in \( H_k(\text{cR}) \), for all \( k < \omega \), and \( \forall \) has a winning strategy in \( F^\beta(\alpha) \) with \( F^\beta \) as in \cite{14, Definition 28}. By the RA analogue of lemma 3.3 proved in \cite{14}, we get that \( \text{cR} \not\in S_tRaCA_5 \). The usual argument of taking an ultrapower of \( \text{cR} \), followed by a downward elementary chain argument, one gets a countable atom structure \( \alpha \), such that \( \exists \alpha \equiv \text{cR} \) and \( \exists \) has a winning strategy in \( H(\alpha) \), so using exactly the same argument in \cite{14} allowing infinite conjunction in \cite{14, Theorem 39}, we get \( \exists \alpha \not\in S_tRaCA_\omega \). In particular, \( \text{cR} \not\in \text{RaCA} \sim \text{CRRA} \) and any algebra having atom structure \( \alpha \) is completely representable.

The last result is an improvement of the result in \cite{15} by \cite{14, Theorem 36}. The integral relation algebra (in which \( \text{Id} \) is an atom) defined next by listing its forbidden
tripples, will be used in the proof of the next theorem.

**Example 4.9.** Take $\mathcal{R}$ to be a symmetric, atomic relation algebra with atoms

$$\text{Id}, r(i), y(i), b(i) : i < \omega.$$ 

Non-identity atoms have colours, $r$ is red, $b$ is blue, and $y$ is yellow. All atoms are self-converse. Composition of atoms is defined by listing the forbidden triples. The forbidden triples are (Peircean transforms) or permutations of $(\text{Id}, x, y)$ for $x \neq y$, and

$$(r(i), r(i), r(j)), (y(i), y(i), y(j)), (b(i), b(i), b(j)) : i \leq j < \omega$$

$\mathcal{R}$ is the complex algebra over this atom structure.

Let $\alpha$ be an ordinal. $\mathcal{R}^{\alpha}$ is obtained from $\mathcal{R}$ by splitting the atom $r(0)$ into $\alpha$ parts $r^k(0) : k < \alpha$ and then taking the full complex algebra. In more detail, we put red atoms $r^k(0)$ for $k < \alpha$. In the altered algebra the forbidden triples are $(y(i), y(i), y(j)), (b(i), b(i), b(j)) : i \leq j < \omega, (r(i), r(i), r(j)) : 0 < i \leq j < \omega, (r^k(0), r^l(0), r^m(0)) : k, l, m < \alpha$. These algebras were used in [26] to show that $\text{RaCA}_k$ for all $k \geq 5$ is not elementary.

**Theorem 4.10.** Any class between $\text{RaCA}_\omega$ and $\text{RaCA}_5$, as well as the class $\text{CRRA}$, is not closed under $\equiv_{\infty, \omega}$.

*Proof.* In $\mathcal{R}^{\alpha}$, we use the following abbreviations: $r(0) = \sum_{k<\alpha} r^k(0)$, $r = \sum_{i<\omega} r(i)$, $y = \sum_{i<\omega} y(i)$ and $b = \sum_{i<\omega} b(i)$. These suprema exist because they are taken in the complex algebras which are complete. The index of $r(i), y(i)$ and $b(i)$ is $i$ and the index of $r^k(0)$ is also 0. Now let $\mathcal{B} = \mathcal{R}^{\omega}$ and $\mathfrak{A} = \mathcal{R}^{\alpha}$. We claim that $\mathcal{B} \in \text{RaCA}_\omega$ and $\mathfrak{A} \equiv \mathcal{B}$. For the first required, it is shown in [26] that $\mathcal{B}$ has a cylindric basis by exhibiting a winning strategy for $\mathfrak{A}$ in the the cylindric-basis game, which is a simpler version of the hyperbasis game [17, Definition 12.26]. Now, let $\mathcal{F}$ be an $\omega$-dimensional cylindric basis for $\mathcal{B}$. Then $\text{CaF} \in \text{CA}_\omega$. Consider the cylindric algebra $\mathcal{C} = \mathcal{S}_0^{\text{CaF}} \mathcal{B}$, the subalgebra of $\text{CaF}$ generated by $\mathcal{B}$. In principal, new two dimensional elements that were not originally in $\mathcal{B}$, can be created in $\mathcal{C}$ using the spare dimensions in $\text{CaF}$. But in fact $\mathcal{B}$ exhausts the 2-dimensional elements of $\text{RaC}_\mathcal{C}$, more concisely, we have $\mathcal{B} = \text{RaC} \mathcal{C}$ [26]. Like the proof of theorem 4.1, we show that $\mathcal{A} \equiv_{\infty, \omega} \mathcal{B}$ by replacing $\text{Id}$ by the newly splitted $r(0)$. We have proved that $\mathcal{B} \in \text{RaCA}_\omega$ and $\mathfrak{A} \equiv \mathcal{B}$. In [26], it is proved that $\mathfrak{A} \notin \text{RaCA}_5$. So we get the first required, namely, that any class $K$, such that $\text{RaCA}_K \subseteq K \subseteq \text{RaCA}_5$ is not closed under $\equiv_{\infty, \omega}$.

Now we show that CRRA is not closed under $\equiv_{\infty, \omega}$, strengthening the result in [16] that only shows that CRRA is not closed under elementary equivalence proving the remaining required. Since $\mathcal{B} \in \text{RaCA}_\omega$ has countably many atoms, then $\mathcal{B}$ is completely representable [14, Theorem 29]. For this purpose, we show that $\mathfrak{A}$ is not completely representable. We work with the term algebra, $\text{TermAtA}$, since the latter is completely representable $\iff$ the complex algebra is. Let $r = \{r(i) : 1 \leq i < \omega\} \cup \{r^k(0) : k < 2^\omega\}$, $y = \{y(i) : i \in \omega\}$, $b^+ = \{b(i) : i \in \omega\}$. It is not hard to check every element of $\text{TermAtA} \subseteq \text{TermAtC}$ has the form $F \cup R_0 \cup B_0 \cup Y_0$, where $F$ is a finite set of atoms, $R_0$ is either empty or a co-finite subset of $r$, $B_0$ is either empty or a co-finite subset of $b$, and $Y_0$ is either empty or a co-finite subset of $y$. We show that the existence of a complete representation necessarily forces a monochromatic triangle, that we avoided at the start when defining $\mathfrak{A}$. Let $x, y$ be points in the representation with $M \models y(0)(x, y)$. For each $i < 2^\omega$, there is a point $z_i \in M$ such that $M \models \text{red}(x, z_i) \land y(0)(z_i, y)$ (some red
Let $Z = \{ z_i : i < 2^{|\omega|} \}$. Within $Z$ each edge is labelled by one of the $\omega$ atoms in $y^+$ or $b^\perp$. The Erdős-Rado theorem forces the existence of three points $z^1, z^2, z^3 \in Z$ such that $M \models y(j)(z^1, z^2) \land y(j)(z^2, z^3) \land y(j)(z^3, z_1)$, for some single $j < \omega$ or three points $z^1, z^2, z^3 \in Z$ such that $M \models b(l)(z^1, z^2) \land b(l)(z^2, z^3) \land b(l)(z^3, z_1)$, for some single $l < \omega$. This contradicts the definition of composition in $A$, since we avoided monochromatic triangles. We have proved that $\text{CRRA}$ is not closed under $\equiv_{\omega, \omega}$, since $A \equiv_{\infty, \omega} B$, $A$ is not completely representable, but $B$ is completely representable.

Fix $2 < n < \omega$. The algebra $A \in \text{NRCA}_n$ used in theorem 4.1 can be viewed as splitting the atoms of the atom structure $A = (n, \forall, D_{ij})_{i,j < n}$ each to $\kappa$-many atoms so $A$ can be denoted $\text{split}(1_{nd}, A, \kappa)$ ($\kappa$ an uncountable cardinal). The algebra $B \notin \text{NRCA}_{n+1}$ can be viewed as splitting the same atom structure, each atom – except for one atom that is split into countably many atoms – is also split into $\kappa$-many atoms, so $B$ can be denoted by $\text{split}(1_{nd}, A, \omega)$. By the same token, for an ordinal $\alpha$, $B^\alpha$ can be denoted by $\text{split}(\alpha(0), A, \alpha)$ after splitting (the red atom) $\alpha(0)$ into $\alpha$ parts.

Example 4.11. Fix $2 < n < \omega$. The usual atomic game $G$ grips $\text{CRCA}_n$. The game $H$ used in theorem 4.1 weakly grips $\text{NRCA}_n$ but $H$ does not grip $\text{NRCA}_\omega$. We devise an $\omega$-rounded non-atomic game $G$ gripping $\text{NRCA}_n$. By non-atomic, we mean that arbitrary elements of the algebra not necessarily atoms are allowed during the play. By example 4.2, $G$ is stronger than $H$.

The game $G$ is played on both $\lambda$-neat hypernetworks as defined in the proof of the first item of theorem 4.1, and complete labelled graphs (possibly non-atoms) with no consistency conditions. The play at a certain point, like in $H$ as in the third part of the first item of theorem 4.1, will be a $\lambda$-neat hypernetwork, call its network part $X$, and we write $X(\bar{x})$ for the atom the edge $\bar{x}$. By network part we mean forgetting $k$-hypedges getting non-atomic labels. An $n$-matrix is a finite complete graph with nodes including $0, \ldots, n-1$ with all edges labelled by arbitrary elements of $B$. No consistency properties are assumed. $G$ can play an arbitrary $n$-matrix $N$, $\exists$ must replace $N(0, \ldots, n-1)$, by some element $a \in B$; this is a non-atomic move. The final move is that $\forall$ can pick a previously played $n$-matrix $N$, and pick any tuple $\bar{x} = (x_0, \ldots, x_{n-1})$ whose atomic label is below $N(0, \ldots, n-1)$. $\exists$ must respond by extending $X$ to $X'$ such that there is an embedding $\theta$ of $N$ into $X'$ such that $\theta(0) = x_0, \ldots, \theta(n-1) = x_{n-1}$ and for all $i_0, \ldots, i_{n-1} \in N$, we have

$$X(\theta(i_0) \ldots, \theta(i_{n-1})) \leq N(i_0, \ldots, i_{n-1}).$$

This ensures that in the limit, the constraints in $N$ really define the element $a$. Assume that $B \in \text{CA}_n$ is atomic and has countably many atoms. If $\exists$ has a winning strategy in $G(B)$, then the extra move involving non-atoms labelling matrices, ensures that that every $n$-dimensional element generated by $B$ in a dilation $D \in \text{RCA}_n$, having base $M$, constructed from a winning strategy in $G$ as the limit of the $\lambda$-neat hypernetworks played during the game (and further assuming without loss that $\forall$ plays every possible move) is already an element of $B$. For $k < \omega$, let $G_k$ be the game $G$ truncated to $k$ rounds, and let $G^\alpha$ and $G^\alpha_k$ be the relation algebra analogue of the game obtained by adding the non-atomic move replacing $n$-matrices by $2 - \text{matrices}$, to the game $H$ as defined for relation algebras [14, Definition 28].

Using the argument in the proof of the first item of theorem 4.1 replacing $H$ by $G$ we get: Assume that $2 < n < m < \omega$. If there exists a countable atom structure $\alpha$ such that $\exists$ has a winning strategy in $G_k(\text{CA}_\omega)$ for all $k \in \omega$ and $\forall$ has a winning strategy in $F^m$, then any class $K$, such that $\text{NRCA}_\omega \subseteq K \subseteq S, \text{NRCA}_m$, is not elementary. We have
already proved the last result. The relation algebra case is more interesting. Undefined notation can be found in [14]; detailed citation is given in the proof.

**Theorem 4.12.** Assume that $2 < m < \omega$. If there exists a countable atom structure $\alpha$ such that $\exists$ has a winning strategy in $G^\alpha_k(\mathcal{C}\mathcal{M}_\alpha)$ for all $k \in \omega$ and $\forall$ has a winning strategy in $F^m$, then any class $K$, such that $\text{RaCA}_\omega \subseteq K \subseteq S_\omega \text{RaCA}_5$, is not elementary.

**Proof.** The analogous result can be obtained for relation algebras for $2 < m < \omega$ obtained by replacing $\text{Nr}_n$ by $\text{Ra}$. One uses the arguments in [14, Theorem 39, 45], but resorting to the game $G^\alpha_k$ in place of $H_k (k < \omega)$, as defined for relation algebras [14, Definition 28]. Now by assumption we have a countable relation algebra atom structure $\alpha$, for which $\exists$ has a winning strategy in $G^\alpha_k(\mathcal{C}\mathcal{M}_\alpha)$, for all $k < \omega$, and $\forall$ has a winning strategy in $F^m(\alpha)$ with $F^m$ as defined in [14, Definition 28]. By the RA analogue of lemma 3.3 proved in [14, Theorem 33], we get that $\mathcal{C}\mathcal{M}_\alpha \notin S_\omega \text{RaCA}_n$. The usual argument of taking an ultrapower of $\mathcal{C}\mathcal{M}_\alpha$, followed by a downward elementary chain argument, one gets a countable $\mathcal{B} \in RA$, such that $\mathcal{B} \equiv \mathcal{C}\mathcal{M}_\alpha$ and $\exists$ has a winning strategy in $G^\alpha_k(\mathcal{B})$, so $\mathcal{B} \in \text{RaCA}_\omega$ because $G^\alpha$ grips $\text{RaCA}_\omega$. Hence for any $K$, such that $\text{RaCA}_\omega \subseteq K \subseteq S_2 \text{RaCA}_5$, we have $\mathcal{C}\mathcal{M}_\alpha \notin K$, $\mathcal{B} \in K$ and $\mathcal{C}\mathcal{M}_\alpha \equiv \mathcal{B}$. □

In the next table we summarize the results obtained on non-first order definability proved in theorems 4.1, 4.8, 4.10. The last column in the second row remains unsettled for RAs.

<table>
<thead>
<tr>
<th>Cylindric algebras</th>
<th>Relation algebras</th>
<th>Elementary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Nr}<em>n \text{CA}</em>\omega \subseteq K \subseteq S_1 \text{Nr}<em>n \text{CA}</em>{n+3}$</td>
<td>$\text{RaCA}_\omega \subseteq K \subseteq S_3 \text{RaCA}_5$</td>
<td>no for CAs</td>
</tr>
<tr>
<td>$\text{AtNr}<em>n \text{CA}</em>\omega \subseteq K \subseteq \text{AtS}_n \text{Nr}<em>n \text{CA}</em>{n+3}$</td>
<td>$\text{AtRaCA}_\omega \subseteq K \subseteq \text{AtS}_3 \text{RaCA}_5$</td>
<td>no</td>
</tr>
<tr>
<td>$S_3 \text{Nr}<em>n \text{CA}</em>\omega \subseteq K \subseteq S_1 \text{Nr}<em>n \text{CA}</em>{n+3}$</td>
<td>$S_1 \text{RaCA}_\omega \subseteq K \subseteq S_3 \text{RaCA}_5$</td>
<td>no</td>
</tr>
<tr>
<td>$S_1 \text{Nr}<em>n \text{CA}</em>\omega \subseteq K \subseteq S_1 \text{Nr}<em>n \text{CA}</em>{n+3}$</td>
<td>$S_1 \text{RaCA}_\omega \subseteq K \subseteq S_3 \text{RaCA}_5$</td>
<td>no</td>
</tr>
<tr>
<td>$\text{AtS}_1 \text{Nr}<em>n \text{CA}</em>\omega \subseteq K \subseteq \text{AtS}_3 \text{Nr}<em>n \text{CA}</em>{n+3}$</td>
<td>$\text{AtS}<em>1 \text{RaCA}</em>\omega \subseteq K \subseteq \text{AtS}_3 \text{RaCA}_5$</td>
<td>no</td>
</tr>
<tr>
<td>$\text{Nr}<em>n \text{CA}</em>\omega \subseteq K \subseteq \text{Nr}<em>n \text{CA}</em>{n+1}$</td>
<td>$\text{RaCA}_\omega \subseteq K \subseteq \text{RaCA}_5$</td>
<td>no</td>
</tr>
</tbody>
</table>

### 4.4 Other algebras

Fix $2 < n < \omega$. Now we investigate the analogues of the results proved in theorem 3.4 for several cylindric–like algebras, like diagonal free CAs (DF), Pinter’s substitution algebras (Sc) and quasi-polyadic equality algebras (QEA). For any class $K$ between DF and QEA, and any ordinal $\alpha$, $K_\alpha$ denotes the class of K–algebras of dimension $\alpha$ and $\text{RK}_\alpha$ denotes the class of representable $K_\alpha$s.

The next lemma will enable to obtain the result (on non–atom canonicity) for RDF$_n$ by bouncing it back to the (known) RCA$_n$ case. It is generally useful to transfer results from RCA$_n$s to their diagonal free reducts. Also, it generalizes a result of Johnson [13, Theorem 5.4.26]. Johnson’s result is the special case when only finite intersections are allowed. Henceforth, we write $\text{DF}_d$ short hand for ‘diagonal free reduct’.

**Lemma 4.13.** Let $2 < n < \omega$. Assume that $\mathcal{A} \in \text{CA}_n$. $\text{DF}_d \mathcal{A}$ is a diagonal free cylindric set algebra (of dimension $n$) with base $U$, and $R \subseteq U \times U$ are as in the hypothesis of [13, Theorem 5.1.49]. Let $E = \{x \in A : (\forall y, y \in nU)(\forall i < n)(x_i R_y \implies (x \in X \iff y \in X))\}$. Then $\{x \in A : \Delta x \neq n\} \subseteq E$ and $E \in \text{CA}_n$ is closed under infinite intersections. In particular, if $\mathcal{A} \in \text{CA}_n$, is such that its diagonal free reduct is representable, and $\mathcal{A}$ is generated by $\{x \in A : \Delta x \neq n\}$ using infinite intersections (together with the other cylindric operations) then $\mathcal{A} \in \text{RCA}_n$.
Proof. [13, Lemma 5.1.50, Theorem 5.1.51]. In the former lemma, using the notation in op.cit, one just has to check that $E$ (as defined above) is closed under infinite intersections. This is completely straightforward following directly from the definition of $E$. In more detail, let $X_j : j \in J$ be in $E$. We will show that $\bigcap_{j \in J} X_j \in E$. Let $x, y \in nU$ such that $x_iR_y$ for all $i < \alpha$, and assume that $x \in \bigcap_{j \in J} X_j$. Then $x \in X_j$ for every $j \in J$. Now fix $i \in J$. Then $x \in X_i$ and $X_i \in E$, so by definition of $E$ we get that $y \in X_i$. Since $i$ was arbitrary, we get that $y \in \bigcap_{j \in J} X_j$. By symmetry we are done. \[\square\]

Lemma 5.1.4. For any ordinal $\alpha > 1$, and any uncountable cardinal $\kappa \geq |\alpha|$, there exist completely representable algebras $\mathfrak{A}, \mathfrak{B} \in \text{QEA}_{\alpha}$, that are weak set algebras, such that $|\mathfrak{A}| = |\mathfrak{B}| = \kappa$, $\mathfrak{A} \in \text{Nr}_n \text{QEA}_{\alpha + \omega}$, $\mathfrak{A} \neq \text{Nr}_n \text{S}_{\alpha + 1}$, $\mathfrak{A} \equiv_{\omega, \omega} \mathfrak{B}$ and $\text{At}_{\mathfrak{A}} \equiv \omega, \omega \text{ At}_{\mathfrak{B}}$.

Proof. Here we consider only finite dimensions. Let $L$ be a signature consisting of the unary relation symbols $P_0, P_1, \ldots, P_{n-1}$ and uncountably many $n$-ary predicate symbols. $M$ is as in [27, Lemma 5.1.3], but the tenary relations are replaced by $n$-ary ones, and we require that the interpretations of the $n$-ary relations in $M$ are pairwise disjoint not only distinct. This can be fixed. In addition to pairwise disjointness of $n$-ary relations, we require their symmetry, that is, permuting the variables does not change their semantics.

For $u \in n^n$, let $\chi_u$ be the formula $\bigwedge_{u \in n^n} P_u(x_i)$. We assume that the $n$-ary relation symbols are indexed by (an uncountable set) $I$ and that there is a binary operation $+$ on $I$, such that $(I, +)$ is an abelian group, and for distinct $i \neq j \in I$, we have $R_i \circ R_j = R_{i+j}$. For $n \leq k \leq \omega$, let $\mathfrak{A}_k = \{\phi^M : \phi \in L_k\}(\subseteq \phi^M)$, where $\phi$ is taken in the signature $L$, and $\phi^M = \{s \in \phi^M \setminus M : M \models \phi[s]\}$.

Let $\mathfrak{A} = \mathfrak{A}_n$, then $\mathfrak{A} \in \text{PES}_n$ by the added symmetry condition. Also $\mathfrak{A} \cong \text{Nr}_n \mathfrak{A}_\omega$; the isomorphism is given by $\phi^M \mapsto \phi^M$. The map is obviously an injective homomorphism; it is surjective, because $M$ (as stipulated in [27, item (1) of lemma 5.1.3]), has quantifier elimination.

For $u \in n^n$, let $\mathfrak{A}_u = \{x \in \mathfrak{A} : x \leq \chi_u^M\}$. Then $\mathfrak{A}_u$ is an uncountable and atomic Boolean algebra (atomicity follows from the new disjointness condition) and $\mathfrak{A}_u \cong \text{Cof}(|I|)$, the finite–cofinite Boolean algebra on $|I|$. Define a map $f : \mathfrak{B} \text{At} \rightarrow \mathfrak{P} \equiv \mathfrak{A}_u$, by $f(a) = \langle a \cdot \chi_u \rangle_{u \in n^n+1}$.

Let $\mathfrak{P}$ denote the structure for the signature of Boolean algebras expanded by constant symbols $1_u, u \in n^n$, $u_{ij}$, and unary relation symbols $s_{[ij]}$ for each $i, j \in n$. Then for each $i < j < n$, there are quantifier free formulas $\eta_i(x, y)$ and $\eta_{ij}(x, y)$ such that $\mathfrak{P} \models \eta_i(f(a), b) \iff b = f(c_{ij}a)$, and $\mathfrak{P} \models \eta_{ij}(f(a), b) \iff b = f(s_{[ij]}a)$.

The one corresponding to cylindrifiers is exactly like the CA case [27, pp.113-114]. For substitutions corresponding to transpositions, it is simply $y = s_{[ij]}x$. The diagonal elements and the boolean operations are easy to interpret. Hence, $\mathfrak{P}$ is interpretable in $\mathfrak{A}$, and the interpretation is one dimensional and quantifier free. For $v \in n^n$, by the Tarski–Skolem downward theorem, let $\mathfrak{B}_v$ be a countable elementary subalgebra of $\mathfrak{A}_v$. (Here we are using the countable signature of PEA$_h$). Let $\mathfrak{S}_u(\subseteq n^n)$ be the set of permutations in $n^n$.

Take $u_1 = (0, 1, 0, \ldots, 0)$ and $u_2 = (1, 0, 0, \ldots, 0) \in n^n$. Let $v = \tau(u_1, u_2)$ where $\tau(x, y) = c_1(c_0 x \cdot s_{[01]}c_1 y) \cdot c_1 x \cdot c_0 y$. We call $\tau$ an approximate witness. It is not hard to show that $\tau(u_1, u_2)$ is actually the composition of $u_1$ and $u_2$, so that $\tau(u_1, u_2)$ is the constant zero map; which we denote by 0; it is also in $n^n$. Clearly for every $i < j < n$, $s_{[ij]}^{-1}n \{0\} \neq \{u_1, u_2\}$.

We can assume without loss that the Boolean reduct of $\mathfrak{A}$ is the following product:

$$\mathfrak{A}_{u_1} \times \mathfrak{A}_{u_2} \times \mathfrak{A}_0 \times \mathfrak{P}_{u \in V \sim J} \mathfrak{A}_u,$$
where \( J = \{ u_1, u_2, 0 \} \). Let

\[
\mathfrak{B} = ((\mathfrak{A}_{u_1} \times \mathfrak{A}_{u_2} \times \mathfrak{B}_0 \times \mathbb{P}_{u \in V \sim J} \mathfrak{A}_u), 1_u, d_{ij}, s_{i,j|x})_{i,j<n},
\]

recall that \( \mathfrak{B}_0 \prec \mathfrak{A}_0 \) and \( |\mathfrak{B}_0| = \omega \), inheriting the same interpretation. Then by the Feferman–Vaught theorem, we get that \( \mathfrak{B} \equiv \mathfrak{A} \). Now assume for contradiction, that \( \mathfrak{R}_w \mathfrak{B} = \mathfrak{N}_K \mathfrak{D} \), where \( \mathfrak{D} \in \mathfrak{S}_n+1 \). Let \( \tau_n(x,y) \), which we call an \( n \)-witness, be defined by \( c_n(s_{i,j|x} \cdot \mathcal{S}(n) c_n y) \). By a straightforward, but possibly tedious computation, one can obtain \( \mathfrak{S}_n+1 \models \tau_n(x,y) \leq \tau(x,y) \) so that the approximate witness dominates the \( n \)-witness. The term \( \tau(x,y) \) does not use any spare dimensions, and it ‘approximates’ the term \( \tau_n(x,y) \) that uses the spare dimension \( n \).

Let \( \lambda = |I| \). For brevity, we write \( 1_u \) for \( \chi_u^{M} \). The algebra \( \mathfrak{A} \) can be viewed as splitting the atoms of the atom structure \( \mathfrak{A}t = (n|, =, \neq, D_{ij})_{i,j<n} \) each to uncountably many atoms. We denote \( \mathfrak{A} \) by \( \text{split}(\mathfrak{A}, 1_0, \lambda) \). On the other hand, \( \mathfrak{B} \) can be viewed as splitting the same atom structure, each atom – except for one atom that is split into countably many atoms – is also split into uncountably many atoms (the same as in \( \mathfrak{A} \)). We denote \( \mathfrak{B} \) by \( \text{split}(\mathfrak{A}, 1_0, \omega) \).

On the ‘global’ level, namely, in the complex algebra of the finite (splitted) atom structure \( n| \), these two terms are equal, the approximate witness is the \( n \)-witness. The complex algebra \( \mathfrak{Cm}(n|) \) does not ‘see’ the \( n \)th dimension. But in the algebras \( \mathfrak{A} \) and \( \mathfrak{B} \) (obtained after splitting), the \( n \)-witness becomes then a genuine witness, not an approximate one. The approximate witness strictly dominates the \( n \)-witness. The \( n \)-witness using the spare dimension \( n \), detects the cardinality twist that \( L_{\infty,\omega}, a \text{ priori} \), first order logic misses out on. If the \( n \)-witness were term definable (in the case we have a full neat reduct of an algebra in only one extra dimension), then it takes two uncountable component to an uncountable one, and this is not possible for \( \mathfrak{B} \), because in \( \mathfrak{B} \), the target component is forced to be countable.

Now for \( x \in \mathfrak{B}_{u_1} \) and \( y \in \mathfrak{B}_{u_2} \), we have

\[
\tau_n^D(x,y) \leq \tau_n^D(\chi_{u_1}, \chi_{u_2}) \leq \tau^D(\chi_{u_1}, \chi_{u_2}) = \chi_{\tau^D(\chi_{u_1}, \chi_{u_2})} = \chi_{\tau(u_1, u_2)} = \chi_0.
\]

But for \( i \neq j \in I \), \( \tau_n^D(R_{ij}^{M_{ij}}, \chi_{u_1}, R_{ij}^{M_{ij}}, \chi_{u_2}) = R_{ij}^{M_{ij}}, \chi_0 \), and so \( \mathfrak{B}_0 \) will be uncountable, which is impossible. It can be shown that \( \mathfrak{A} \equiv \mathfrak{B} \).

**Theorem 4.15.** Let \( K \) be a class between QEA and \( \mathfrak{S}c \) and \( 2 < n < \omega \). Then the following hold:

1. The variety \( \text{SNR}_n \mathfrak{K}_{n+3} \) is not atom–canonical,
2. The variety \( \text{RDF}_n \) is not atom–canonical,
3. Any class between \( \text{NR}_n \mathfrak{K}_\omega \cap \text{CRK}_n \) and \( \mathfrak{S}_c \text{NR}_n \mathfrak{K}_{n+3} \) is not elementary.

**Proof.** The rough idea is that one works with QEA\(_n\)s and then take reducts. The changes we need are:

- For QEA\(_n\), the definition of networks should be changed adding a symmetry condition; and accordingly also the definition of the ‘network part’ of \( \lambda \)-net hypernetworks. An \( n \)-dimensional network on an atomic \( \mathfrak{A} \in \mathfrak{CA}_n \) is a map \( N : n\Delta \rightarrow \mathfrak{At}\mathfrak{A} \) that is a cylindric network with the addition that \( \bar{x} \in n\text{nodes}(N) \) and \( i < j < n \), then \( s_{[i,j]} N(\bar{x}) = N(\bar{x} \circ [i,j]) \).

For \( \mathfrak{S}c \) one replaces the condition involving diagonal elements \( i \) the CA case in a network \( N : n\Delta \rightarrow \mathfrak{At}\mathfrak{A} \) by \( s_{[i]} N(\bar{x}) = N(\bar{x} \circ [i]) \), for all \( \bar{x} \in n\text{nodes}(N) \) and \( i < j < n \).
• One uses lemma 3.3 replacing \( C_n \) by \( K_n \). The same implication holds: For \( 2 < n < \omega < m, \ \mathfrak{A} \in S_n, N_r, K_m \Rightarrow \exists \) has a winning strategy in \( G''(At\mathfrak{A}) \) restricted to atomic networks in the given signature.

• Accessibility relations corresponds to \( s_{i,j} \) should be defined for \( C_n \) rainbow atom structures giving \( QE_n \) atom structures. The accessibility binary relation corresponding substitution operator \((i < j < n)\) is defined by \( [a]S_{i,j}[b] \iff a = b \circ [i,j] \), where \( a : n \to \Delta \) and \( b : n \to \Gamma \), and \( \Delta \) and \( \Gamma \) are coloured graphs in the given rainbow signature. In this case given a rainbow atom structure \( At \) of dimension \( n \), with \( S_{i,j} \) \((i < j < n)\) defined as above, \( \mathfrak{C} At \mathfrak{A} \in RQE_n \). The \( n \)-dimensional \( QE_n \) such that if \( A \subseteq K \) taking \( \mathfrak{C} \) and \( \mathfrak{D} \) will now be denoted by \( PEA_{G,c} \).

1. One blows up and blurs the \( QE_n, PEA_{n+1,n} \) giving a polyadic atom structure \( At \). The polyadic operations on the formula algebra \( \mathfrak{A} \) and its Dedekind-MacNeille completion are defined by swapping variables. The embedding of \( PEA_{n+1,n} \) to \( \mathfrak{C} At \mathfrak{A} \) is as before. One proves that \( \forall \) has a winning strategy in \( G''(At(\mathfrak{D}^{+}_c)PDA_n(n,n+1,n)) \) for some \( r < \omega \) specifying the number of rounds, implying that \( \mathfrak{D} \) \( ((\mathfrak{D})^{+}_c) \mathfrak{C} At \mathfrak{A} \notin S_N r, S_{n+3} \). This proves the required.

2. One applies lemma 4.13 to \( \mathfrak{D}^{+}_c \mathfrak{A} \) and its Dedekind-MacNeille completion \( \mathfrak{D}^{+}_c \mathfrak{C} (At \mathfrak{A}) \) where \( \mathfrak{A} \) is the formula \( S_n \) as defined in theorem 3.4.

3. One shows that there is an \( \omega \) rounded (atomic) game \( H_\omega (\alpha) \) as defined in lemma 4.6 played on atomic \( \lambda \)-neat hypernetworks, whose network part is modified as above, such that if \( \alpha \) is a countable atom structure, and \( \exists \) has a winning strategy in \( H_\omega (\alpha) \), then any algebra \( \mathfrak{A} \) having atom structure \( \alpha \) is completely representable as a \( QE_n \), and like in the \( C_n \) case, there will exist a complete \( \mathfrak{D} \in QE_\omega \) such that \( \mathfrak{C} \alpha \equiv N_r \mathfrak{D} \) and \( \alpha \equiv At_N \mathfrak{D} \). Then the hitherto defined game \( \mathfrak{H} \) will be applied to \( At(C_n,n) \) expanded with the polyadic operations, which we denote by a slight abuse of notation, also by \( C_n \). From a winning strategy of \( \exists \) in \( H_k(At(C_n,n)) \) \((\text{where } H_k \text{ is } \mathfrak{H} \text{ truncated to } k \text{ rounds})\) for all \( k \in \omega \), it will follow that \( C_{n,n} \equiv \mathfrak{C} \mathfrak{D} \) for some completely representable atom structure \( \alpha \in At(N_r QE_\omega) \), for which \( \mathfrak{C} \alpha \in N_r QE_\omega \). The \( K_n \) reduct of term algebra \( \mathfrak{D} \mathfrak{A} \) will be used to show that any class between \( S_d, N_r K_n \cap CRK_n \) and \( S_n, N_r K_{n+3} \) is not elementary as follows: One first shows, using the modified lemma 3.3, that \( \mathfrak{D} \mathfrak{A} \mathfrak{C} \mathfrak{C} \mathfrak{A} \notin S_{d}, N_r S_{n+3} \). Then as in the \( C_n \) case one shows that there exists a \( QE_n \) atom structure \( \alpha \) such that \( \mathfrak{C} \alpha \in S_d, N_r QE_\omega \cap CRQEA_n \) such that \( \alpha \) satisfies the above two conditions, and \( C_{n,n} \equiv \mathfrak{C} \mathfrak{D} \). In forming the dilation \( \mathfrak{D} \) based on the weak model \( \mathcal{M} \) as in the proof of theorem 4.6, one adds a clause for \( s_{i,j} \) viewed as a unary atomic whose semantics is swapping the \( i \)th and \( j \)th variables. Thus any class between \( S_d, N_r K_n \cap CRK_n \) and \( S_n, N_r K_{n+3} \) is not first order definable.

To get the stronger result of removing the \( S_d \) one uses the construction in lemma 4.14 as follows: We identify notationally \( \mathfrak{A} \) and \( \mathfrak{B} \) with their \( K \) reducts. We have already excluded any first order definable class between \( S_d, N_r K_n \cap CRK_n \) and \( S_n, N_r K_{n+3} \). For contradiction, we assume that there is a class \( \mathfrak{M} \) between \( N_r K_n \cap CRK_n \) and \( S_d, N_r K_n \cap CRK_n \) that is first order definable. Then, like before, \( E(\mathfrak{M}, N_r K_n \cap CRK_n) \subseteq \mathfrak{M} \subseteq S_d, N_r K_n \cap CRK_n \). We have \( \mathfrak{B} \equiv \mathfrak{A} \) and \( \mathfrak{A} \in N_r K_n \cap CRK_n \), hence \( \mathfrak{B} \in E(\mathfrak{D}(N_r K_n \cap CRK_n)) \subseteq S_d, N_r K_n \cap CRK_n \). Also like before, we show that \( \mathfrak{D} \mathfrak{B} \) is in fact outside \( S_d, N_r K_n \cap CRK_n \). Take the cardinality of the signature of \( \mathfrak{M} \) on which \( \mathfrak{A} \) and \( \mathfrak{B} \) are based, to be \( \kappa = 2^{2^\omega} \) and \( |\mathfrak{B}| = \omega \). Assume (for contradiction) that \( \mathfrak{B} \subseteq \mathfrak{A} \), for some \( \mathfrak{D} \in K_n \) and \( N_r K_n \) is atomic. For brevity, let \( \mathfrak{E} = N_r K_n \) and denote \( \mathfrak{I}_x \) by \( x \). Then \( \mathfrak{R} \mathfrak{B} \subseteq \mathfrak{A} \mathfrak{C} \). Since \( \mathfrak{C} \) is atomic, we have \( \mathfrak{R} \mathfrak{C} \) is also atomic. Using the same reasoning as in the proof of the second item of theorem 4.2, we get that \( |\mathfrak{R} \mathfrak{E}| > 2^{\omega} \), because \( \mathfrak{R} \mathfrak{B} \) and \( \mathfrak{R} \mathfrak{E} \) where \( u = 1_{01} \) and \( v = 1_{10} \) are
Let \( \mathcal{C} \), and \( |\mathcal{R}_x\mathcal{B}| = |\mathcal{R}_y\mathcal{B}| = \lambda \). Hence \( |\text{At}\mathcal{R}_x\mathcal{C}| > \omega \). By density, observing that \( \mathcal{R}_x\mathcal{B} \) is also atomic, we get \( |\mathcal{R}_y\mathcal{B}| > \omega \), because we have \( \text{At}\mathcal{R}_x\mathcal{C} \subseteq \text{At}\mathcal{R}_y\mathcal{B} \). But \( |\mathcal{R}_x\mathcal{B}| = |\text{At}\mathcal{R}_y\mathcal{B}| = \omega \), which is a contradiction.

**Corollary 4.16.** Let \( 2 < n < \omega \) and \( k \geq 3 \). Then the following classes, together with the intersection of any two of them, the last four taken at the same \( k \), are not elementary: \( \text{CRK}_n \) [16], \( \text{Nr}_n \text{K}_{n+k} \) [27, Theorem 5.4.1], [32, 29, 30] \( \text{S}_d \text{Nr}_n \text{K}_{n+k} \) and \( \text{S}_c \text{Nr}_n \text{K}_{n+k} \).

### 5 Relativized notions of complete and strong representability via squareness and flatness

Fix \( 2 < n < \omega \). The chapter [18] is devoted to studying the following inclusions between various types of atom structures:

\[
\text{CR}_n \subseteq \text{LC}_n \subseteq \text{SR}_n \subseteq \text{WR}_n.
\]

The first is the class of completely representable atom structures, the second is the class of atom structures satisfying the Lyndon conditions and the third is the class of strongly representable atom structures, and the last is the class of weakly representable atom structures, all of dimension \( n \) where it is shown that all inclusions are proper. The first and third classes are elementary, while the second and fourth are elementary.

We lift such notions from working on the frame level to working on the algebra level restricting our attention to atomic ones; furthermore we consider also relativized versions of such notions of representability, like \( m \)-square and \( m \)-flat, where \( 2 < n < m < \omega \). (Such notions will be defined below). We start with Tarskian ‘non-relativized’ representations.

#### 5.1 Complete representations

Fix a finite ordinal \( n > 2 \). Let \( \text{GS}_n \) denote the class of generalized set algebra of dimension \( n \); \( \mathcal{C} \in \text{GS}_n \), if \( \mathcal{C} \) has top element \( V \) a disjoint union of cartesian squares, that is \( V = \bigcup_{i \in I} n_i U_i \), \( I \) is a non-empty indexing set, \( U_i \neq \emptyset \) and \( U_i \cap U_j = \emptyset \) for all \( i \neq j \). The operations of \( \mathcal{C} \) are defined like in cylindric set algebras of dimension \( n \) relativized to \( V \).

In the next theorem we approach the class \( \text{CRCA}_n \) of completely representable \( \text{CA}_n \)s.

Recall that \( \mathfrak{A} \in \text{CRCA}_n \iff \) there exists \( C \in \text{GS}_n \), and an isomorphism \( f : \mathfrak{A} \to \mathcal{C} \) such that for all \( X \subseteq \mathfrak{A} \), \( f(\Sigma X) = \bigcup_{x \in X} f(x) \), whenever \( \Sigma X \) exists in \( \mathfrak{A} \). In this case, we say that \( \mathfrak{A} \) is completely representable via \( f \). It can be easily proved that \( \mathfrak{A} \) is completely representable via \( f : \mathfrak{A} \to \mathcal{C} \), where \( \mathcal{C} \in \text{GS}_n \) has top element \( V \) say \( \iff \) \( \mathfrak{A} \) is atomic and \( f \) is atomic in the sense that \( f(\Sigma \text{At}\mathfrak{A}) = \bigcup_{x \in \text{At}\mathfrak{A}} f(x) = V \), where \( \text{At}\mathfrak{A} \) denotes the set of atoms of \( \mathfrak{A} \).

Define the class \( \text{LCA}_n \) as follows: \( \mathfrak{A} \in \text{LCA}_n \iff \mathfrak{A} \) is atomic and \( \exists \) has a winning strategy in \( G_k(\text{At}\mathfrak{A}) \) for all \( k < \omega \). Then this class is elementary because a winning strategy for \( \exists \) in \( G_k \) can be coded in first order sentence \( \rho_k \). Hirsch and Hodkinson study the class of atom structures of this class, that they call atom structures satisfying ‘Lyndon conditions’ [18]. In our context, working now on the algebra level, the Lyndon conditions are just the \( \rho_k \)s. Let \( \text{F}_n = \{ \mathfrak{A} \in \text{GS}_n : A = \varphi(U) \text{ some non-empty set } U \} \).

**Theorem 5.1.** For \( 2 < n < \omega \) the following hold:

1. \( \text{CRCA}_n \subseteq \text{S}_c \text{Nr}_n (\text{CA}_n \cap \text{At}) \cap \text{At} \subseteq \text{S}_c \text{Nr}_n \text{CA}_n \cap \text{At} \). Furthermore, \( \text{CRCA}_n \subseteq \text{S}_c \text{Nr}_n \text{CA}_n \cap \text{At} \).
2. If \( \mathfrak{A} \in \text{CRCA}_n \), then \( \exists \) has a winning strategy in \( G_\omega(\text{At}\mathfrak{A}) \) and \( G^\omega(\text{At}\mathfrak{A}) \).

3. All reverse inclusions and implications in the previous two items hold, if algebras considered have countably many atoms.

4. \( \text{CRCA}_n = S_n PFS_n \).

5. Non of the classes in the first item are elementary; at least two are distinct but the elementary closure of all three coincide. In more detail, \( \text{EICRCA}_n = \text{El}(S_n \text{Nr}_n(\text{CA}_\omega \cap \text{At}) \cap \text{At}) = \text{El}(S_n \text{Nr}_n \text{CA}_\omega \cap \text{At}) = \text{El}(S_n \text{Nr}_n(\text{CA}_\omega \cap \text{At}) = \text{LCA}_n \).

6. \( \text{LCA}_n \) is not finitely axiomatizable.

Proof. Second item follows from the first item, together with lemma 3.3. Third item follows by observing that the class \( \text{CRCA}_n \) coincides with the class of atomic algebras in \( S_n \text{Nr}_n \text{CA}_\omega \) on algebras having countably many atoms [27, Theorem 5.3.6], together with [18, Theorem 3.3.3]. In the fourth item, the inclusion \( \subseteq \) is straightforward. Conversely, assume that \( \mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C} \subseteq \mathfrak{D} \subseteq \mathfrak{E} \). Then \( \mathfrak{E} = \bigcup_{i \in I} \mathfrak{E}(a U_i) \), where \( V \) is the disjoint union of the \( U_i \), is clearly completely representable. We show that \( \mathfrak{A} \) is completely representable, too. We identify set algebras with their domain. Let \( f : \mathfrak{B} \to \mathfrak{B}(V) \) be a complete representation of \( \mathfrak{B} \), where \( V \) is a disjoint union of cartesian squares. We claim that \( g = f \mid \mathfrak{A} \) is a complete representation of \( \mathfrak{A} \). Let \( X \subseteq \mathfrak{B} \) be such that \( \sum_{i \in I} X = 1 \). Then by \( \mathfrak{A} = \mathfrak{B} \), we have \( \sum_{i \in I} X = 1 \). Furthermore, for all \( x \in X(\subseteq \mathfrak{A}) \) we have \( f(x) = g(x) \), so that \( \bigcup_{x \in X} g(x) = \bigcup_{x \in X} f(x) = V \), since \( f \) is a complete representation of \( V \), and we are done.

Non of the three classes follows from the construction in theorem 4.1. The algebra \( \mathfrak{C} \) constructed in op.cit is outside all three classes while the algebra \( \mathfrak{B} \) constructed in the same proof is in all three classes (by the first item) since it is completely representable, and \( \mathfrak{C} \equiv \mathfrak{B} \). At least two of the aforementioned classes are distinct follows from the second part of the the first item (to be proved in a while). We now prove the required equalities of the given classes. We show, as claimed, that all the given classes coincide with \( \text{LCA}_n \). Assume that \( \mathfrak{A} \in \text{LCA}_n \). Take a countable elementary subalgebra \( \mathfrak{C} \) of \( \mathfrak{A} \). Since \( \text{LCA}_n \) is elementary, then \( \mathfrak{C} \in \text{LCA}_n \), so for \( k < \omega \), \( \exists \) has a winning strategy \( \rho_k \), in \( G_k(\text{At}\mathfrak{C}) \). Using the same technique in theorem 4.1 of forming ultrapowers followed by an elementary chain argument, we get that \( \exists \) has a winning strategy in \( G_\omega(\text{At}\mathfrak{B}) \), for some countable \( \mathfrak{B} \), with \( \mathfrak{B} \equiv \mathfrak{C} \) and (by [18, Theorem 3.3.3]) \( \mathfrak{B} \) is completely representable. Thus \( \mathfrak{A} \equiv \mathfrak{B} \), hence \( \mathfrak{A} \in \text{EICRCA}_n \). Now if \( \mathfrak{A} \in S_n \text{Nr}_n \text{CA}_\omega \cap \text{At} \), then by lemma 3.3, \( \exists \) has a winning strategy in \( F^\omega(\text{At}\mathfrak{A}) \), hence in \( G_\omega(\text{At}\mathfrak{A}) \), a fortiori, in \( G_k(\text{At}\mathfrak{A}) \) for all \( k < \omega \), so \( \mathfrak{A} \in \text{LCA}_n \). Since \( \text{LCA}_n \) is elementary, we get that \( \text{El}(S_n \text{Nr}_n \text{CA}_\omega \cap \text{At}) \subseteq \text{LCA}_n \). But \( \text{CRCA}_n \subseteq S_n \text{Nr}_n \text{CA}_\omega \cap \text{At} \), hence \( \text{LCA}_n = \text{EICRCA}_n \subseteq \text{El}(S_n \text{Nr}_n \text{CA}_\omega \cap \text{At}) \subseteq \text{LCA}_n \).

Now \( S_n \text{Nr}_n \text{CA}_\omega \cap \text{At} \subseteq \text{EIS}_n \text{Nr}_n \text{CA}_\omega \cap \text{At} \), and the latter class is elementary (if \( K \) is elementary, then \( K \cap \text{At} \) is elementary), so \( \text{El}(S_n \text{Nr}_n \text{CA}_\omega \cap \text{At}) \subseteq \text{EIS}_n \text{Nr}_n \text{CA}_\omega \cap \text{At} \). Conversely, if \( \mathfrak{C} \) is in the last class, then \( \mathfrak{C} \) is atomic and \( \mathfrak{C} \equiv \mathfrak{D} \), for some \( \mathfrak{D} \in S_n \text{Nr}_n \text{CA}_\omega \). Hence \( \mathfrak{D} \) is atomic, so \( \mathfrak{D} \in S_n \text{Nr}_n \text{CA}_\omega \cap \text{At} \), thus \( \mathfrak{C} \in \text{El}(S_n \text{Nr}_n \text{CA}_\omega \cap \text{At}) \). We have shown that \( \text{EIS}_n \text{Nr}_n \text{CA}_\omega \cap \text{At} = \text{El}(S_n \text{Nr}_n \text{CA}_\omega \cap \text{At}) = \text{LCA}_n = \text{EICRCA}_n \).

Finally, by lemma 3.3, \( S_n \text{Nr}_n \text{CA}_\omega \cap \text{At} \cap \text{At} \subseteq \text{LCA}_n \), so \( \text{El}(S_n \text{Nr}_n \text{CA}_\omega \cap \text{At}) \subseteq \text{LCA}_n \). The other inclusion follows from \( \text{CRCA}_n \subseteq S_n \text{Nr}_n \text{CA}_\omega \cap \text{At} \cap \text{At} \), so \( \text{LCA}_n = \text{EICRCA}_n \subseteq \text{El}(S_n \text{Nr}_n \text{CA}_\omega \cap \text{At}) \cap \text{At} \). We have shown that all classes coincide with \( \text{LCA}_n \), which is the elementary closure of \( \text{CRCA}_n \), and we are done.

For non–finite axiomatizability, fix \( 2 < n < \omega \). For each \( 2 < n < l < \omega \), let \( \mathfrak{R}_l \) be the finite Maddux algebra \( \mathfrak{C}_{f(l)}(2,3) \) with strong \( l \)--blur \( \langle J_l, E_l \rangle \) and \( f(l) \geq l \) as specified.
in the second item of the proof of theorem [4]. Let \( \mathcal{R}_l = \text{split}(\mathcal{R}_l, J_l, E_l) \in \text{RRA} \) and let \( \mathfrak{A}_l = \mathcal{N}_l \text{split}(\mathcal{R}_l, J_l, E_l) \in \text{RCA}_n \). Then \((\mathcal{A} \mathcal{R}_l : l \in \omega \sim n)\), and \((\mathcal{A}_l : l \in \omega \sim n)\) are sequences of weakly representable atom structures that are not strongly representable with a completely representable ultraprodut. The (complex algebra) sequences \((\text{CmAt}{\mathcal{R}_l} : l \in \omega \sim n)\), \((\text{CmAt}{\mathcal{A}_l} : l \in \omega \sim n)\) are typical examples of what Hirsch and Hodkinson call ‘bad Monk (non-representable) algebras’ converging to ‘good (representable) one’, namely, their (non-trivial) ultraprodut witnessing that the varieties \( \text{RCA}_n \) and \( \text{RRA} \) are not finitey axiomatizable. The first sequence witnesses non-finite axiomatizability of \( \text{LCA}_n \), because \( \text{CmAt}{\mathfrak{A}_l} \not\subseteq \text{RCA}_n (\supseteq \text{LCA}_n) \), but \( \Pi_{l \in \omega \sim n} \text{CmAt}{\mathfrak{A}_l} / F \in \text{CRCA}_n \subseteq \text{LCA}_n \), since \( \Pi_{l \in \omega \sim n} \text{CmAt}{\mathfrak{A}_l} / F \cong \text{Cm}[\Pi_{l \in \omega \sim n} \text{At}{\mathfrak{A}_l} / F] \), and \( \Pi_{l \in \omega \sim n} \text{At}{\mathfrak{A}_l} / F \) is a completely representable atom structure.

Analogous results hold for relation algebras by replacing \( \mathcal{N}_n \text{CA}_\omega \) by \( \text{RaCA}_\omega \), cf. [14, Theorem 29] and [17, Proposition 17.4]. To the best of our knowledge, the characterizion \( \text{LCRA} = \text{ElS}_n \text{RaCA}_\omega \cap \text{At} \), where \( \text{LCRA} \) is the class of (atomic) relation algebras satifying the Lyndon conditions, is new in the context of \( \text{RAS} \), too. This equality can be proved exactly like the CA case, using the RA analogue of Lemma 3.3 proved in [14].

If \( \mathcal{V} \) is a variety of BAOs, then it is often desirable to find an elementary class of atom structures \( \mathcal{S} \) that generates \( \mathcal{V} \) in the sense that that \( \text{HSP}\mathcal{S} = \mathcal{V} \). The obvious candidate if \( \mathcal{V} \) happens to be completely additive is \( \text{At}\mathcal{V} \), for in this case \( \text{At}\mathcal{V} \) is an elementary class, a result of Venema’s available as [17, Theorem 2.84]. But by theorem 3.4, \( \text{CmAt}\text{RCA}_n \not\subseteq \text{RCA}_n \), so \( \text{At}\text{RCA}_n \) does not work for \( \text{RCA}_n \).

We say that \( \mathfrak{A} \in \text{CA}_n \) is strongly representable if it is atomic and \( \text{CmAt}\mathfrak{A} \) is representable. We let \( \text{SRC}_{\mathcal{N}_n} \) denote the class of strongly representable \( \text{CA}_n \)'s. While, by definition, \( \text{Str}(\text{RCA}_n) = \{ \mathfrak{A} \in \text{SRC}_{\mathcal{N}_n} : \text{CmAt}\mathfrak{A} \in \text{RCA}_n \} \) generates \( \text{RCA}_n \) in the stronger sense, \( \text{CmAt}\text{Str}(\text{RCA}_n) = \text{RCA}_n \), \( \text{Str}(\text{RCA}_n) \) is not elementary, because \( \text{SRC}_{\mathcal{N}_n} \) is not elementary [18]. The class \( \text{AtLCA}_n \) is denoted by \( \text{LCAS}_{\mathcal{N}_n} \) on [18, p.73]. In the third item of the next theorem, it is shown that \( \text{LCAS}_{\mathcal{N}_n} \) works, giving generation even in the aforementioned stronger sense. In the fourth item it is shown that the (smaller) class \( \text{AtEIN}_{\mathcal{N}_n} \text{CA}_\omega \) works but in the weaker sense (though \( \mathcal{H} \) is not needed). In the following theorem \( \text{Up} \) denotes the operation of forming ultraproduts, and \( \text{Ur} \) denotes the operation of forming ultraroots.

**Corollary 5.2.** Let \( 2 < n < \omega \).

1. \( \text{CmAtRCA}_n \not\subseteq \text{SN}_{\mathcal{N}_n} \text{CA}_{n+3} \).

2. For all \( k < 0 \), \( \text{CmAt(RCA}_n \cap \text{SN}_{\mathcal{N}_n} \text{CA}_{n+k}) \not\subseteq \text{RCA}_n \).

3. \( \text{CmAtLCAS}_{\mathcal{N}_n} = \text{RCA}_n \).

4. \( \text{SP}\text{CmAt(N}_{\mathcal{N}_n} \text{CA}_\omega) = \text{RCA}_n \). In fact for any class \( \mathcal{L} \) such that \( \text{AtN}_{\mathcal{N}_n} \text{CA}_\omega \subseteq \mathcal{L} \subseteq \text{LCAS}_{\mathcal{N}_n} \), \( \text{SP}\text{CmL} = \text{RCA}_n \).

5. Let \( 2 < n < \omega \). Then neither of the classes \( \text{S}_{\mathcal{N}_n} \text{N}_{\mathcal{N}_n} \text{CA}_\omega \cap \text{At} \) and \( \text{CRCA}_n \) is contained in the other.

6. The following strict inclusions hold: \( \text{N}_{\mathcal{N}_n} \text{CA}_\omega \subseteq \text{S}_{\mathcal{N}_n} \text{N}_{\mathcal{N}_n} \text{CA}_\omega \subseteq \text{S}_{\mathcal{N}_n} \text{N}_{\mathcal{N}_n} \text{CA}_\omega \subseteq \text{ElS}_{\mathcal{N}_n} \text{N}_{\mathcal{N}_n} \text{CA}_\omega \subseteq \text{SN}_{\mathcal{N}_n} \text{CA}_\omega \).

7. \( \text{N}_{\mathcal{N}_n} \text{CA}_\omega \cap \text{At} \subseteq \text{EIN}_{\mathcal{N}_n} \text{CA}_\omega \cap \text{At} \subseteq \text{ElS}_{\mathcal{N}_n} \text{N}_{\mathcal{N}_n} \text{CA}_\omega \cap \text{At} \subseteq \text{EIN}_{\mathcal{N}_n} \text{N}_{\mathcal{N}_n} \text{CA}_\omega \cap \text{At} = \text{LCA}_n \subseteq \text{SRC}_{\mathcal{N}_n} \subseteq \text{UpSRCA}_n = \text{URCA}_n = \text{EISRCA}_n \subseteq \text{SN}_{\mathcal{N}_n} \text{CA}_\omega \cap \text{At} \).
8. **EIL** for any L of the classes in the fifth item is an elementary subclass of RCA<sub>n</sub> that is not finitely axiomatizable.

**Proof.** item (1) follows from theorem 3.4, and item (2) follows from [4], where for each k > 0 a countable atomic \( \mathfrak{A} \in RCA_n \cap Nr_{n+k} \) having no complete representation, is constructed, cf. the proof of item (3) of theorem 3.1 for an explicit description of such an algebra.

Item (3): If \( \mathfrak{A} \in RCA_n \), then \( \mathfrak{A}^+ \) is completely representable, so \( At\mathfrak{A}^+ \in LCAS_n \). By \( \mathfrak{A} \subseteq \mathfrak{A}^+ = \mathfrak{A}mAt\mathfrak{A}^+ \), and \( \mathfrak{A}mAt\mathfrak{A}^+ \in \mathfrak{A}mLCAS_n \), we are done.

Item (4) follows from that the class of full \( C_s \) s denoted above by \( F_s \) is contained in \( \mathfrak{A}mAtRCA \). Indeed, suppose that \( \mathfrak{A} \in F_s \), then \( \mathfrak{A} \in Sr_{n}CA_{\omega} \), hence \( At\mathfrak{A} \in AtSr_{n}CA_{\omega} \) and \( \mathfrak{A} = \mathfrak{A}mAt\mathfrak{A} \in \mathfrak{A}mAtSr_{n}CA_{\omega} \). Thus \( RCA_n = SPF_s \subseteq SPCmAtSr_{n}CA_{\omega} \subseteq SP\mathfrak{A}mLCAS_n \subseteq RCA_n \).

Item (5): For \( S_d Sr_{n}CA_{\omega} \cap At \), we use the construction in [27, Lemma 5.1.3, Theorem 5.1.4] as modified in theorem ?? The algebra \( \mathfrak{B} \) so constructed is atomic and is outside \( S_d Sr_{n}CA_{\omega} \).

Furthermore, \( \mathfrak{B} \in CRCA_{n} \), because \( \mathfrak{B} \in S_d Sr_{n}CA_{\omega} \cap At \), we are done. Thus the identity may establishes a complete representation of \( \mathfrak{B} \).

Item (6) The algebra \( \mathfrak{B} \) constructed in the second part of item (6) of theorem 5.1 witnesses that \( Nr_{n}CA_{\omega} \subseteq S_d Nr_{n}CA_{\omega} \), that \( S_d Nr_{n}CA_{\omega} \subseteq S_d Nr_{n}CA_{\omega} \) follows from that the former class is included in the latter class. The strictness of the inclusion before the last follows from that \( S_d Nr_{n}CA_{\omega} \) is not elementary. To deal with the last strict inclusion, we give an example of an atomic \( \mathfrak{B} \in RCA_n \sim LCA_n \), cf. [17]. Then by theorem 5.1, \( \mathfrak{B} \not\in EIS\mathfrak{B}mSr_{n}CA_{\omega} \).

For item (7): The strictness of the first inclusion is witnessed by the algebra \( \mathfrak{B} \) in theorem 4.1 since \( \mathfrak{B} \in EINr_{n}CA_{\omega} \cap At \), but \( \mathfrak{B} \not\in Nr_{n}CA_{n+1} \). The second strictness is witnessed by the algebra (denoted also by) \( \mathfrak{B} \) in example 4.2, for this case, \( \mathfrak{B} \not\in EINr_{n}CA_{n+1} \), but \( \mathfrak{B} \in S_d Nr_{n}CA_{\omega} \cap At \). LCA_n \subseteq RCAS_n \), because if \( \mathfrak{A} \in CA_n \) and \( \exists \) has a winning strategy in \( G_k(At\mathfrak{A}) \) for all \( k < \omega \). The inclusion is proper, because the first class is elementary by definition, while the second is not [18]. It is known [17, Proposition 2.9] that \( U_{\mathfrak{R}CA_n} = U_{\mathfrak{R}CA_n} = EIS\mathfrak{R}CA_n \).

The strictness of inclusion \( EIS\mathfrak{R}CA_n \subseteq RCA_n \cap At \) is not trivial to show.

Proving non–finitely axiomatizability alleged in the last item, one uses [13, Construction 3.2.76, pp.94]. In op. cit non–representable finite Monk algebras outside \( RCA_n \supseteq EIS\mathfrak{R}CA_n \supseteq LCA_n \) are constructed, such that any (atomic) non–trivial ultraproduct of such algebras is in \( Nr_{n}CA_{\omega} \cap At \subseteq EINr_{n}CA_{\omega} \cap At \subseteq EIS\mathfrak{B}mSr_{n}CA_{\omega} \cap At = LCA_n \subseteq EIS\mathfrak{R}CA_n \). (Witness too the proof of the last item in theorem 5.1).

Recall that for relation algebras \( \mathfrak{A}mAtRRA \not\subseteq SRaCA_n \), theorem 3.2. For relation algebras, the analogues of (3) and (4) hold, as well. The analogue of the inclusions of item (5) involves some open questions and the last item holds for RAs.

In the next theorem \( \mathfrak{H} \) denotes the operation of forming homomorphic images, \( \mathfrak{U} \) denotes the operation of forming ultraproducts and \( \mathfrak{U} \) denotes the operator of forming ultraroots. For a class \( K \), recall that \( EIK = \mathfrak{U}k \).

**Theorem 5.3.** Let \( 2 < n < \omega \). Then the following hold.

1. The classes \( CRCA_n \) and \( Nr_{n}CA_m \) for \( n < m \) are pseudo–elementary but not elementary, nor pseudo–universal. Furthermore, their elementary theory is recursively enumerable.
2. The class \( \text{CRCA}_n \) is closed under \( S_d \), a fortiori it is closed under \( S_{d'} \). Both classes addressed in the previous item are closed under \( P \) and \( U_p \) but are not closed under \( S \) nor \( U_r \).

3. Entirely analogous results (to those in the previous two items) hold for relation algebras.

4. The classes \( \text{Nr}_n \text{CA}_m \) and \( \text{RaCA}_m \) \( (m > 2) \) are closed under \( H \); the former class is not closed under \( S_d \), a fortiori it is not closed under \( S_c \).

5. The class \( \text{RaCA}_\omega \) is not closed under \( S_c \) [14, Theorem 36].

Proof. We have shown that the class \( \text{CRCA}_n \) is not elementary, hence it is not pseudo-universal. It is also not closed under \( S \). Take any \( \mathfrak{A} \in \text{RCA}_n \) that is not completely representable, for example take any non-atomic \( \mathfrak{A} \in \text{RCA}_n \). Other atomic examples are the algebras \( \mathcal{C} \) and the term algebra \( \mathbb{M} \text{At} \) dealt with in theorem 3.4. Call such an algebra lacking a complete representation \( \mathfrak{A} \). Then \( \mathfrak{A}^+ \) is completely representable, a classical result of Monk’s [16] and \( \mathfrak{A} \) embeds into \( \mathfrak{A}^+ \). We have shown that for \( 1 < n < m \cap \omega \), the class \( \text{Nr}_n \text{CA}_m \) is not elementary [27, Theorem 5.1.4]. For pseudo-elementarity for the class \( \text{Nr}_n \text{CA}_\beta \) for any \( 2 < n < \beta \) one easily adapts [14, Theorem 21] by defining \( \text{Nr}_n \text{CA}_\beta \) in a two-sorted theory, when \( 1 < n < \beta < \omega \), and a three-sorted one, when \( \beta = \omega \). Recursive enumerability follows from [17, Theorem 9.37]. That \( \text{Nr}_n \text{CA}_m \) is not closed under \( S_d \) follows by using the algebra \( \mathcal{B} \) in example 4.2. \( \text{CRCA}_n \) is closed under \( S_c \) from the characterization established in item (3) of theorem 5.1 (or its proof). Closure under \( P \) for both classes is straightforward. Both classes are closed under \( U_p \) because they are psuedo-elementary, and they are not closed under \( U_r \), because they are not elementary. That \( \text{Nr}_n \text{CA}_m \) is closed under \( H \) is proved in [28].

5.2 Locally well-behaved relativized representations via dilations and basis

Clique guarded semantics for \( \text{CA}_n \)s can be defined similarly to relation algebras. We consider (the locally well-behaved) \( m \)-square, \( m \)-flat and infinitary \( m \)-flat [17, Definitions 13.4, 13.6] representation of \( \mathfrak{A} \in \text{CA}_n \) with \( 2 < n < m \leq \omega \) [17, Chapter 13]. We characterize the existence of such representations, using dilations and so called basis and hyperbasis.

Assume that \( 1 < n < m < \omega \). Let \( \mathcal{M} \) be a relativized representation of \( \mathfrak{A} \in \text{CA}_n \) witnessed by an injective homomorphism \( f : \mathfrak{A} \rightarrow \wp(V) \), where \( V \subseteq {}^n\mathbb{M} \) and \( \bigcup_{s \in V} \text{rng}(s) = \mathcal{M} \). Here we identify notionally the set algebra with universe \( \wp(V) \) with its universe \( \wp(V) \). We write \( \mathcal{M} \models a(s) \) for \( s \in f(a) \). Let \( \mathcal{L}(\mathfrak{A})^m \) be the first order signature using \( m \) variables and one \( n \)-ary relation symbol for each element in \( A \) and \( \mathcal{L}(\mathfrak{A})^m_{n, \omega} \) be the infinitary extension of \( \mathcal{L}(\mathfrak{A})^m \) allowing infinite conjunctions. Then an \( n \)-clique is a set \( C \subseteq \mathcal{M} \) such that \( (a_1, \ldots, a_{n-1}) \in V = 1^m \) for distinct \( a_1, \ldots, a_n \in C \). Let \( C^m(\mathcal{M}) = \{ s \in {}^n\mathbb{M} : \text{rng}(s) \text{ is an } n \text{-clique} \} \). Then \( C^m(\mathcal{M}) \) is called the \( n \)-Gaifman hypergraph of \( \mathcal{M} \), with the \( n \)-hyperedge relation \( 1^m \).

(1) The clique guarded semantics \( \models_c \) are defined inductively. For atomic formulas and Boolean connectives they are defined like the classical case and for existential quantifiers (cylindrifiers) they are defined as follows: for \( \bar{s} \in {}^m\mathbb{M} \), \( i < m \), \( \mathcal{M}, \bar{s} \models_c \exists x \phi \iff \) there is a \( \bar{t} \in C^m(\mathcal{M}) \), \( \bar{t} \equiv_i \bar{s} \) such that \( \mathcal{M}, \bar{t} \models \phi \).
We say that $M$ is an \emph{m-square representation} of $\mathfrak{A}$, if $\bar{s} \in C^m(M)$, $a \in \mathfrak{A}$, $i < n$, and $l : n \to m$ is an injective map, $M \models \varphi(l(0), \ldots, l(n-1))$, $\iff$ there is a $\bar{t} \in C^m(M)$ with $\bar{t} \equiv \bar{s}$, and $M \models a(l(0), \ldots, l(n-1))$.

$M$ is said to be \emph{(infinite) m-flat representation} of $\mathfrak{A}$ if it is m-square and for all $\phi \in (\mathfrak{A}_m)_{\infty, \omega} \Sigma(\mathfrak{A})^m$, for all $\bar{s} \in C^m(M)$, for all distinct $i, j < m$, we have $M \models \varphi(\bar{s} \setminus i) = \varphi(\bar{s} \setminus j)$.

Complete m-relativized representations are defined analogous to the classical case. Let $\mathfrak{A} \in \mathcal{C}\mathcal{A}_n$, $M$ and $f$ be as above. We say that $M$ is a \emph{complete m-square representation} of $\mathfrak{A}$ via $f$, or simply a complete representation of $\mathfrak{A}$ if $f(\bigcup X) = \bigcup_{x \in X} f(x)$, for all $X \subseteq \mathfrak{A}$ for which $\bigcup X$ exists. It is straightforward to show, like in the classical case, that $\mathfrak{A}$ has a complete m-square complete representation $M$ via $f$ $\iff$ $\mathfrak{A}$ is atomic and $f$ is \emph{atomic} in the sense that $\bigcup_{x \in \mathfrak{A}} f(x) = 1^M$. One can define m-square representations of an algebra $\mathfrak{A} \in \mathcal{C}\mathcal{A}_n$ as in [17, Definition 13.12]. But like infinitary m-flat representations, they really do not add much to m-flatness as far as (ordinary as opposed to complete) representations are concerned.

We also need the notion of m-dimensional hyperbasis. This hyperbasis is made up of m-dimensional hypernetworks. An \emph{m-dimensional hypernetwork} on the atomic algebra $\mathfrak{A}$ is an $n$-dimensional network $N$, with \textbf{nodes} $\subseteq \mathfrak{A}$, endowed with a set of labels $\Lambda$ for hyperedges of length $\leq m$, not equal to $n$ (the dimension), such that $\Lambda \cap \mathfrak{A} = \emptyset$. We call a label in $\Lambda$ a non-unary label. Like in networks, $n$-hyperedges are labelled by atoms. In addition to the consistency properties for networks, an $m$-dimensional hypernetwork must satisfy the following additional consistency rule involving non-atomic labels: If $\bar{x}, \bar{y} \in \mathcal{L}^m \mathfrak{A}$, $|\bar{x}| = |\bar{y}| \neq n$ and $\exists \bar{z}$, such that $\forall i < |\bar{x}|$, $N(x_i, y_i, z) \leq d_{M}$, then $N(\bar{x}) = N(\bar{y}) \in \Lambda$. (Compare with the definition of hypernetworks with ‘hyperedges of unbounded length’ used in the proof of theorem 4.1).

\textbf{Definition 5.4.} Let $2 < n < m < \omega$ and $\mathfrak{A} \in \mathcal{C}\mathcal{A}_n$ be atomic.

\begin{enumerate}
\item An \emph{m-dimensional basis} $B$ for $\mathfrak{A}$ consists of a set of $n$-dimensional networks whose \textbf{nodes} $\subseteq m$, satisfying the following properties:
\begin{itemize}
\item For all $a \in \mathfrak{A}$, there is an $N \in B$ such that $N(0, 1, \ldots, n-1) = a$,
\item The \emph{cylindrifier property}: For all $N \in B$, all $i < n$, all $\bar{x} \in n$-\textbf{nodes} $(\subseteq n)$, all $a \in \mathfrak{A}$, such that $N(\bar{x}) \leq c_i a$, there exists $M \in B$, $M \equiv_i N$, $\bar{y} \in n$-\textbf{nodes} $(M)$ such that $\bar{y} \equiv_j \bar{x}$ and $M(\bar{y}) = a$. We can always assume that $\bar{y}i$ is a new node else one takes $M = N$.
\end{itemize}
\item An \emph{m-dimensional hyperbasis} $H$ consists of \emph{m-dimensional hypernetworks}, satisfying the above two conditions reformulated the obvious way for hypernetworks, in addition, $H$ has an amalgamation property for overlapping hypernetworks off of at most $2$ nodes; this property corresponds to commutativity of cylindrifiers:
\begin{itemize}
\item For all $M, N \in H$ and $x, y < m$, with $M \equiv_{xy} N$, there is $L \in H$ such that $M \equiv_x L \equiv_y N$. Here $M \equiv_S N$, means that $M$ and $N$ agree off of $S$ [17, Definition 12.11].
\end{itemize}
\end{enumerate}

The main ideas used in the next theorem needed to perform a transfer from results relating neat embedding properties to relativized representations from RAs to CAs can be found in [17, Definitions 12.1, 12.9, 12.10, 12.25, Propositions 12.25, 12.27]. In all cases, the m-dimensional dilation stipulated in the statement of the theorem, will have
top element $C^m(M)$, where $M$ is the $m$–relativized representation of the given algebra, and the (completely additive) operations of the dilation are induced by the $n$-clique–guarded semantics. But first we need to recall the definition of certain non–commutative set algebras. Let $n < \omega$. The class $D_n(G_n)$ is a class of set algebras having the same signature as $\mathcal{C}A_n$. If $\mathfrak{A} \in D_n(G_n)$, then the top element of $\mathfrak{A}$ is a set $V \subseteq ^nU$ (some non–empty set $U$), such that if $s \in V$, and $i < j < n$ ($\tau : n \rightarrow \omega$), then $s \circ [i,j](s \circ \tau) \in V$.

**Theorem 5.5.** [17, Theorems 13.45, 13.36]. Assume that $2 < n < m < \omega$ and let $\mathfrak{A} \in \mathcal{C}A_n$. Then the following hold:

1. $\mathfrak{A} \in SNr_n\mathcal{C}A_m \iff \mathfrak{A}$ has an $m$–smooth representation $\iff \mathfrak{A}$ has an infinitary $m$–flat representation $\iff \mathfrak{A}$ has an $m$–flat representation $\iff \mathfrak{A}^+$ has an $m$–dimensional hyperbasis.

2. $\mathfrak{A} \in SNr_nD_m \iff \mathfrak{A} \in SNr_nG_m \iff \mathfrak{A}$ has an $m$–square representation $\iff \mathfrak{A}^+$ has an $m$–dimensional basis

3. If $\mathfrak{A}$ is atomic, then: $\mathfrak{A}$ has a complete infinitary $m$–flat representation $\iff \mathfrak{A}$ has a complete $m$–smooth representation $\iff \mathfrak{A} \in SNr_n(\mathcal{C}A_m \cap \mathcal{A}t) \iff \mathfrak{A}$ has an $m$–dimensional hyperbasis.

**Lemma 5.6.** Assume that $2 < n < m < \omega$ and assume that $\mathfrak{A} \in \mathcal{C}A_n$ is atomic. The algebra $\mathfrak{A}$ has a complete $m$–square representation $\iff \exists \exists$ has a winning strategy in $G^m(\mathcal{A}t\mathfrak{A})$.

Given any cardinal $\kappa$, possibly infinite, $\mathfrak{A} \in \mathcal{C}A_m$, with $m < \omega$, then (complete) $\kappa$–square representations of $\mathfrak{A}$ can be easily defined [17, Definition 17.22]. If $\omega \leq \kappa < \lambda$, an algebra having a complete $\lambda$–square representation, may not have a complete $\kappa$–square one. The rainbow algebra of dimension $n$, for any $2 < n < \omega$, $\mathfrak{A} = \mathfrak{A}_\lambda, n$, witnesses this. Any complete $\kappa$–square representation of $\mathfrak{A}$ will force a `$\kappa$ red clique’ indexed by the $\lambda$ greens which is impossible because the indices of reds must match within the red clique.

**Theorem 5.7.** If $\mathfrak{A} \in \mathcal{C}A_n$ is atomic and $\exists \exists$ has a winning strategy in $G^m(\mathcal{A}t\mathfrak{A})$, then $\mathfrak{A}$ has a complete $\omega$–square representation. In particular, if $\mathfrak{A} \in NR_n\mathcal{C}A_\omega$ is atomic, then it has a complete $\omega$–square representation, and if $\mathfrak{A}$ has countably many atoms, then it is completely representable.

**Proof.** Observe that the last part is already proved in the third item of theorem 5.1 and in [18, Theorem 3.3.3]. For the first part, we build an $\omega$–dimensional basis for $\mathfrak{A}$ denoted below by $\mathcal{H}$, from which the $\omega$–square complete representation of $\mathfrak{A}$ will be built. For a network $N$ and function $\theta$, the network $N\theta$ is the complete labelled graph with nodes $\theta^{-1}(\text{nodes}(N)) = \{x \in \text{dom}(\theta) : \theta(x) \in \text{nodes}(N)\}$, and labelling defined by

$$(N\theta)(i_0, \ldots, i_{n-1}) = N(\theta(i_0), \theta(i_1), \ldots, \theta(i_{n-1})),\)$$

for $i_0, \ldots, i_{n-1} \in \theta^{-1}(\text{nodes}(N))$.

Let $\mathcal{H}$ be the set of all $N\theta$ where $N$ occurs in some play of $G^m(\mathcal{A}t\mathfrak{A})$ which $\exists \exists$ uses her winning strategy and $\theta : \omega \rightarrow \text{nodes}(N)$. The $\omega$–dimensional basis $\mathcal{H}$ for $\mathfrak{A}$, consists of a set of $n$–dimensional networks on $\mathfrak{A}$ where for $N \in \mathcal{H}$, $\text{nodes}(N) \subseteq \omega$. It can be checked that $\mathcal{H}$, like a(n ordinary) basis, satisfies

- For all $a \in \mathcal{A}t\mathfrak{A}$, there is an $N \in \mathcal{H}$ such that $N(0,1,\ldots,n-1) = a$,

- The cylindrifier property: For all $N \in \mathcal{H}$, all $i < n$, all $\bar{x} \in \text{"nodes}(N)(\subseteq \omega)$, all $a \in \mathcal{A}t\mathfrak{A}$, such that $N(\bar{x}) \leq c_i a$, there exists $M \in \mathcal{H}$, $M \equiv_i N$, $\bar{y} \in \text{"nodes}(M)$ such that $\bar{y} \equiv_i \bar{x}$ and $M(\bar{y}) = a$. 33
We consider $n$–hypergraphs on $\mathfrak{A}$ as approximation to a complete square $\omega$–representation. By such an $n$–hypergraph, we understand a hypergraph (not necessarily complete) such that some of its $n$–hyperedges are labelled by atoms of $\mathfrak{A}$. Given such a hypergraph with $n$–hyperedge relation $E$, then $C \subseteq M$ is a clique if for all injective map $s : n \to C$, we have $s \in E$. We build a complete $\omega$–square representation of $\mathfrak{A}$ from $\mathcal{H}$ in a step–by–step way, requiring inductively in step $t < \omega$, that the constructed hypergraph $M_t$ satisfies that for any finite clique $C \mid |C| < \omega$, there is $N \in S$ and an embedding $\theta : N \to M_t$ (of hypergraphs) such that $\text{rng} \theta \supseteq C$. Also we require that $M_t$ satisfies the first three items (by replacing $M$ by $M_t$) and the limit hypergraph, which is the base required of the representation will be fixed (along the way) to satisfy some more conditions [17, Proposition 13.37, Lemma 17.24]:

1. each $n$–hyperedge of of $M$ is labelled by an atom of $\mathfrak{A},$
2. $M(\bar{x}) \leq d_{ij} \iff x_i = x_j,$
3. for any clique $\bar{x} \in M$ of arbitrary finite length $> m$, there is a unique $N \in \mathcal{H}$, such that $\bar{x}$ is labelled by $N$, and we write this as $M(\bar{x}) = N,$
4. if $l \geq n$ is finite, $x_0, \ldots, x_{l-1} \in M$ and $M(\bar{x}) = N \in \mathcal{H}$, then for all $i_0, \ldots, i_{n-1} < l,$ we have $(x_{i_0}, \ldots, x_{i_{n-1}})$ is a hyperedge, and
5. $M$ is symmetric (closed under substitutions: $N \in M \implies \lambda \theta \in M$, any $\theta$),
6. if $\bar{x}$ is a clique of arbitrary length, $k < |\bar{x}|$ and $N \in \mathcal{H}$, then $M(\bar{x}) \equiv_k N \iff$ there is a $y \in M$ such that $M(x_0, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_l) = N$ (with $y$ in the $k$th place),
7. for every $N \in S$, for every finite $l \geq n$, there are $x_0, \ldots, x_{l-1} \in M$, $M(\bar{x}) = N$.

Then $M$ will be an $\omega$–square complete representation of $\mathfrak{A}$, defined for $r \in \mathfrak{A}$ via

\[ M \models r(\bar{x}) \iff \bar{x} \text{ an } n \text{–hyperedge of } M \text{ and } M(\bar{x}) \leq r. \]

$M$ is complete because every $n$–hyperedge is labelled by an atom, so that $M$ is an atomic, hence complete $\omega$–square representation. \hfill \Box

\textbf{Corollary 5.8.} For any infinite cardinal $\kappa$, there exists a relation algebra $\mathfrak{R} \in \text{EICRRA}$ with $\text{At} \mathfrak{A} = 2^\kappa$, and an atomless $\mathfrak{C} \in \text{CA}_\omega$ such that for any $2 < n < \omega$, $\text{N}_{n} \mathfrak{C} \in \text{EICRCA}_n$, $\mathfrak{R} = \text{Ra} \text{Nr}_{n} \mathfrak{C} = \text{Ra} \mathfrak{C}$, $\mathfrak{R}$ and $\text{N}_{n} \mathfrak{C}$ have no complete representations, but each has an $\omega$–square representation.

\textbf{Proof.} From the construction in [26] and the previous theorem. \hfill \Box

\textbf{Corollary 5.9.} (Hirsch–Hodkinson) Let $2 < n < \omega$. Then the classes CRRA and CRCA are not elementary.

In the next definition we define certain structures that have ‘two parts’. The first is a complete irreflexive graph, and the second is a linear order.

\textbf{Definition 5.10.} [17, pp. 524]. For any cardinal $\kappa$, $K_\kappa$ will denote the complete irreflexive graph with $\kappa$ nodes. Let $p < \omega$, and $I$ a linearly irreflexive ordered set, viewed as model to a signature containing a binary relation $\prec$. $M[p, I]$ is the disjoint union of $I$ and the complete graph $K_p$ with $p$ nodes. $\prec$ is interpreted in this structure as follows $\prec \cup \prec K_p \cup I \times K_p \cup (K_p \times I)$ where the order on $K_p$ is the edge relation.

\textbf{Theorem 5.11.} Let $k \geq 3$ be finite and $\kappa$ be a cardinal $\geq n + 3$. \hfill 34
1. The classes of $\text{CA}_n$s having complete $n+k$–smooth, complete $n+k$ infinitary flat representations, and complete $\kappa$–square representations are not elementary.

2. The classes of $\text{CA}_n$s having $n+k$–flat and $\kappa$–square representations are not atom–canonical.

3. Analogous results hold for relation algebras [17].

Proof. Non atom-canonicity for $\text{CA}_n$s follows from theorems 3.4, ?? and theorem 5.5. Non–first order definability for $\text{CA}_n$s of ‘complete $\kappa$–squareness’ for $\kappa \geq \omega$, follows from the fact that, given an infinite cardinal $\kappa$, an algebra $\mathfrak{A} \in \text{CA}_n$ having countably many atoms is completely representable $\iff$ $\mathfrak{A}$ has a complete $\kappa$–square representation, and that the algebra $\mathfrak{C}$ constructed in theorem 4.1 has countably many atoms, $\mathfrak{C}$ is not in $\text{Sr}, \text{Nr}_n, \text{CA}_{n+3}$ hence $\mathfrak{C}$ is not completely representable, but it is elementarily equivalent to a countable completely representable algebra $\mathfrak{B}$, say. The ‘no’s for ‘flatness and smoothness’ follow using the same $\mathfrak{C}$ and $\mathfrak{B}$ as above together with theorem 5.5. The same reasoning is valid for RAs using the construction in theorem 4.10, [14, 15].

We show non–elementarity of the notion of $\kappa$–squareness, for finite $\kappa \geq n + 3$. (Recall that the constructions in theorems 4.1 and 4.10 settles $\kappa$–flatness). For RAs this is given in [17, Exercise 1(ii) p.524 ]. For $\text{CAs}$ on deals with the $\text{CAs}$ analogue of the aforementioned exercise. Let $3 \leq k < \omega$. Take $G$ and $R$ to be $M[\kappa - 4, \mathbb{Z}]$ and $R = M[\kappa - 4, \mathbb{N}]$, respectively, as defined above. Let $\mathfrak{D} = \text{CA}_{G,R}$ be the $n$–dimensional rainbow algebra based on $G$ and $R$. For $m, r \leq \omega$, $G_r^{m}$ denotes the $r$ rounded game $G_r$ restricted to $m$ nodes.

Then it can be shown that $\exists$ has a winning strategy in the $r$–rounded game $G_r^m$ for all $r < \omega$ but $\forall$ can win $G_r^m$. From the last property $\mathfrak{D}$ will not have an $\kappa$–square representation. From the first, using the usual technique of forming ultrapowers followed by an elementary chain argument, we get a countable algebra $\mathfrak{B}$ that is elementary equivalent to $\mathfrak{D}$, but $\mathfrak{B}$ has a complete $\kappa$–square representation.

We describe winning strategies in the private Ehrenfeucht–Fraïssé game. Assume that $\kappa < \omega$ and $\kappa \geq n + 3$. Let $A = M[\kappa - 4, \mathbb{Z}]$ and $B = M[\kappa - 4, \mathbb{N}]$. In his private game, $\forall$ always places the pebbles on distinct elements of $\mathbb{Z}$. She uses rounds $0, \ldots, \kappa - 3$, to cover $\kappa - 4$ and two elements of $\mathbb{Z}$. Because at least two out of three distinct colours are related by $<$, $\exists$ must respond by pebbling $\kappa - 4 \cup \{e, e'\}$ for some $e, e' \in \mathbb{N}$. Assuming that $\forall$ has not won, then he has at least arranged that two elements of $\mathbb{Z}$ are pebbled, the corresponding pebbles in $B$ being in $\mathbb{N}$. Then $\forall$ can force $\exists$ to play a two pebble game of length $\omega$ on $\mathbb{Z}$, $\forall$ which he can win.

In her private game, $\exists$ picks up a spare pebble pair and place the first pebble of it on $a \in A$. By the rules of the game, $a$ is not currently occupied by a pebble. $\exists$ has to choose which element of $B$ to put the pebble on. $\exists$ chooses an unoccupied element in $\kappa - 4$, if possible. If they are all already occupied, she chooses $b$ to be an arbitrary element $x \in \mathbb{N}$. Because there are only $\kappa - 3$ pebble pairs, $\exists$ can always implement this strategy and win. 

5.3 Relativizing various notions of representability

We have already dealt with the notion of relativized complete representability. We continue to relativize the notions of ‘satisfying the Lyndon conditions’ and ‘strong representability.’

We introduce classes defined via $\text{Nr}_n$ and $\mathfrak{C}m$ (for CAs) and $\text{Ra}$ and $\mathfrak{C}m$ (for RAs):
Definition 5.12. Let $O \in \{S, S^d, S_c, I\}$. Let $2 < n < m \leq \omega$. An atomic algebra $A \in CA_n$ has the strong $O$ neat embedding property up to $m$, if $\mathsf{EmAt}A \in ONr_nCA_m$. Analogously an atomic relation algebra $R$ has the strong $O$ neat embedding property up to $m$, if $\mathsf{EmAt}R \in ORaCA_m$.

We denote the class of $CA_n$'s having the strong $O$ neat embedding property up to $m$ by $SNPCA^O_{n,m}$. We denote the class of RAs having the strong $O$ neat embedding property up to $m$ by $SNPRA^O_{m}$. We let $RCA^O_{n,m} := SNPCA^O_{n,m} \cap RCA_n$ and $RRA^O_{m} := SNPRA^O_{m} \cap RA$.

Observe that $RCA^S_{n,\omega}$ coincides with the previously introduced class $SRCA_n$ of strongly representable $CA_n$'s. From now on, we use the notation $RCA^S_{n,\omega}$ for the class of strongly representable $CA_n$'s. By the same token, the class $RRA^S$ denotes the class of strongly representable RAs, defined like the $CA$ case. It is well known that for both $CA$s and RAs, an algebra is strongly representable $\iff$ its atom structure is strongly representable.

Seminal results of Hirsch and Hodkinson show that $RCA^S_{n,\omega}$ and $RRA^S$ are not elementary, cf. [19] and [17, Corollary 14.14]. Their proof addresses the atom structures but it is straightforward to lift it to the algebra level.

In the following theorem we need:

Definition 5.13. Let $M$ be the base of a representation of $A \in CA_n$. Then $M$ is $n$-homogeneous if for any partial isomorphism $\theta$ having size $n$ or less and any finite subset $X$ of $M$, there is a partial isomorphism $\psi$ extending $\theta$ with $X$ contained within rng($\psi$).

Theorem 5.14. Let $2 < n \leq l < m \leq \omega$ and $O \in \{S, S^d, S_c, I\}$. Then the following hold:

1. $RCA^O_{n,m} \subseteq RCA^O_{n,l}$ and $RCA^I_{n,l} \subseteq RCA^S_{n,l} \subseteq RCA^S_{n+1,l} \subseteq RCA^S_{n,l+1}$. The last inclusion is proper for $l \geq n + 3$. If there exists a finite $RCA_n$ with no $n$-homogeneous representation, then $RCA^S_{n,\omega} \not\subseteq RCA^S_{n,\omega}$.

2. For $O \in \{S, S^d, S_c\}$, $SNPCA^O_{n,l} \subseteq ONr_nCA_l$ and for $O = S$, the inclusion is proper for $l \geq n + 3$. But $SNPCA^I_{n,l} \not\subseteq Nr_nCA_l$.

3. For any positive $k$, $SNPCA^O_{n,n+k+1} \subseteq SNPCA^O_{n,n+k}$, and $SNPCA^O_{n,\omega} \subseteq RCA_n$.

4. For $O \in \{I, S_c, S_d\}$, $l \geq n + 3$ and $K$ any class such that $RCA^O_{n,l} \subseteq K \subseteq SNPCA^O_{n,l}$, $K$ is not first order definable.

Proof. The inclusions in the first item follows from the definition and the strictness of the last inclusion in this item is witnessed by the algebra $C$ used in the first part of theorem 4.1, since $C$ satisfies $C = \mathsf{EmAt}C \in RCA_n$ but $C \notin S_cNr_nCA_l$ for $l \geq n + 3$. Let $O \in \{S, S^d, S_c\}$. If $\mathsf{EmAt}A \in ONr_nCA_l$, then $A \subseteq d \mathsf{EmAt}A$, so $A \in S_dONr_nCA_l \subseteq ONr_nCA_l$. This proves the first part of item (2). The strictness of the last inclusion follows from theorem 3.4 on squareness, since the atomic countable algebra $A$ constructed in op.cit is in $RCA_n$, but $\mathsf{EmAt}A \notin SNr_nCA_l$ for any $l \geq n + 3$.

We prove the last part of the first item. Let $2 < n < \omega$. Assume that $A \in RCA_n$ is complete and atomic having countably many atoms. Here our assumption is me general, the algebra $A$ we address can be infinite. If $A$ has no $n$-homogeneous representation, then we claim that $AtA \notin AtNr_nCA_\omega$ and $A \notin S_dNr_nCA_\omega$. We first show that $\alpha = AtA \notin AtNr_nCA_\omega$. Assume for contradiction that $\alpha \in AtNr_nCA_\omega$. Let $G_{ca}$ be a game that is like $H$ used in the proof of theorem 4.1, having a cylinderier move and two amalgamation moves, but it is played on networks not hypernetworks. Also, in amalgamation moves
∀'s choice is restricted by choosing networks that overlap only on at most n nodes. Having at hand the assumption that \( \text{At}\mathfrak{A} \in \text{AtN}_{r_\omega} \text{CA}\), one proves that \( \exists \) has a winning strategy in \( G_{\text{ca}}(\alpha) \). Next one builds a sequence of networks \( N_0 \subseteq \ldots \subseteq N_r \subseteq \omega \), such that \( N_0 \) is \( \exists \)'s response to \( \forall \)'s move choosing \( a \) in the initial round. By construction this sequence of networks satisfies:

(a) if \( N_r(\tilde{x}) \leq c \) for \( \tilde{x} \in \text{nodes}(N_r) \), then there exists \( N_s \supseteq N_r \) and a node \( k \in \omega \sim N_r \) such that \( N_r(\tilde{y}) = b \); where \( \tilde{y} \equiv \iota \tilde{x} \) and \( \tilde{y}_i = k \).

(b) if \( \tilde{x}, \tilde{y} \in \text{nodes}(N_r) \) such that \( N_r(\tilde{x}) = N_r(\tilde{y}) \), then there is a finite surjective map \( \theta \) extending \( \{(x_i, y_i) : i < n\} \) mapping onto \( \text{nodes}(N_r) \) such that \( \text{dom}(\theta) \cap \text{nodes}(N_r) = \tilde{y} \).

(c) if \( N_r \) is in the sequence and \( \theta \) is any partial isomorphism of \( N_r \), then there is \( s \geq r \) and a partial isomorphism \( \theta^+ \) of \( N_s \) extending \( \theta \) such that \( \text{rng}(\theta^+) \supseteq \text{nodes}(N_r) \).

Let \( N_\alpha \) be the limit of such networks. Define a representation \( \mathcal{N} \) of \( \mathfrak{A} \) having domain \( \bigcup_{a \in A} \text{nodes}(N_a) \), by \( S^\mathcal{N} = \{ \tilde{x} : \exists a \in A, \exists s \in S, N_\alpha(\tilde{x}) = s \} \), for any subset \( S \) of \( \alpha \). Then this can be checked to be by construction (using (a) and (b) and (c)) to be a complete \( n \)-homogeneous representation of \( \mathfrak{A} \), contradiction, so \( \alpha \notin \text{AtN}_{r_\omega} \text{CA}_\omega \).

A fortiori \( \mathfrak{A} \notin \text{N}_{r_\omega} \text{CA}_\omega \), but \( \mathfrak{A} \) is complete and atomic so \( \mathfrak{A} \notin \text{S}_{d} \text{N}_{r_\omega} \text{CA}_\omega \). Now if \( \mathfrak{A} \in \text{RCA}_n \) is finite having no homogeneous representation, then by what we just saw, \( \mathfrak{A} \notin \text{S}_{d} \text{N}_{r_\omega} \text{CA}_\omega \), but by the first item of theorem 5.1, \( \mathfrak{A} \in \text{S}_{d} \text{N}_{r_\omega} \text{CA}_\omega \), because it is certainly completely representable. Since \( \mathfrak{A} \) is finite, then \( \mathfrak{A} = \text{cmAt}\mathfrak{A} \) and we are done.

For the last non-inclusion in item (2), we use the same algebras in example 4.2. Let \( V = n\mathbb{Q} \) and let \( \mathfrak{A} \) be the \( \text{Cs}_n \) with universe \( \varphi(V) \). Then \( \mathfrak{A} \in \text{N}_{r_\omega} \text{CA}_\omega \). Like before, let \( y \) denote the \( n \)-ary relation \( \{ s \in V : s_0 + 1 = \sum_{i > 0} s_i \} \) and let \( \mathfrak{B} = \mathfrak{A}_G^\alpha \{(y) \cup \{\{s\} : s \in V\} \}. \) Now \( \mathfrak{B} \) and \( \mathfrak{A} \) are in \( \text{Cs}_n \) and they share the same atom structure, namely, the singletons, so \( \mathfrak{B} \) is a dense subalgebra of \( \mathfrak{A} \), and clearly \( \text{cmAt}\mathfrak{B} = \mathfrak{A} \in \text{N}_{r_\omega} \text{CA}_\omega \). As proved in [28], \( \mathfrak{B} \notin \text{Eln}_{r_\omega} \text{CA}_{n+1} \), so \( \mathfrak{B} \notin \text{N}_{r_\omega} \text{CA}_{n+1} \subset \text{N}_{r_\omega} \text{CA}_\omega \). But \( \text{cmAt}\mathfrak{B} \in \text{N}_{r_\omega} \text{CA}_\omega \), hence \( \mathfrak{B} \in \text{RCA}^1_{n,k} \). We have shown that \( \mathfrak{B} \in \text{RCA}^1_{n,k} \sim \text{N}_{r_\omega} \text{CA}_1 \), and we are through with the last required in item (2).

Item (3) follows by definition observing that if \( \mathfrak{A} \) is finite then \( \mathfrak{A} = \text{cmAt}\mathfrak{A} \). The strictness of the first inclusion follows from the construction in [21] where it shown that for an positive \( k \), there is a finite algebra \( \mathfrak{A} \) in \( \text{N}_{r_\omega} \text{CA}_{n+k} \sim \text{SN}_{r_\omega} \text{CA}_{n+k+1} \). The inclusion \( \text{SN}_{n,\omega} \text{CA}_{n,\omega} \subseteq \text{RCA}_n \) holds because if \( \mathfrak{B} \in \text{SN}_{n,\omega} \text{CA}_{n,\omega} \), then \( \mathfrak{B} \subseteq \text{cmAt}\mathfrak{B} \in \text{ON}_{r_\omega} \text{CA}_\omega \subseteq \text{RCA}_n \). The \( \mathfrak{A} \) used in theorem 3.4 witnesses the strictness of the last inclusion.

For the last item, we use the construction (and notation adopted) in theorem 4.1. In particular, \( n \) is finite \( > 2 \). Fix \( k \geq 3 \). We have \( \exists \) has a winning strategy in \( H_\omega(\alpha) \) for some countable atom structure \( \alpha \), \( \exists \text{ma}_n \subseteq_d \text{cmo} \in \text{N}_{r_\omega} \text{CA}_\omega \) and \( \exists \text{mo} \in \text{CRCA}_n \). On the other hand the rainbow like algebra \( \mathfrak{C} \) based on \( \mathbb{Z} \) and \( \mathbb{N} \) is not in \( \text{S}_{i} \text{N}_{r_\omega} \text{CA}_{n+3} \).

Hence \( \mathfrak{C} = \text{cmAt}\mathfrak{C} \notin \text{RCA}^1_{n,k} \), \( \mathfrak{C} \equiv \exists \text{mo} \) and \( \exists \text{mo} \in \text{RCA}^1_{n,\omega} \subseteq \text{RCA}^1_{n,n+k} \) because \( \exists \text{mo} \in \text{N}_{r_\omega} \text{CA}_\omega \), which proves the required.

Entirely analogous results to all of the results obtained for \( \text{CA} \) in the previous theorem 5.14 except for one, hold for relation algebras obtained by undergoing the expected replacements. On the other hand, for relation algebras there are known examples of integral representable non-permutational finite relational algebras; such algebras have no homogeneous representations, so in the RA case, using the above argument, we know that \( \text{S}_{i} \text{RaCA}_\omega \subseteq \text{S}_{i} \text{RaCA}_\omega \). As for the RA analogue of some of the results obtained for \( \text{CA} \) above. For the first item one replaces \( \text{RCA}^O_{n,l} \) by \( \text{RRA}^O_{l} \) and \( l \geq n+3 \) by \( l \geq 5 \) getting the complete RA analogue. For the second and third one replaces \( \text{SN}_{r_\omega} \text{CA}_\omega \) by \( \text{SN}_{r_\omega} \text{RA}_\omega \) and \( \text{ON}_{r_\omega} \text{CA}_\omega \) by \( \text{OR}_{r_\omega} \text{CA}_\omega \). The strictness of the inclusions follow either from the relation algebras constructed in [21], or the relation algebras used in the proof of [17, Theorem 15.1] (and below in the proof of theorem 7.1). But here we do not
know whether $\text{SNPRA}_I \not\subseteq \text{RA}_\omega$. The last item for relation algebras, follows from the proof of theorem 4.8; we have, using the notation in op.cit, $\exists m, \beta \not\in \text{RA}_\omega(\supseteq \text{RA}_m)$, $\exists m(\alpha) \in \text{RA}_\omega \subseteq \text{RA}_m$ and $\exists m(\beta) \equiv \exists m(\alpha)$.

Let $2 < n < m < \omega$. Let $\text{CRCA}_n^{m,s}$ be the class of algebras having complete $m$–square representations. We know that for $m \geq n + 3$, $\text{CRCA}_n^{m,s}$ is not elementary. Now we specify its elementary closure using games. The class $\text{LCA}_n^{m,s}$ is defined as follows: $\exists \in \text{LCA}_n^{m,s} \iff \exists$ is atomic and $\exists$ has a winning strategy in $G_k^m(\text{At}\exists)$ for all $k < \omega$. Then $\text{LCA}_n^{m,s} = \text{EILCA}_n^{m,s} = \text{El}((\text{CA}_m \cap S_{n} N_{n}(D_m \cap \text{At})) = \text{El}(\text{CA}_m \cap S_n N_{n}(G_m \cap \text{At})))$.

An entirely analogous situation hold for relation algebras $\text{RA}_s$. Define $\exists \in \text{LC}_{m,s} \iff \exists$ is atomic and $\exists$ has a winning strategy in $G_k^m(\text{At}\exists)$ for all $k < \omega$. Then for $m \geq 4$, $\text{LC}_{m+1,s}$ is not finitely axiomatizable over $\text{LC}_{m,s}$. To see why, fix $m \geq 4$. For $r \geq 4$, define $\exists^m_r$ as in [17, p. 521] replacing $n$ by $m$. Then by [17, Lemma 7.15], for all $r \geq 4$, $\exists$ has a winning strategy in $G_k^m(\exists^m_r)$ for all $k < \omega$, hence $\exists^m_r \in \text{LC}_{m,s}$. But $\exists$ has a winning strategy in $G_k^{m+1}(\Pi_{4\leq i \leq \omega} \exists^m_{/D})$, cf. [17, Theorem 17.18], a fortiori, $\exists$ has a winning strategy in $G_k^{m+1}(\Pi_{4\leq i \leq \omega} \exists^m_{/D})$ for all $k < \omega$, so $\Pi_{4\leq i \leq \omega} \exists^m_{/D} \in \text{LC}_{m+1,s}$. But for each $r \geq 4$, $\exists$ has a winning strategy in $G_k^{m+1} \exists^m_r$, and $\exists^m_r$ is finite, so $\exists^m_r \not\in \text{LC}_{m+1,s}$, proving the required.

Fix $2 < n < m \leq \omega$. We can also define the elementary closure of the class of algebras having complete $m$-flat representations, for both $\text{RA}_s$ and $\text{CA}_s$ using for $\text{CA}_s$ ‘a hyperbasis game’, denote it by $H\text{ca}^m_k$ analogous to $H^m_k$ the last formulated for relation algebras [17, Definition 12.26].

For an atomic $\exists \in \text{CA}_n$, $k < \omega$, and $m \leq \omega$, $H\text{ca}^m_k(\text{At}\exists)$ is a $k$–rounded game played on hypernetworks of $\exists$ having $m$ nodes. Testing the existence of an $m$–dimensional hyperbasis, it has one cylindrifier move and one amalgamation move.

Letting $\text{CRCA}_n^{m,f}$ be the class of $\text{CA}_n$ having complete $m$–flat representations, and $\text{LCA}_n^{m,f}$ be its elementary closure: $\exists \in \text{LCA}_n^{m,f} \iff \exists$ is atomic and $\exists$ has a winning strategy in $H_k^m(\text{At}\exists)$ for all $k < \omega$, we get $\text{LCA}_n^{m,f} = \text{EILCA}_n^{m,f} = \text{El}(S_n N_{n}(\text{CA}_m \cap \text{At})))$.

one can express a winning strategy for $H_k^m$ by a first order sentence using the idea of an elementary view to $m$–flat representations, cf [17].

For $n < m \leq \omega$, let $\text{RCA}_{m,n,s} = \{\exists \in \text{RCA}_n \cap \text{At} : \exists \text{At}\exists \text{ has an } m \text{–square representation}\}$. Then the following hold:

**Theorem 5.15.** Let $2 < n < m \leq \omega$. Then $\text{RCA}_n \subseteq \text{RCA}_m$, and equality holds $\iff m = \omega$. Furthermore, the following inclusions are valid:

$\text{CRCA}_n^{m,s} \subseteq \text{LCA}_n^{m,s} \subseteq \text{RCA}_m \cap \text{SN}_{n} D_{m} \cap \text{At}$,

$\text{CRCA}_n^{m,f} \subseteq \text{LCA}_n^{m,f} \subseteq \text{RCA}_m \cap \text{SN}_{n} \text{CA} \cap \text{At}$.

The first inclusion in both displayed statements are proper for $m \geq 3$, and there exists a finite $m > n$, such that all inclusions are proper.

**Proof.** The classes $\text{CRCA}_n^{m,s}$ and $\text{CRCA}_n^{m,f}$ are not elementary, while (their elementary closure) $\text{LCA}_n^{m,s}$ and $\text{LCA}_n^{m,s}$ are elementary, hence the strictness of the first inclusion in both statements. The last part follows from the fact that $\text{RCA}_n^{s}$ is not elementary and that $\text{RCA}_n = \bigcap_{k \geq n} \text{SN}_{n} \text{CA}_{n+k} = \bigcap_{k \geq n} (\text{SN}_{n} D_{n+k} \cap \text{CA}_n)$.
Let $m > n$. Let $V_{n,m} = SN_{r_m}CA_m$. Then $AtV_{n,m}$ is an elementary class. We have seen that for $m \geq n+3$, $SCmAtV_{n,m} \not\subseteq V_{n,m}$. Let $At_m$ be the elementary closure of the class of atom structures having complete $m$–flat representations.

**Theorem 5.16.** For $2 < n < \omega$, $V_{n,m} = SCmAt_m$. Furthermore, for $m \geq n+3$, $At_m$ is not finitely axiomatizable.

**Proof.** Let $A \in V_{n,m}$. Then $A^+$ is atomic, and has a complete $m$–flat representation. This can be proved, using ideas of Hirsch and Hodkinson, by taking an $\omega$–saturated model of the consistent theory stipulating the existence of an $m$–flat representation for $A$, as the base of the complete $m$– square representation for $A^+$. In more detail, let $M$ be an $\omega$–saturated model, of this theory, then we show that it is a complete $m$–flat representation. One defines an injective complete embedding $h : A^+ \to \wp(1^M)$. First note that the set $f_x = \{a \in A : a(\bar{x})\}$ is an ultrafilter in $A$, whenever $x \in M$ and $M \models 1(\bar{x})$. Now $A^+ = \mathfrak{CM}(\mathcal{UF}A)$. For $S \subseteq \mathcal{UF}A$, let $h(S) = \{x \in 1^M : f_x \in S\}$. We check only injectivity using saturation. For the (ideas used in) rest of the proof the reader is referred to [17, Corollary 13.18]. It suffices to show that for any ultrafilter $F$ of $A$ which is an atom in $A^+$, we have $h(F) \neq 0$. Let $p(\bar{x}) = \{a(\bar{x}) : a \in F\}$. Then this type is finitely satisfiable. Hence by $\omega$ saturation $p$ is realized in $M$ by $\bar{y}$, say. Now $M \models 1(\bar{y})$ and $F \subseteq f_x$, since these are both ultrafilters, equality holds. Thus $AtA^+ \subseteq At_m$, and $A \subseteq \mathfrak{CM}AtA^+ = A^+$.

Fix $m \geq n + 2$. For $r \geq 4$, define $CA_r^n$ to be the $CA_n$ based on the greens and reds on which the rainbow relation algebra $A^n_r$ is based. That is using the notation in definition 5.10, we take $G = M[m - 3, 2^{r-1}]$ and $R = M[m - 3, 2^{r-1} - 1]$. Then it can be proved that for each $r \geq 4$, $\forall$ has a winning strategy in $G_r^{\omega+1}(CA_r^n)$. Since $CA_r^n$ is finite, then $CA_r^n \not\subseteq LCA_n^{m+1,s}$ for each $r \geq 4$. But it can also be proved that $\exists$ has a winning strategy in $G_r^{\omega+1}(CA_r^n / D)$ for all $k < \omega$, hence $LCA_r^{m+1,s}$ is not finitely axiomatizable. The winning strategies in the Ehrenfeucht–Fraïssé private games $EF_r^\omega(G, R)$ and $EF_r^{m-1}(G, R)$ are given in the proofs of [17, Lemmata 17.16, 17.17].

As a relativized version of corollary 5.2, we have:

**Corollary 5.17.** Let $2 < n < m \leq \omega$. Then $N_r^nCA_m \cap At \subseteq ELN_{r_n}CA_m \cap At \subseteq ELS_{dN_r^nCA_m \cap At} \subseteq ELS_{N_r^nCA_m \cap At} = ESL_{N_r^nCA_m \cap At} = LCA_n^{m,f} \subseteq RCA_n^{S,m,n} \subseteq UrCA_n^{S,m,n} = UrCA_n^{S,m,n} = EIRCA_n^{S,m,n} = SN_{r_n}CA_m \cap At$.

**Proof.** The strictness of the first inclusion and second inclusion use the same algebras in the first item of theorem 5.2 addressing the case $m = \omega$: the algebras work for any $m \geq n + 1$. Here one can also show, cf. [17, Proposition 2.90], that $UrCA_n^{S,m,n} = UrCA_n^{S,m,n} = EIRCA_n^{S,m,n}$. 

6 Omitting types in clique guarded fragments

$L_n$ denotes $n$ variable first order logic. Fix $2 < n \leq l < m \leq \omega$. Consider the statement $\Psi(l, m)$: There is an atomic, countable and complete $L_n$ theory $T$, such that the type $\Gamma$ consisting of co–atoms is realizable in every $m$– square model, but any formula isolating this type has to contain more than $l$ variables. By an $m$–square model $M$ of $T$ we understand an $m$–square representation of the algebra $\mathfrak{S}m_T$ with base $M$. $\Psi(l, m)_f$ is the formula obtained from $\Psi(l, m)$ by replacing square by flat.

Let $VT(l, m) = \neg\Psi(l, m)$, short for Vaught’s theorem holds `at the parameters $l$ and $m$’ where by definition, we stipulate that $VT(\omega, \omega)$ is just Vaught’s theorem.
Let $L_{\omega, \omega}$: Countable atomic theories have countable atomic models. For $2 < n \leq l < m \leq \omega$ and $l = m = \omega$, it is likely and plausible that (**) $\forall \ell, m \leftrightarrow l = m = \omega$.

In other words: Vaught’s theorem holds only in the limiting case when $l \to \infty$ and $m = \omega$ and not ‘before’. This was proved on the ‘paths’ $(l, \omega)$, $n \leq l < \omega$ ($x$ axis) and $(n, n+k)$, $k \geq n+3$ (y axis) using two different blow up and blur constructions, given in theorems 3.1, 3.4, respectively.

One can view the limit as $l \to \infty$ on the $x$ axis algebraically using ultraproducts like in the proof of the last item in theorem 5.1. If $R_l = \text{split}(R_{l1}, R_{l2}, E_l) \in \text{RRA}$ and $\mathcal{A}_l = \text{Nr}_l \text{split}(\mathcal{A}_{l1}, \mathcal{A}_{l2}, E_l) \in \text{RCA}_n$ are as defined in theorem 4, where $\mathcal{A}_{l}$ is the finite Maddux algebra $\mathcal{E}_{l}(2,3)$ having strong $l$-blur, with $k$ the number of non identity atoms depending on $l$, then $(\text{At}\mathcal{A}_l : l \in \omega \sim n)$ is a sequence of weakly representable atom structures that are not strongly representable, a fortiori not completely representable, with a completely representable ultraproduct.

**Corollary 6.1.** Let $2 < n < \omega$.

1. Assume that $n \leq l < \omega$ and that $m \geq n + 3$. Then $\Psi(l, \omega)$ and $\Psi(n, m)$ are true. In particular, $\Psi(l, \omega)_f$ and $\Psi(n, m)_f$ and $\Psi(n, \omega)$ are true.

2. Let $m$ be finite $> n$. If there is a finite relation algebra $\mathcal{A}_m$ having a strong $m-1$-blur $(J, E)$ and no $m$-square representation, then $\Psi(m-1, m)$ is true. If such an $\mathcal{A}_m$ exists for every $m > n$, then (**) would be true.

**Proof.** For item (1) we use the second item of theorem 3.1 and theorem 3.4. $\Psi(n, m)_f$ can be also proved using the algebra $\overline{\mathcal{A}}\mathcal{M}\mathcal{C}$, where $\mathcal{C}$ is the algebra used in theorem 4.1, since $\overline{\mathcal{A}}\mathcal{M}\mathcal{C}$ is countable, and $\mathcal{M}\mathcal{A}t\overline{\mathcal{A}}\mathcal{M}\mathcal{C} = \mathcal{M}\mathcal{A}t\mathcal{C} = \mathcal{C} \notin S_n\mathcal{N}_n\mathcal{C}_{A_{n+3}}$, a fortiori, $\mathcal{C} \notin S_n\mathcal{N}_n((\mathbb{C}_{A_{n+3}} \cap \text{At})$, hence $\overline{\mathcal{A}}\mathcal{M}\mathcal{C}$ does not have a complete $m$–flat representation (any $m \geq n + 3$.)

The second item: $\Psi(m, m-1)$ follows from that the algebra $\mathcal{C} = \text{split}_n(\mathcal{A}_l, J, E)$ will be in $\text{Nr}_n\mathcal{C}_{A_{m-1}} \cap \text{RCA}_n$ but it will have no complete $m$–square representation. \( \square \)

We will show in theorem 7.1 that for $2 < m \leq l < n < \omega$, $\Psi(l, n)$ is ‘infinitely stronger’ than $\Psi(l, n)_f$ for $n \geq m + 2$. We denote the statement ‘any countable atomic $\mathcal{L}_l$ theory has an $m$–square atomic model’ by $\forall \ell (\mathcal{L}_l, m)$. For any finite $j$, ‘$j$-hyp’ is short hand for infinite $j$–dimensional hyperbasis, and $j$–basis is short hand for $j$–dimensional relational basis.

<table>
<thead>
<tr>
<th>$\Psi(n, \omega)$</th>
<th>yes, [4]</th>
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<tr>
<td>$\Psi(n, n+3)$</td>
<td>yes, thm 3.4</td>
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<tr>
<td>$\Psi(n, n+2)_f$</td>
<td>yes, if there is $\mathcal{R}$ with $n$–blur and no $n + 2$-hyp, thm 3.1</td>
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<tr>
<td>$\Psi(l, \omega)$</td>
<td>yes, $\mathcal{E}_{f(l)}(2,3)$ has strong $l$-blur, and no $\omega$-hyp, thm 3.1, cor 6.1</td>
</tr>
<tr>
<td>$\Psi(l, m)_f, l \leq m - 1$</td>
<td>yes, if there exists $\mathcal{R}$ with strong $l$-blur, and no $m$-hyp, thm 3.1</td>
</tr>
<tr>
<td>$\Psi(l, m), l \leq m - 1$</td>
<td>yes, if there exists $\mathcal{R}$ with strong $l$-blur, and no $m$-bases, 3.1, cor 7.6</td>
</tr>
<tr>
<td>$\Psi(\omega, \omega)$</td>
<td>no, Vaught’s theorem for $L_{\omega, \omega}$</td>
</tr>
</tbody>
</table>

For a class of algebras $\mathbf{K}$, let $\mathbf{K} \cap \text{Count}$ denote the class of countable algebras in $\mathbf{K}$.

Continue to fix $2 < n \leq l < m \leq \omega$. In analogy to the class $\text{RCA}_{n,m}$ defined above, or more precisely its complement, define $\text{RCA}_{l,n,m} = \{ \mathcal{A} \in \text{Nr}_n\mathcal{C}_{A_l} \cap \text{RCA}_n \cap \text{At} : \mathcal{M}\mathcal{A}t\mathcal{A} \notin \mathcal{S}_n\mathcal{N}_n\mathcal{C}_{A_m} \}$. Then $\text{RCA}_{l,n,m}$ consists of atomic algebras with weakly representable atom structures that are not strongly representable. Furthermore, $\text{RCA}_{l,n,m} \cap \text{Count} \neq \emptyset$ implies the following: $\Psi(l, m)$, $\neg \forall \ell (\mathcal{L}_l, m)$ and $\text{Sr}_{n}\mathcal{C}_m$ is not atom–canonical, cf. theorem ???. For $2 < n < \omega$, allowing $l = \omega$, we have $(\forall l \leq \omega)(\text{RCA}_{l,n,\omega} \cap \text{Count} = \emptyset) \iff$
$l = \omega$). Blowing up and blurring the Maddux algebras $E_l(2, 3)$ with $k$ as specified in the second item of theorem 3.1 provides an algebra in $\text{RCA}_n^{\omega, \omega} \cap \text{Count}$ when $l < \omega$. To prove the $\iff$ in the last equivalence assume that $l = \omega$, and that $\mathfrak{A} \in \text{Nr}_n \text{CA}_\omega$ is countable. Then $\mathfrak{A}$ will be completely representable, so $\mathfrak{CmAt}\mathfrak{A} \in \text{RCA}_n = \text{SNr}_n \text{CA}_\omega$. Thus $\text{RCA}_n^{\omega, \omega} \cap \text{Count} = \emptyset$, and we are done.

7 The neat embedding problem

Sometimes Monk-like algebras, without a rainbow intervention using the independent parameter (of greens) G are efficient in controlling 'excluding a pre-assigned number of spare dimensions ' in a certain construction. (Cf. theorem 3.4 and the remark after it).

Let us elaborate some more: Hirsch and Hodkinson [17, Theorem 15.1] solve [13, Problem 2.12] which was referred to in the literature (originally by Monk) : The neat embedding problem. In [21], the analogue of the neat embedding problem is approached for diagonal free cylindric algebras Dfs, Pinter’s substitution algebras Scs, polyadic algebras PAs, quasi–polyadic algebras QA, PAs with equality PEAs, and QAs with equality QEAs

It is proved in op. cit that (like the CA case recalled in theorem 7.1 and settled by Hirsch and Hodkinson) for any class $K$ between Sc and QEA, for any positive k, and for any ordinal $\alpha > 2$, the variety $\text{SNr}_n K_{\alpha + k + 1}$ is not axiomatizable by a finite schema over $\text{SNr}_n K_{\alpha + k}$. The case $\text{CA}_n$ for infinite $\alpha$ was not tackled Hirsch and Hodkinson.

Recall that for an atomic relation algebra $\mathfrak{A}$ and $l > 3$, recall that we denote by $\text{Mat}(\text{At}\mathfrak{A})$ the set of all $l$–dimensional basic matrices on $\mathfrak{A}$. For $\tau : l \to l$ we write $(f\tau)$ for the function defined by $(f\tau)(x, y) = f(x, \tau(y))$. It is always the case that $f\tau \in \text{Mat}(\text{At}\mathfrak{A})$ for any $f \in \text{Mat}(\text{At}\mathfrak{A})$ and any $\tau : l \to l$, so if $\text{Mat}(\text{At}\mathfrak{A})$ is an $l$–dimensional cylindric basis, then $\text{CmMat}(\text{At}\mathfrak{A})$ can be expanded to a QEA$_l$, by defining for $X \subseteq \text{Mat}(\text{At}\mathfrak{A})$ and transposition $\tau : l \to l$: $s_\tau(X) = \{ f \in \text{Mat}(\text{At}\mathfrak{A}) : f\tau \in X \}$.

Fix $2 < m < n < \omega$. Let $\mathfrak{C}(m, n, r)$ be the algebra $\mathfrak{C}(\mathfrak{H})$ where $\mathfrak{H} = H_{m+1}(\mathfrak{A}(n, r), \omega)$, is the $\text{CA}_m$, atom structure consisting of all $n + 1$–wide $m$–dimensional wide $\omega$ hyper-networks [17, Definition 12.21] on $\mathfrak{A}(n, r)$ as defined in [17, Definition 15.2]. Then $\mathfrak{C}(m, n, r) \in \text{CA}_m$, and it can be easily expanded to a QEA$_m$, since $\mathfrak{C}(m, n, r)$ is 'symmetric', in the sense that it allows a polyadic equality expansion by defining substitution operations corresponding to transpositions. This follows by observing that $\mathfrak{H}$ is obviously symmetric in the following exact sense: For $\theta : m \to m$ and $N \in \mathfrak{H}$, $\mathfrak{N}\theta \in \mathfrak{H}$, where $\mathfrak{N}\theta$ is defined by $(\mathfrak{N}\theta)(x, y) = (\theta(x), \theta(y))$. Hence, the binary polyadic operations defined on the atom structure $\mathfrak{H}$ the obvious way (by swapping co–ordinates) lifts to polyadic operations of its complex algebra $\mathfrak{C}(m, n, r)$. In more detail, for a transposition $\tau : m \to m$, and $X \subseteq \mathfrak{H}$, define $s_\tau(X) = \{ N \in \mathfrak{H} : N\tau \in X \}$.

Furthermore, for any $r \in \omega$ and $3 \leq m \leq n < \omega$, $\mathfrak{C}(m, n, r) \in \text{Nr}_m \text{QEA}_n$, $\mathfrak{N}\theta\mathfrak{C}(m, n, r) \notin \text{SNr}_m \text{CA}_{m+1}$ and $\Pi_{/\ell} \mathfrak{C}(m, n, r) \in \text{RQEA}_m$ by easily adapting [17, Corollaries 15.7, 5.10, Exercise 2, pp. 484, Remark 15.13] to the QEA context.

**Theorem 7.1.** Let $2 < m < n < \omega$.

(1) For any $K$ any class between $\text{CA}$ and $\text{QEA}$, any positive $k \geq 1$, and any finite $l \geq 0$, the variety $\text{SNr}_m K_{m+k+1}$ is not finitely axiomatizable over the variety $\text{SNr}_m K_{m+k}$ and $\text{RK}_m$ is not finitely axiomatizable over $\text{SNr}_m \text{CA}_{m+1}$ for any $0 < l < \omega$.
(2) The variety of $\text{CA}_m$s having $n$–flat representations is not finitely axiomatizable over the variety of $\text{CA}_m$s having $n$–square representations.

Proof. Item (5) follows from the discussion preceding the theorem. We prove item (2). Write $\mathfrak{C}_r$ for $\mathfrak{C}(m,n,r) \in \text{CA}_m$ (used in the proof of theorem 7.1) not to clutter notation. The parameters $m$ and $n$ will be clear from context. Given positive $k$, then for any $r \geq k^2$, $\exists$ has a winning strategy in $G^k_r(\text{At}(\mathfrak{A}(n,r)))$ [17, Remark 15.13]. This implies using ultraproducts and an elementary chain argument that $\exists$ has a winning strategy in the $\omega$–rounded game, in an elementary substructure of $\Pi_{r/U}\mathfrak{A}(n,r)/F$, hence the former is representable, and then so is the latter because RRA is a variety. Now $\exists$ has a winning strategy in $G^k_r(\mathfrak{A}(n,r))$ when $r \geq k^2$, hence, $\mathfrak{A}(n,r)$ embeds into a complete atomic relation algebra having a $k$–dimensional relational basis by [17, Theorem 12.25]. But this induces a winning strategy for $\exists$ in the game $G^k_r(\text{At}(\mathfrak{C}_r))$ with $k'$ nodes and $\omega$ rounds, for $k' \geq k$, $k' \in \omega$ so that $\mathfrak{C}_r$ has a $k'$–square representation, when $r \geq k^2$. So if $n \geq m + 2, k \geq 3$, and $r \geq k^2$, then $\mathfrak{C}_r$ has an $(n+1)$–square representation, an $n$–flat one but does not have an $n+1$–flat one. But $\Pi_{r/U}\mathfrak{C}_r/F \in \text{RCA}_m(\bigodot \text{SNR}_m\text{CA}_{n+1})$ by [17, Corollaries 15.7, 5.10, Exercise 2, pp. 484, Remark 15.13] and we are done. □

We now outline a similar construction; but due to the absence of diagonal elements here (the analogue of) $r$ appearing in $\mathfrak{A}(n,r)$ above is not merely a number, but it carries a linear order. Things here are a little bit more complicated technically but the idea in essence is very similar to that used for $\text{CA}$. But the above result for $\text{CA}$ and $\text{QEA}$s is not re–obtained in its full stength for diagonal free algebras. In this respect, the third item in our coming theorem 7.2, which is (the main theorem) [21, Theorem 1.1] is strictly weaker than the result obtained in theorem 7.1, namely (using the notation preceding op.cit), that $\Pi_{r/U}\mathfrak{C}(m,n,r) \in \text{RQEA}_m$ (upon replacing $\mathfrak{C}(m,n,r)$ by $\mathfrak{D}(m,n,r)$).

Theorem 7.2. For $3 \leq m \leq n$ and $r < \omega$ there exists finite algebras $\mathfrak{D}(m,n,r) \in \text{QEA}_m$.

I. $\mathfrak{D}(m,n,r) \in \text{Nr}_m\text{QEA}_n$,

II. $\mathfrak{R}_n\mathfrak{D}(m,n,r) \notin \text{SNR}_m\text{SC}_{n+1}$,

III. $\Pi_{r/U}\mathfrak{D}(m,n,r)$ is elementarily equivalent to a polyadic equality algebra $\mathfrak{C} \in \text{Nr}_m\text{QEA}_{n+1}$.

We define the algebras $\mathfrak{D}(m,n,r)$ for $3 \leq m \leq n < \omega$ and $r$ and then give a sketch of (II) given in detail in [21, p. 211–215]. We start with.

Definition 7.3. Define a function $\kappa : \omega \times \omega \to \omega$ by $\kappa(x,0) = 0$ (all $x < \omega$) and $\kappa(x,y+1) = 1 + x \times \kappa(x,y)$ (all $x, y < \omega$). For $n, r < \omega$ let

$$\psi(n,r) = \kappa((n-1)r, (n-1)r) + 1.$$ 

This is to ensure that $\psi(n,r)$ is sufficiently big compared to $n, r$ for the proof of non-embeddability to work. The second parameter $r < \omega$ may be considered as a finite linear order of length $r$. For any $n < \omega$ and any linear order $r$, let

$$\mathfrak{B}(n,r) = \{1d\} \cup \{a^k(i,j) : i < n - 1; j \in r, k < \psi(n,r)\}$$

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where $ld, a^k(i, j)$ are distinct objects indexed by $k, i, j$. (So here every atom $a(i, j)$ is split into $\psi(n, r)$ subatoms). The forbidden triples are:

$$\{(ld, b, c) : b \neq c \in \mathcal{B}(n, r)\} \cup \{(a^k(i, j), a^{k'}(i, j), a^{k''}(j, i')) : k, k', k^* < \psi(n, r), i < n - 1, j' \leq j \in r\}.$$ 

Let $3 \leq m \leq n < \omega$. The set of $m$–basic matrices on $\mathcal{R}$ is is a $\text{QEA}_m$ atom structure $\text{Mat}_m(\mathcal{R})$. $\mathcal{D}(m, n, r)$ is defined to be the complex algebra of the $m$–dimensional atom structure $\text{Mat}_m(\mathcal{R})$, that is, $\mathcal{D}(m, n, r) = \mathcal{C}\text{mMat}_m(\mathcal{R})$.

Unlike the algebras $\mathcal{C}(m, n, r)$ proved to be used to prove theorem 7.1, the algebras $\mathcal{D}(m, n, r)$ are now finite. It is not hard to see that $3 \leq m$, $2 \leq n$ and $r < \omega$ the algebra $\mathcal{D}(m, n, r)$. Furthermore, if $3 \leq m \leq m'$, then $\mathcal{D}(m, n, r) \cong \mathcal{N}_m, \mathcal{D}(m', n, r)$ via $X \mapsto \{f \in \text{Mat}_{m'}(\mathcal{R}) : f|_{m \times m} \in X\}$.

We give a sketch of proof of 7.2(II), which is the heart and soul of the proof. Assume hoping for a contradiction that $\mathcal{R} \mathcal{D}_s, \mathcal{D}(m, n, r) \cong \mathcal{N}_m, \mathcal{C}$ for some $\mathcal{C} \in \mathcal{C}_{n+1}$, some finite $m, n, r$. Then for $1 \leq t \leq n + 1$, it can be shown inductively that there must be a 'large' set $S_t$ of distinct elements of $\mathcal{C}$, satisfying certain inductive assumptions, which we outline next. Here largeness depends on $t$ and weakens as $t$ increases; for example $S_n$ has only two elements. For each $s \in S_t$ and $i, j < n + 2$ there is an element $a(s, i, j) \in \mathcal{B}(n, r)$ obtained from $s$ by cylindifying all dimensions in $(n + 1) \setminus \{i, j\}$, then using substitutions to replace $i, j$ by $0, 1$. It can be shown that the triple $(a(s, i, j), a(s, j, k), a(s, i, k))$ is consistent (not forbidden). The induction hypothesis says chiefly that $c_{n, s}$ is constant, for $s \in S_t$, and for $l < n$ there are fixed $i < n - 1, j < r$ such that for all $s \in S_t$, $\alpha(s, l, n) \leq a(i, j)$. This defines, like in the proof of theorem 15.8 in [18] p.471, two functions $I : n \to (n - 1)$, $J : n \to r$ such that $\alpha(s, l, n) \leq a(I(l), J(l))$ for all $s \in S_t$. The rank $rk(I, J)$ of $(I, J)$ (as defined in definition 15.9 in [18]) is the sum (over $i < n - 1$) of the maximum $j$ with $J(k) = i, J(l) = j$ (some $l < n$) or $-1$ if there is no such $j$.

From $S_t$ one constructs a set $S_{t+1}$ with index functions $(I', J')$, still relatively large (large in terms of the number of times we need to repeat the induction step) where the same induction hypotheses hold but where $rk(I', J') > rk(I, J)$. By repeating this enough times (more than $nr$ times) we obtain a non-empty set $T$ with index functions of rank strictly greater than $(n - 1) \times (r - 1)$, an impossibility. We sketch the induction step. Since $I$ cannot be injective there must be distinct $l_1, l_2 < n$ such that $I(l_1) = I(l_2)$ and $J(l_1) \leq J(l_2)$. We may use $l_1$ as a "spare dimension" (changing the index functions on $l$ will not reduce the rank). Since $c_{n, s}$ is constant, we may fix $s_0 \in S_{t-1}$ and choose a new element $s'$ below $c_{s_0} \cdot s^* s_{s_0}$, with certain properties. Let $S_{t+1} = \{s' : s \in S_t \setminus \{s_0\}\}$. Re-establishing many of the induction hypotheses for $S_{t+1}$ is not too hard. Also, it can be shown that $J'(l) \geq J(l)$ for all $l < n$. Since $(\alpha(s, i, j), \alpha(s, j, k), \alpha(s, i, k))$ is consistent and by the definition of the forbidden triples either $\text{rng}(I')$ properly extends $\text{rng}(I)$ or there is $l < n$ such that $J'(l) > J(l)$, hence $rk(I', J') > rk(I, J)$.

The idea of constructing $S_{t+1}$ from $S_t$ is given pictorially on [20, Figure 2, p. 8] in the context of $\mathcal{C}$As. The essence of the ideas used in [20, 21] is the same. Suppose we are at stage $t$. Then every $x \in S_t$ gives a set of colours (atoms) denoted in [20] by $x(i, t)(i < t)$. One gets $S_{t+1}$ from $S_t$ by first 'glueing together' any two elements $x, z$ of $S_t$, using $t + 1$ as a spare dimension, first moving the $t$th co-ordinate of $x$ to $t + 1$ forming $s_{x+z}^t x$. By fixing $z$ and varying $x$ one gets a huge number of different elements. Their $(t, t + 1)$th colours cannot be controlled yet; they may not be the same. To get
over this hurdle, one uses the pigeon-hole principal to pick the still large set \( S_{t+1} \) in which the \((t, t+1)\)th colour is fixed to be the same. ‘Largness’ enables one to do so.

The following bolded paragraph (slightly paraphrased) was communicated to the author by Professor Roger Maddux (personnel communication): It summarizes the essence of the idea used in the solution of [13, Problem 2.12]:

In figure 2 in [20] there is a top element that is connected by coloured edges to the intermediate elements that are all connected to a bottom element. The number of elements (in this figure) is the number of colours plus one. So one gets the same control as rainbow algebras provided by (the second independent parameter) \( G \). The key idea here is that the proof of Ramsey in this context does not require an uncontrollable Ramsey number of ‘spare dimensions’, which were the versions used by Monk and Maddux before proving non-finite axiomatizability [23, 24], but only one more than the number of colours used.

For the above non-representable Monk-style algebras denoted by \( \mathfrak{A}(n, r) \), \( 3 \leq m < n < \omega \) and \( r \in \omega \), it is easy to see that \( \exists \) cannot win the usual infinite atomic game. But this time one can use ‘a hyperbasis game’ denoted by \( G_r^{m,n+1} \) in [17] with \( r \) denoting the number of rounds, to pin point the least \( k > n \) for which \( \mathfrak{A}(n, r) \) ‘stops to be representable’ getting the sharper result we want. The game \( G_r^{m,n+1} \) is stronger than \( G_\omega \), involving additional amalgamation moves played on \( n+1 \)-dimensional \( m \)-wide hypernetworks. One can show that \( \forall \) has a winning strategy in \( G_r^{m,n+1}(\mathfrak{A}(n, r)) \), using exactly \( n+1 \) nodes (for any \( r < \omega \)), getting the same control we get from rainbows using the parameter \( G \), and in fact the best possible. This is the approach adopted in [18]. Here \( \mathfrak{A}(n, r) \) has an \( n \)-dimensional cylindric basis, but no \( n+1 \)-dimensional hyperbasis. Worthy of note, is that the last condition is strictly stronger than ‘not having an \( n+1 \)-dimensional cylindric basis’. Relation algebras having \( n \)-dimensional cylindric basis but no \( n+1 \)-dimensional cylindric basis were constructed by Maddux. We refer to [20] for more.

In the proof of theorem 7.1, one uses that \( \Pi_{r/U} \mathfrak{C}(m,n,r) \in \text{RQEA}_m \). As stated in the last item of theorem 7.2, we do not guarantee that the ultraproduct on \( r \) of the \( \mathfrak{D}(m,n,r) \)s (\( 2 < m < n < \omega \)) is representable. A standard L"{o}s argument shows that \( \Pi_{r/U} \mathfrak{C}(m,n,r) \cong \mathfrak{C}(m,n,\Pi_{r/U}r) \) and \( \Pi_{r/U}r \) contains an infinite ascending sequence. Here one extends the definition of \( \psi \) by letting \( \psi(n,r) = \omega \), for any infinite linear order \( r \). The infinite algebra \( \mathfrak{D}(m,n,J) \in \text{ElNr}_n \text{QEA}_{n+1} \) when \( J \) is the infinite linear order as above. Since \( \Pi_{r/U}r \) is such, then we get \( \Pi_{r/U} \mathfrak{D}(m,n,r) \in \text{ElNr}_n \text{QEA}_{n+1}(\subseteq \text{SNr}_m \text{QEA}_{n+1}) \), cf. [21, pp.216-217]. This suffices to show that for any \( K \) having signature between \( \mathfrak{S} \) and \( \text{QEA} \), for any \( 2 < m < \omega \), and for any positive \( k \), the variety \( \text{SNr}_m K_{m+k} \) is not finitely axiomatizable over the variety \( \text{SNr}_m K_{m+k} \).

The technique used in [21, Theorem 3.1] can be used to show the following by lifting the result in theorem 7.1 to the transfinite.

**Theorem 7.4.** Let \( \alpha \) be any ordinal \( > 2 \) possibly infinite. Then for any \( r \in \omega \), and \( k \geq 1 \), there exists \( \mathfrak{A}_r \in \text{SNr}_m \text{QEA}_{\alpha+k} \) such that \( \mathfrak{R} \mathfrak{A}_r \notin \text{SNr}_\alpha \text{CA}_{\alpha+k+1} \) and \( \Pi_{r/U} \mathfrak{A}_r \in \text{RQEA}_\alpha \) for any non-principal ultrafilter \( U \) on \( \omega \).

**Proof.** Let \( 2 < m < n < \omega \). We use the algebras \( \mathfrak{C}(m,n,r) = \mathfrak{C}(H) \) where \( H = H_r^{n+1}(\mathfrak{A}(n, r), \omega) \), is the \( \text{CA}_m \) atom structure consisting of all \( n+1 \)-wide \( m \)-dimensional \( \omega \) hypernetworks [17, Definition 12.21] on \( \mathfrak{A}(n , r) \) as defined in [17, Definition 15.2]. Take

\[
x_n = \{ f \in H_r^{n+k+1}(\mathfrak{A}(n, r), \omega); m \leq j < n \rightarrow \exists i < m, f(i,j) = \text{Id} \}.
\]
Then $x_n \in C(n, n + k, r)$ and $c_i x_n \cdot c_j x_n = x_n$ for distinct $i, j < m$. Furthermore (*), $I_n : C(m, m + k, r) \cong \mathfrak{A}_x \mathfrak{A}_m C(n, n + k, r)$ via the map, defined for $S \subseteq H^{m+k+1}_m(\mathfrak{A}(m+k, r), \omega))$, by

$$I_n(S) = \{ f \in H^{m+k+1}_m(\mathfrak{A}_x(n, r), \omega) : f \upharpoonright \subseteq m+k+1, m \in S, \forall j \leq j < n \rightarrow \exists i < m, f(i, j) = 1d) \}.$$

Let $\alpha$ be an infinite ordinal. Let $I = \{ \Gamma : \Gamma \subseteq \alpha, |\Gamma| < \omega \}$. For each $\Gamma \in I$, let $M_\Gamma = \{ \Delta \in I : \Gamma \subseteq \Delta \}$, and let $F$ be an ultrafilter on $I$ such that $\forall \Gamma \in I, M_\Gamma \in F$. For each $\Gamma \in I$, let $\rho_F$ be an injective function from $|\Gamma|$ onto $\Gamma$. Let $C_\Gamma$ be an algebra similar to $\mathcal{QA}_\alpha$, such that $\mathfrak{A}_\alpha C_\Gamma = C(|\Gamma|, |\Gamma| + k, r)$ and let $B^\Gamma = \Pi_{\Gamma/F \rho_F} C_\Gamma$. Then we have $B^\Gamma \in N_{\alpha} \mathcal{QA}_\alpha$ and $\mathfrak{A}_\alpha \mathfrak{A}_\alpha B^\Gamma \subseteq S N_{\alpha} \mathcal{CA}_\alpha$. These can be proved exactly like the proof of the first two items in [21, Theorem 3.1]. The second part uses that the element $x_n$ is $m$-rectangular and $m$-symmetric, in the sense that for all $i \neq j < m$, $c_i x_n \cdot c_j x_n = x_n$ and $s^j x_n \cdot s^i x_n = x_n$ (This last two conditions are not entirely independent [13]). This is crucial to guarantee that in the algebra obtained after relativizing to $x_n$, we do not lose commutativity of cylinders. The relativized algebra stays inside $\mathcal{QA}_\alpha$. We know from the finite dimensional case that $\Pi_{\rho_F} \mathfrak{A}_\alpha C_\Gamma = \Pi_{\rho_F} C(|\Gamma|, |\Gamma| + k, r) \subseteq N_{\alpha} \mathcal{QA}_\alpha$, for some $C_\Gamma \in \mathcal{QA}_\alpha C_{\alpha+\omega} = \mathcal{QA}_\omega$. Let $\lambda : \omega \rightarrow \alpha + \omega$ extend $\rho_F : |\Gamma| \rightarrow \Gamma (\subseteq \alpha)$ and satisfy $\lambda_i (|\Gamma| + i) = \alpha + i$ for $i < \omega$. Let $\mathfrak{A}_\Gamma$ be a $\mathcal{QA}_{\alpha+\omega}$ type algebra such that $\mathfrak{A}_\lambda x \mathfrak{A}_\lambda = \mathfrak{A}_\alpha$. Then $\Pi_{\rho_F} \mathfrak{A}_\Gamma \subseteq \mathcal{QA}_{\alpha+\omega}$, and we have proceeding like in the proof of item 3 in [21, Theorem 3.1].

$$\Pi_{\rho_F} B^\Gamma = \Pi_{\rho_F} \Pi_{\rho_F} C_\Gamma \subseteq \Pi_{\rho_F} C(|\Gamma|, |\Gamma| + k, r) \subseteq \Pi_{\rho_F} N_{\alpha} \mathcal{QA}_\alpha = \Pi_{\rho_F} N_{\alpha} \mathfrak{A}_\alpha \mathfrak{A}_\alpha = N_{\alpha} \Pi_{\rho_F} \mathfrak{A}_\Gamma. \text{ But } B = \Pi_{\rho_F} B^\Gamma \subseteq S N_{\alpha} \mathcal{QA}_{\alpha+\omega} \text{ because } \mathfrak{A}_\Gamma = \Pi_{\rho_F} \mathfrak{A}_\Gamma \subseteq \mathcal{QA}_{\alpha+\omega} \text{ and } B \subseteq N_{\alpha} \mathfrak{A}_\Gamma, \text{ hence it is representable (here we use the neat embedding theorem). The rest follows using a standard Los argument}.$$

On the other hand, including diagonal free algebras like $\mathcal{Sc}_\alpha$, in [21, Theorem 3.1] only the following is proved lifting the (weaker) result in theorem 7.2 to the transfinite:

Theorem 7.5. Let $\alpha > 2$. Then for any $r \in \omega$, for any finite $k \geq 1$, there exist $B^\Gamma \in S N_{\alpha} \mathcal{QA}_{\alpha+k}$, and $\mathfrak{A}_\alpha \mathfrak{A}_\alpha B^\Gamma \subseteq S N_{\alpha} \mathcal{Sc}_\alpha+k+1$ such $\Pi_{\rho_F} B^\Gamma \subseteq S N_{\alpha} \mathcal{QA}_{\alpha+k+1}$.

It is open whether we can replace $S N_{\alpha} \mathcal{QA}_{\alpha+k+1}$ in the conclusion by $\mathcal{RO}_{\alpha+k}$, like we did in theorem 7.4 when dealing only with $\mathcal{CA}_\alpha$ and $\mathcal{QA}_\alpha$ where we have the diagonal constants.

The following table collects the results on the famous neat embedding problem, $\mathcal{NEP}$ for short, first posed by Monk in 1969 [23] for cylindric algebras, for various cylindric-like algebras starting from $\mathcal{Sc}_\alpha$ all the way to $\mathcal{PEA}_\alpha$ for both finite and infinite dimensions ($> 2$). Monk's $\mathcal{NEP}$ formulated only for $\mathcal{CA}_\alpha$s was solved by Hirsch and Hodkinson [20] [17, Theorem 15.1].

<table>
<thead>
<tr>
<th>Algebras</th>
<th>Status of the $\mathcal{NEP}$ for $3 \leq \alpha, k &lt; \omega$</th>
<th>Citation</th>
</tr>
</thead>
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<tr>
<td>$\mathcal{Sc}_\alpha$</td>
<td>$\mathcal{Sc}<em>\alpha \subseteq \mathcal{Sc}</em>{\alpha+k+1}$ is n.f.a. over $\mathcal{Sc}_\alpha$</td>
<td>[21, Cor 1.2, Cor 3.3]</td>
</tr>
<tr>
<td>$\mathcal{CA}_\alpha$</td>
<td>$\mathcal{CA}<em>\alpha \subseteq \mathcal{CA}</em>{\alpha+k+1}$ is n.f.a. over $\mathcal{CA}_\alpha$</td>
<td>[21, Cor 1.2, Cor 3.3], thm 7.4</td>
</tr>
<tr>
<td>$\mathcal{QA}_\alpha$</td>
<td>$\mathcal{QA}<em>\alpha \subseteq \mathcal{QA}</em>{\alpha+k+1}$ is n.f.a. over $\mathcal{QA}_\alpha$</td>
<td>[21, Cor 1.2, Cor 3.3]</td>
</tr>
<tr>
<td>$\mathcal{QEA}_\alpha$</td>
<td>$\mathcal{QEA}<em>\alpha \subseteq \mathcal{QEA}</em>{\alpha+k+1}$ is n.f.a. over $\mathcal{QEA}_\alpha$</td>
<td>[21, Cor 1.2, Cor 3.3], thm 7.4</td>
</tr>
</tbody>
</table>

Next we formulate the logical counterpart of [13, Problem 2.12] and theorem 6.1 against each other.

Theorem 7.6. (1) For any $m > 3$ there exists a 3 variable formula that cannot be proved using $m - 1$ variables, but can be proved using $m$ variables. 

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Let $m > 3$. If $R_m$ as in corollary 6.1 exists, then there is an $L_3$ atomic countable theory $T$ such that the non-principal type consisting of co-atoms is realizable in every $m$-flat model, of $T$ but cannot be isolated using $m - 1$ variables. That is any witness must contain more than $m$ variables.

Both items depend on the existence of certain finite relation algebras. For the first item certain finite relation algebras, namely the algebra $A(m - 1, r)$ ($r \in \omega$) with an $m - 1$-dimensional hyperbasis but no $m$-dimensional hyperbasis were used, cf. [17, Theorem 15.1] and [20] (the latter reference for drawing the metalogical conclusion). For the second item, one still needs to find a relation algebra $R_m$ with a strong $m - 1$ blur $(J, E)$, so that split$(R_m, J, E)$ has an $m-1$-dimensional cylindric basis, but $CmAtsplit(R_m, J, E)$ does no have an infinite $m$-dimensional hyperbasis, seeing as how $R_m$ embeds into $CmAtsplit(R_m, J, E)$. If there is such an $R_m$, then the algebra split$_3(R_m, J, E) \in RCA_3$ will give the required theory. Any atomic and countable (consistent) theory $T$ such that $\mathfrak{m}_T \cong \text{split}_3(R_m, J, E)$ will fit the bill. So here modolu the existence of $R_m$, we have the metalogical conclusion, which can be seen (using metalogical jargon) an ‘omitting types analogue’ of [13, Problem 2.12] by replacing the number of variables needed for provability to the number of variables needed to isolate certain types realizable in (locally) relativized models.

References


[31] 