Non Elementary Classes of Relation and Cylindric Algebras

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Abstract

For any pair of ordinals $\alpha < \beta$, $\text{CA}_\alpha$ denotes the class of cylindric algebras of dimension $\alpha$, $\text{RCA}_\alpha$ denotes the class of representable $\text{CA}_\alpha$s and $\text{Nr}_\alpha \text{CA}_\beta$ ($\text{RaCA}_\beta$) denotes the class of $\alpha$-neat reducts (relation algebra reducts) of $\text{CA}_\beta$. We show that any class $K$ such that $\text{RaCA}_\omega \subseteq K \subseteq \text{RaCA}_5$, $K$ is not elementary, i.e not definable in first order logic. Let $2 < n < \omega$. It is also shown that any class $K$ such that $\text{Nr}_n \text{CA}_\omega \cap \text{CRCA}_n \subseteq K \subseteq S_c \text{Nr}_n \text{CA}_{n+3}$, where $\text{CRCA}_n$ is the class of completely representable $\text{CA}_n$s, and $S_c$ denotes the operation of forming complete subalgebras, is proved not to be elementary. Finally, we show that any class $K$ such that $S_d \text{RaCA}_\omega \subseteq K \subseteq S_d \text{RaCA}_5$ is not elementary. It remains to be seen whether there exist elementary classes between $\text{RaCA}_\omega$ and $S_d \text{RaCA}_\omega$. In particular, for $m \geq n + 3$, the classes $\text{Nr}_n \text{CA}_m$, $\text{CRCA}_n$, $S_d \text{Nr}_n \text{CA}_m$, where $S_d$ is the operation of forming dense subalgebras are not first order definable.

Mathematics Subject Classification: 03B45, 03G15

Keywords: neat reducts, complete representations, first order definability

1 Introduction

We follow the notation of [2] which is in conformity with the notation in the monograph. Relation algebras $\text{RAs}$ are abstractions of algebras whose universe consists of binary relations, together with the Boolean operations of intersection, union and complementation, and additional binary operation of composition.
of relations, the unary one forming converses of relations and the identity relation as a constant element in the signature. We consider relation algebras as algebras of the form $\mathcal{R} = \langle R, +, \cdot, -, 1', \sim, ; \rangle$, where $\langle R, +, \cdot, -, 1' \rangle$ is a Boolean algebra $1' \in R$, $\sim$ is the unary operation of forming converses and $;$ is the binary operation of composition. A relation algebra is representable $\iff$ it is isomorphic to a subalgebra of the form $\langle \wp(X), \cup, \cap, \sim, \circlearrowleft, \circlearrowright, \sim, \circlearrowleft, \circlearrowright, \oslash, \odot \rangle$, where $X$ is an equivalence relation, $1'$ is interpreted as the identity relation, $\sim$ is the operation of forming converses, and $;$ is interpreted as composition of relations. Following standard notation, $(R)\mathcal{R}A$ denotes the class of (representable) relation algebras. The class $\mathcal{R}A$ is a discriminator variety that is finitely axiomatizable, cf. [8, Definition 3.8, Theorems 3.19]. Algebras consisting of relations of possibly higher arity $\alpha$, $\alpha$ an arbitrary ordinal are represented by cylindric algebras of dimension $\beta$. In this case the concrete versions of such algebras are Boolean algebras with operators where the extra Boolean operations are the projections or cylindrifications $C_k (k < \beta)$ and so-called diagonal elements reflecting equality. For a set $V$, $\mathcal{B}(V)$ denotes the Boolean set algebra $\langle \wp(V), \cup, \cap, \sim, \oslash, V \rangle$. Let $U$ be a set and $\alpha$ an ordinal; $\alpha$ will be the dimension of the algebra. For $s, t \in ^{\alpha}U$ write $s \equiv t$ if $s(j) = t(j)$ for all $j \neq i$. For $X \subseteq ^{\alpha}U$ and $i, j < \alpha$, let

$$C_i X = \{s \in ^{\alpha}U : (\exists t \in X)(t \equiv_i s)\}$$

and

$$D_{ij} = \{s \in ^{\alpha}U : s_i = s_j\}.$$  

$\langle \mathcal{B}(^{\alpha}U), C_i, D_{ij} \rangle_{i,j<\alpha}$ is called the full cylindric set algebra of dimension $\alpha$ with unit (or greatest element) $^{\alpha}U$. Any subalgebra of the latter is called a set algebra of dimension $\alpha$. Examples of subalgebras of such set algebras arise naturally from models of first order theories. Indeed, if $M$ is a first order structure in a first order signature $\mathcal{L}$ with $\alpha$ many variables, then one manufactures a cylindric set algebra based on $M$ as follows, cf. [4, §4.3]. Let

$$\phi^M = \{s \in ^{\alpha}M : M \models [s] \},$$

(here $M \models [s]$ means that $s$ satisfies $\phi$ in $M$), then the set $\{\phi^M : \phi \in Fm^L\}$ is a cylindric set algebra of dimension $\alpha$, where $Fm^L$ denotes the set of first order formulas taken in the signature $L$. To see why, we have:

$$\phi^M \cap \psi^M = (\phi \land \psi)^M,$$

$$^{\alpha}M \sim \phi^M = (\neg \phi)^M,$$

$$C_i(\phi^M) = (\exists \alpha_i \phi)^M,$$

$$D_{ij} = (x_i = x_j)^M.$$
Following [4], \( C_\alpha \) denotes the class of all subalgebras of full set algebras of dimension \( \alpha \). The (equationally defined) \( CA_\alpha \) class is obtained from cylindric set algebras by a process of abstraction and is defined by a finite schema of equations given in [4, Definition 1.1.1] that holds of course in the more concrete set algebras.

**Definition 1.1.** Let \( \alpha \) be an ordinal. By a cylindric algebra of dimension \( \alpha \), briefly a \( CA_\alpha \), we mean an algebra

\[
\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, d_{ij} \rangle_{\kappa, \lambda < \alpha}
\]

where \( \langle A, +, \cdot, -, 0, 1 \rangle \) is a Boolean algebra such that 0, 1, and \( d_{ij} \) are distinguished elements of \( A \) (for all \( j, i < \alpha \)), - and \( c_i \) are unary operations on \( A \) (for all \( i < \alpha \)), + and \( . \) are binary operations on \( A \), and such that the following equations are satisfied for any \( x, y \in A \) and any \( i, j, \mu < \alpha \):

\[
\begin{align*}
(C_1) & \quad c_i 0 = 0, \\
(C_2) & \quad x \leq c_i x \ (i.e., x + c_i x = c_i x), \\
(C_3) & \quad c_i (x \cdot c_i y) = c_i x \cdot c_i y, \\
(C_4) & \quad c_i c_j x = c_j c_i x, \\
(C_5) & \quad d_{ii} = 1, \\
(C_6) & \quad \text{if } i \neq j, \mu, \text{ then } d_{j\mu} = c_i (d_{ji} \cdot d_{i\mu}), \\
(C_7) & \quad \text{if } i \neq j, \text{ then } c_i (d_{ij} \cdot x) \cdot c_i (d_{ij} \cdot -x) = 0.
\end{align*}
\]

Let \( \alpha \) be an ordinal and \( \mathfrak{A} \in CA_\alpha \). For any \( i, j, l < \alpha \), let \( s_i^j x = x \) if \( i = j \) and \( s_i^j x = c_j (d_{ij} \cdot x) \) if \( i \neq j \). Let \( \mathfrak{A} = \langle A, +, \cdot, -, 0, 1, s_0^1 \rangle \) of \( A \) and in the next definition, in its first item we define the notion of forming \( \alpha \)-neat reducts of \( CA_\beta \)s with \( \beta > \alpha \), in symbols \( \mathfrak{A} \in \mathfrak{A} \), and in the second item we define relation algebras obtained from cylindric algebras using the operator \( \mathfrak{R}_2 \).

**Definition 1.2.** 1. Assume that \( \alpha < \beta \) are ordinals and that \( \mathfrak{B} \in CA_\beta \).

Then the \( \alpha \)-neat reduct of \( \mathfrak{B} \), in symbols \( \mathfrak{R}_\alpha \mathfrak{B} \), is the algebra obtained from \( \mathfrak{B} \), by discarding cylindrifiers and diagonal elements whose indices are in \( \beta \setminus \alpha \), and restricting the universe to the set \( Nr_\alpha B = \{ x \in \mathfrak{B} : \forall i \in \beta : c_i x \neq x \} \subseteq \alpha \} \).

2. Assume that \( \alpha \geq 3 \). Let \( \mathfrak{A} \in CA_\alpha \). Then \( \mathfrak{R}_\alpha \mathfrak{A} = \langle Nr_\alpha \mathfrak{A}, +, \cdot, -, 0, 1, s_0^1 \rangle \) where for any \( x, y \in Nr_\alpha \mathfrak{A}, x; y = c_2 (s_0^1 x \cdot s_0^1 y) \) and \( x =_2 s(0.1) x \).
If \( \mathcal{A} \in \text{CA}_n \), \( \text{Ra}\mathcal{A} \), having the same signature as RA may not be a relation algebra as associativity of the (abstract) composition operation may fail, but for \( \alpha \geq 4 \), \( \text{RaCA}_\beta \subseteq \text{RA} \). It is straightforward to check that \( \mathcal{N}_{\alpha}\mathcal{B} \in \text{CA}_\alpha \).

Let \( \alpha < \beta \) be ordinals. If \( \mathcal{A} \in \text{CA}_\alpha \) and \( \mathcal{A} \subseteq \mathcal{N}_{\alpha}\mathcal{B} \), with \( \mathcal{B} \in \text{CA}_\beta \), then we say that \( \mathcal{A} \) neatly embeds in \( \mathcal{B} \), and that \( \mathcal{B} \) is a \( \beta \)-dilation of \( \mathcal{A} \), or simply a dilation of \( \mathcal{A} \) if \( \beta \) is clear from context. For \( \mathcal{K} \subseteq \text{CA}_\beta \), we write \( \mathcal{N}_{\alpha}\mathcal{K} \) for the class \( \{ \mathcal{N}_{\alpha}\mathcal{B} : \mathcal{B} \in \mathcal{K} \} \). Following [4], \( \mathcal{G}_{n} \) denotes the class of generalized cylindric set algebra of dimension \( n \); \( \mathcal{C} \in \mathcal{G}_n \), if \( \mathcal{C} \) has top element \( V \) a disjoint union of cartesian squares, that is \( V = \bigcup_{i \in I} U_i \), \( I \) is a non-empty indexing set, \( U_i \neq \emptyset \) and \( U_i \cap U_j = \emptyset \) for all \( i \neq j \). The operations of \( \mathcal{C} \) are defined like in cylindric set algebras of dimension \( n \) relativized to \( V \). By the same token the variety of representable relation algebras is denoted by \( \text{RRA} \). It is known that \( \mathcal{G}_{n} = \text{RCA}_n \), that \( \mathcal{S}_{\mathcal{N}_{\alpha}} \mathcal{C}_n \mathcal{A}_\omega = \bigcap_{k \in \omega} \mathcal{S}_{\mathcal{N}_{\alpha}} \mathcal{C}_{n+k} \) and that \( \text{RRA} = \mathcal{S}_{\text{RaCA}_\omega} = \bigcap_{k \in \omega} \mathcal{S}_{\text{RaCA}_{\omega+k}} \). We often identify set algebras with their domain referring to an injection \( f: \mathcal{A} \rightarrow \wp(V) \) \( (\mathcal{A} \in \text{CA}_n) \) as a complete representation of \( \mathcal{A} \) (via \( f \)) where \( V \) is a \( \mathcal{G}_{n} \) unit.

**Definition 1.3.** An algebra \( \mathcal{A} \in \text{CA}_n \) is completely representable \( \iff \) there exists \( \mathcal{C} \in \mathcal{G}_n \), and an isomorphism \( f: \mathcal{A} \rightarrow \mathcal{C} \) such that for all \( X \subseteq \mathcal{A} \), \( f(\sum X) = \bigcup_{x \in X} f(x) \), whenever \( \sum X \) exists in \( \mathcal{A} \). In this case, we say that \( \mathcal{A} \) is completely representable via \( f \).

It is known that \( \mathcal{A} \) is completely representable via \( f: \mathcal{A} \rightarrow \mathcal{C} \), where \( \mathcal{C} \in \mathcal{G}_n \) has top element \( V \) say \( \iff \mathcal{A} \) is atomic and \( f \) is atomic in the sense that \( f(\sum \text{At}(\mathcal{A})) = \bigcup_{x \in \text{At}(\mathcal{A})} f(x) = V \) [7]. We denote the class of completely representable \( \text{CA}_n \)s by \( \text{CRCA}_n \). Complete representations for \( \text{RAs} \) are defined analogously. The class of completely representable \( \text{RA} \) is denoted by \( \text{CRRA} \).

Unless otherwise indicated, \( n \) will be a finite ordinal \( > 2 \). In [5] it is proved that any class between \( \mathcal{K} \) between \( \mathcal{S}_{\text{c}} \text{RaCA}_\omega \cap \text{CRRA} \) and \( \mathcal{S}_{\text{c}} \text{RaCA}_5 \), \( \mathcal{K} \) is not elementary. Here we strengthen this result by replacing the first \( \mathcal{S}_{\text{c}} \) by \( \mathcal{S}_{\text{d}} \) (the operation of forming dense subalgebras). We recall that for BAOs \( \mathcal{A} \) and \( \mathcal{B} \) having the same signature, \( \mathcal{B} \) is dense in \( \mathcal{A} \), written \( \mathcal{B} \subseteq_d \mathcal{A} \), if \( \mathcal{B} \) is subalgebra of \( \mathcal{A} \) such that for all non-zero \( a \in \mathcal{A} \), there exists a non-zero \( b \in \mathcal{B} \) with \( b \leq a \). It is known that for a class \( \mathcal{K} \) of BAOs \( \mathcal{K} \subseteq \mathcal{S}_d \mathcal{K} \subseteq \mathcal{S}_\text{c} \mathcal{K} \); and the inclusion are proper for Boolean algebras (without operators). For a class \( \mathcal{K} \) of BAO, we let \( \mathcal{S}_d \mathcal{K} = \{ \mathcal{B} : \exists \mathcal{A} \in \mathcal{K} \} \) \( (\forall X \subseteq \mathcal{A}) \) \( (\exists a \in \mathcal{A} \) \( \sum a \mathcal{X} = 1 \Rightarrow \sum X \) exists in \( \mathcal{B} \) and \( \sum a \mathcal{X} = 1 \} \). It is known that for any \( \mathcal{K} \) of BAOs, \( \mathcal{K} \subseteq \mathcal{S}_d \mathcal{K} \subseteq \mathcal{S}_\text{c} \mathcal{K} \). If \( \mathcal{K} \) happens to be the class of Boolean algebras (without operators) then these inclusions are proper and in the specific case we address the inclusion is also strict thereby obtaining a stronger result than that announced in [6]. More importantly, we show that any class \( \mathcal{K} \), such that \( \text{RaCA}_\omega \cap \text{CRRA} \subseteq \mathcal{K} \subseteq \text{RaCA}_5 \) is not elementary. In the first case, we use so-called Rainbow algebras, in the second case we use so-called Monk algebra where a particular combinatorial form of Ramsey’s theorem plays an essential role. In [18] it is proved that
for any pair of infinite ordinals $\alpha < \beta$, the class $\text{Nr}_\alpha \text{CA}_\beta$ is not elementary. A different model theoretic proof for finite $\alpha$ is given in [20, Theorem 5.4.1]. This result is extended to many cylindric like algebras like Halmos’ polyadic algbras with and without equality, and Pinter’s substitution algebras in [16, 17]. The class $\text{CRCA}_n$ is proved not be elementary by Hirsch and Hodkinson in [7]. Neat embeddings and complete representations are linked in [21, Theorem 5.3.6] where it is shown that $\text{CRCA}_n$ coincides with the class $\text{S}_c \text{Nr}_n \text{CA}_\omega$ on atomic algebra having countably many atoms. Below it is proved that this characteriztion does not generalize to atomic algebras having uncountably many atoms. Completely analogous results are obtained for $\text{RAs}$, that is to say, $\text{S}_c \text{RaCA}_\omega$ and $\text{CRRA}$ coincide on atomic algebras with countably many atoms, and this characterization does not generalize to the case of atomic $\text{RAs}$ having uncountably many atoms. In fact, we shall prove that there exists an atomless $\mathcal{C} \in \text{CA}_\omega$, such that for all $n < \omega$, $\text{Nr}_n \mathfrak{A}$ and $\text{RaA}$ are atomic algebras having uncountably many atoms, but do not have a complete representation.

In this paper, we use combinatorial game theory combined with basic graph theory resorting to (as mentioned above) Rainbow construction which is extremely efficient and flexible in constructing subtle delicate counterexamples. Rainbow constructions are based on two player deterministic games and as the name suggests they involve ‘colours’. Such games happen to be simple Ehrenfeucht–Fraïssé forth games where the two players $\exists$illoise and $\forall$belard, between them, use pebble pairs outside the board, each player pebbling one of the two structures which she/he sticks to it during the whole play. In the number of rounds played (that can be transfinte), $\exists$ tries to show that two simple relational structures $G$ (the greens) and $R$ (the reds) have similar structures while $\forall$ tries to show that they are essentially distinct. Such structures may include ordered structures and complete irreflexive graphs, such as finite ordinals, $\omega_1$, $N$, $Z$ or $R$. A winning strategy for either player in the Ehrenfeucht–Fraïssé game can be lifted to winning strategy in a rainbow game played on so–called atomic networks on a rainbow atom structure (for both $\text{CAs}$ and $\text{RAs}$ ) based also on $G$ and $R$. Once $G$ and $R$ are specified, the rainbow atom structure is uniquely defined. Though more (rainbow) colours (like whites and shades of yellow) are involved in the rainbow atom structure, the crucial thing here is that the number of rounds and nodes in networks used in the rainbow game, depend recursively on the number of rounds and pebble pairs in the simple Ehrenfeucht–Fraïssé forth two player game played on $G$ and $R$. Due to the control on winning strategy’s in terms of the relational structures ($G$ and $R$) chosen in advance, and the number of pebble pairs used outside the board, rainbow constructions have proved highly effective in providing subtle counterexamples to really bewildering ‘yes or no’ assertions for both $\text{CAs}$ and $\text{RAs}$ (relation algebras) cf. [7, 8, 10, 23]
2 Preliminaries

From now on, unless otherwise indicated, \( n \) is fixed to be a finite ordinal > 2. Let \( i < n \). For \( n \)-ary sequences \( \bar{x} \) and \( \bar{y} \), we write \( \bar{x} \equiv_i \bar{y} \iff \bar{y}(j) = \bar{x}(j) \) for all \( j \neq i \). For \( i, j < n \) the replacement \([i/j] \) is the map that is like the identity on \( n \), except that \( i \) is mapped to \( j \) and the transposition \([i, j] \) is the like the identity on \( n \), except that \( i \) is swapped with \( j \).

**Definition 2.1.** 1. An \( n \)-dimensional atomic network on an atomic algebra \( \mathfrak{A} \in \text{CA}_n \) is a map \( N : \Delta \to \text{At}\mathfrak{A} \), where \( \Delta \) is a non-empty finite set of nodes, denoted by \( \text{nodes}(N) \), satisfying the following consistency conditions for all \( i < j < n \):

   (i) If \( \bar{x} \in \text{nodes}(N) \) then \( N(\bar{x}) \leq d_{ij} \iff \bar{x}_i = \bar{x}_j \),

   (ii) If \( \bar{x}, \bar{y} \in \text{nodes}(N) \), \( i < n \) and \( \bar{x} \equiv_i \bar{y} \), then \( N(\bar{x}) \leq c_i N(\bar{y}) \).

Let \( i < n \). For \( n \)-dimensional atomic networks \( M \) and \( N \), we write \( M \equiv_i N \iff M(\bar{y}) = N(\bar{y}) \) for all \( \bar{y} \in \text{nodes}(N) \setminus \{ \bar{i} \} \).

2. Assume that \( \mathfrak{A} \in \text{CA}_n \) is atomic and that \( m, k \leq \omega \). The atomic game \( G^m_k(\text{At}\mathfrak{A}) \), or simply \( G^m_k \), is the game played on atomic networks of \( \mathfrak{A} \) using \( m \) nodes and having \( k \) rounds [10, Definition 3.3.2], where \( \forall \) is offered only one move, namely, a cylindrifier move:

Suppose that we are at round \( t > 0 \). Then \( \forall \) picks a previously played network \( N_t \) \( (\text{nodes}(N_t) \subseteq m) \), \( i < n \), \( a \in \text{At}\mathfrak{A} \), \( \bar{x} \in \text{nodes}(N_t) \), such that \( N_t(\bar{x}) \leq c_i a \). For her response, \( \exists \) has to deliver a network \( M \) such that \( \text{nodes}(M) \subseteq m \), \( M \equiv_i N_t \), and there is \( \bar{y} \in \text{nodes}(M) \) that satisfies \( \bar{y} \equiv_i \bar{x} \) and \( M(\bar{y}) = a \).

We write \( G_k(\text{At}\mathfrak{A}) \), or simply \( G_k \), for \( G^m_k(\text{At}\mathfrak{A}) \) if \( m \geq \omega \).

3. The \( \omega \)-rounded game \( G^m(\text{At}\mathfrak{A}) \) or simply \( G^m \) is like the game \( G^m_\omega(\text{At}\mathfrak{A}) \) except that \( \forall \) has the option to reuse the \( m \) nodes in play.

**Proposition 2.2.** Suppose that \( \mathfrak{A} \in \text{CA}_n \) is atomic having countably many atoms. Then \( \exists \) has a winning strategy in \( G_\omega(\text{At}\mathfrak{A}) \iff \exists \) has a winning strategy in \( G^m(\text{At}\mathfrak{A}) \iff \mathfrak{A} \in \text{CRCA}_n \). In particular, if \( \mathfrak{A} \) is finite, then \( \exists \) has a winning strategy in \( G_\omega(\text{At}\mathfrak{A}) \iff \mathfrak{A} \) is representable.

**Proof.** [10, Theorem 3.3.3], together with observing that the game \( G_\omega(\text{At}\mathfrak{A}) \) is equivalent to the game \( G^\omega(\text{At}\mathfrak{A}) \), in the sense that \( \exists \) has a winning strategy in \( G^\omega(\text{At}\mathfrak{A}) \iff \exists \) has a winning strategy in \( G_\omega(\text{At}\mathfrak{A}) \) whenever \( \mathfrak{A} \) is atomic with countably many atoms (the converse implication is trivial). The rest of the cases are analogous.

**Lemma 2.3.** Assume that \( 2 < n < \omega \). Let \( m \) be an ordinal > \( n \). If \( \mathfrak{A} \in \text{S}_n \text{N} \text{R}_n \text{CA} \) is atomic, then \( \exists \) has a winning strategy in \( G^m(\text{At}\mathfrak{A}) \).
Proof. First a piece of notation. Let $m$ be a finite ordinal $> 0$. An $s$ word is a finite string of substitutions $(s^i) (i, j < m)$, a $c$ word is a finite string of cylindrifications $(c_i), i < m$; an $sc$ word $w$, is a finite string of both, namely, of substitutions and cylindrifications. An $sc$ word induces a partial map $\hat{w} : m \to m$: $\hat{c} = Id, \hat{w^i_j} = \hat{w} \circ [i,j]$ and $\hat{w}c_i = \hat{w} \restriction (m \setminus \{i\})$. If $a \in \langle m \rangle_1$, we write $s_a$, or $s_{a_0...a_{k-1}}$, where $k = |a|$, for an arbitrary chosen $sc$ word $w$ such that $\hat{w} = \bar{a}$. Such a $w$ exists by [8, Definition 5.23 Lemma 13.29].

Fix $2 < n < m$. Assume that $\mathcal{C} \in \mathcal{CA}_m$, $\mathfrak{A} \subseteq \mathfrak{At}_{\mathfrak{C}} \mathcal{C}$ is an atomic $\mathcal{CA}_n$ and $N$ is an $\mathfrak{A}$-network with nodes $(\mathfrak{C}) \subseteq m$. Define $N^+ \in \mathfrak{C}$ by

$$N^+ = \prod_{i_0, \ldots, i_{n-1} \in \text{nodes}(N)} s_{i_0, \ldots, i_{n-1}} N(i_0, \ldots, i_{n-1}).$$

For a network $N$ and function $\theta$, the network $N\theta$ is the complete labelled graph with nodes $\theta^{-1}(\text{nodes}(N)) = \{x \in \text{dom}(\theta) : \theta(x) \in \text{nodes}(N)\}$, and labelling defined by

$$(N\theta)(i_0, \ldots, i_{n-1}) = N(\theta(i_0), \theta(i_1), \ldots, \theta(i_{n-1})), $$

for $i_0, \ldots, i_{n-1} \in \theta^{-1}(\text{nodes}(N))$. Then the following hold:

1. for all $x \in \mathfrak{C} \setminus \{0\}$ and all $i_0, \ldots, i_{n-1} < m$, there is $a \in \text{At}_\mathfrak{A}$, such that $s_{i_0, \ldots, i_{n-1}} a \cdot x \neq 0$,
2. for any $x \in \mathfrak{C} \setminus \{0\}$ and any finite set $I \subseteq m$, there is a network $N$ such that $\text{nodes}(N) = I$ and $x \cdot N^+ \neq 0$. Furthermore, for any networks $M, N$ if $M^+ \cdot N^+ \neq 0$, then $M |_{\text{nodes}(M) \cap \text{nodes}(N)} = N |_{\text{nodes}(M) \cap \text{nodes}(N)}$,
3. if $\theta$ is any partial, finite map $m \to m$ and if $\text{nodes}(N)$ is a proper subset of $m$, then $N^+ \neq 0 \to (N\theta)^+ \neq 0$. If $i \not\in \text{nodes}(N)$, then $c_i N^+ = N^+$.

Since $\mathfrak{A} \subseteq \mathfrak{At}_{\mathfrak{C}} \mathcal{C}$, then $\sum \mathfrak{At}_{\mathfrak{A}} = 1$. For (1), $s_j^i$ is a completely additive operator (any $i, j < m$), hence $s_{i_0, \ldots, i_{n-1}}$ is, too. So $\sum \mathfrak{C} \{s_{i_0, \ldots, i_{n-1}} a : a \in \text{At}_\mathfrak{A}\} = s_{i_0, \ldots, i_{n-1}} \mathfrak{At}_{\mathfrak{A}} = s_{i_0, \ldots, i_{n-1}} 1 = 1$ for any $i_0, \ldots, i_{n-1} < m$. Let $x \in \mathfrak{C} \setminus \{0\}$. Assume for contradiction that $s_{i_0, \ldots, i_{n-1}} a \cdot x = 0$ for all $a \in \text{At}_\mathfrak{A}$. Then $1 - x$ will be an upper bound for $\{s_{i_0, \ldots, i_{n-1}} a : a \in \text{At}_\mathfrak{A}\}$. But this is impossible because $\sum \mathfrak{C} \{s_{i_0, \ldots, i_{n-1}} a : a \in \text{At}_\mathfrak{A}\} = 1$.

To prove the first part of (2), we repeatedly use (1). We define the edge labelling of $N$ one edge at a time. Initially, no hyperedges are labelled. Suppose $E \subseteq \text{nodes}(N) \times \text{nodes}(N) \times \text{nodes}(N)$ is the set of labelled hyperedges of $N$ (initially $E = \emptyset$) and $x \cdot \prod_{e \in E} s_{e} N(\bar{e}) \neq 0$. Pick $\bar{d}$ such that $\bar{d} \notin E$. Then by (1) there is $a \in \text{At}_\mathfrak{A}$ such that $x \cdot \prod_{e \in E} s_{e} N(\bar{e}) \cdot s_{\bar{e}} a \neq 0$. Include the hyperedge $\bar{d}$ in $E$. We keep on doing this until eventually all hyperedges will be labelled, so we obtain a completely labelled graph $N$ with $N^+ \neq 0$. It is easily checked that $N$ is a network.

For the second part of (2), we proceed contrapositively. Assume that there is $\bar{c} \in \text{nodes}(M) \cap \text{nodes}(N)$ such that $M(\bar{c}) \neq N(\bar{c})$. Since edges are labelled by atoms, we have $M(\bar{c}) \cdot N(\bar{c}) = 0$, so $0 = s_{\bar{c}} 0 = s_{\bar{c}} M(\bar{c}) \cdot s_{\bar{c}} N(\bar{c}) \geq M^+ \cdot N^+$. A
Proof. Throughout the proof fix \( 2 < m \leq \omega \). \( \text{FCs} \cap \text{At} \subseteq \text{LCA}_n \). Furthermore, for any elementary class \( K \) between \( \text{ElCN}_{n}\text{CA}_\omega \cap \text{At} \) and \( \text{LCA}_n \), \( \text{RCA}_n \) is generated by \( \text{AtK} \).

**Proof.** Throughout the proof fix \( 2 < n < \omega \).

It suffices to show that \( \text{Nr}_n\text{CA}_\omega \cap \text{At} \subseteq \text{LCA}_n \), since the last class is elementary. This follows from Lemma 2.3, since if \( \mathfrak{A} \in \text{Nr}_n\text{CA}_\omega \) is atomic, then \( \exists \) has a winning strategy in \( G^\omega(\text{At}\mathfrak{A}) \), hence in \( G_\omega(\text{At}\mathfrak{A}) \), a fortiori, \( \exists \) has a winning strategy in \( G_k(\text{At}\mathfrak{A}) \) for all \( k < \omega \), so (by definition) \( \mathfrak{A} \in \text{LCA}_n \).

To show strictness of the last inclusion, let \( V = {}^n\mathbb{Q} \) and let \( \mathfrak{A} \in \text{Cs}_n \) have universe \( \wp(V) \). Then \( \mathfrak{A} \in \text{Nr}_n\text{CA}_\omega \). Let \( y = \{ s \in V : s_0 + 1 = \sum_{i>0}s_i \} \) and \( \mathcal{E} = \mathcal{G}^\mathfrak{A}(\{ y \} \cup X) \), where \( X = \{ \{ s \} : s \in V \} \). Now \( \mathcal{E} \) and \( \mathfrak{A} \) having same top element \( V \), share the same atom structure, namely, the singletons, so \( \text{EmAt}\mathcal{E} = \mathfrak{A} \). Furthermore, plainly \( \mathfrak{A}, \mathcal{E} \in \text{CRCA}_n \). So \( \mathcal{E} \in \text{CRCA}_n \subseteq \text{LCA}_n \), and as proved in [24], \( \mathcal{E} \notin \text{ElCN}_{n}\text{CA}_n \), hence \( \mathcal{E} \) witnesses the required strict inclusion.

Now we show that \( \text{AtElCN}_{n}\text{CA}_\omega \) generates \( \text{RCA}_n \). Let \( \text{FCs}_n \) denote the class of full \( \text{Cs}_n \)s, that is \( \text{Cs}_n \)s having universe \( \wp(\wp(U)) \) (\( U \) non-empty set). First we show that \( \text{FCs}_n \subseteq \text{EmAt}\text{Nr}_n\text{CA}_\omega \). Let \( \mathfrak{A} \in \text{FCs}_n \). Then \( \mathfrak{A} \in \text{Nr}_n\text{CA}_\omega \cap \text{At} \). Throughout all of this, we have that \( \mathfrak{A} \models \theta \) is a theory with the required properties. For \( i < m \), let \( \text{Id}_{-i} \) be the partial map \( \{ (k, k) : k \in m \setminus \{ i \} \} \).

For the first part of (3) (cf. [8, Lemma 13.29] using the notation in op.cit), since there is \( k \in m \setminus \text{nodes}(N) \), \( \theta \) can be expressed as a product \( \sigma_0 \sigma_1 \cdots \sigma_t \) of maps such that, for \( s \leq t \), there are either \( \sigma_s = \text{Id}_{-i} \) for some \( i < m \) or \( \sigma_s = [i/\cdot] \) for some \( i, j < m \) and where \( i \notin \text{nodes}(N\sigma_0 \cdots \sigma_{s-1}) \). But clearly \( (NI\text{Id}_{-i})^+ \geq N^+ \) and if \( i \notin \text{nodes}(N) \) and \( j \in \text{nodes}(N) \), then \( N^+ \neq 0 \rightarrow (N[i/j])^+ \neq 0 \). The required now follows. The last part is straightforward.

Using the above proven facts, we are now ready to show that \( \exists \) has a winning strategy in \( G^m \). She can always play a network \( N \) with \( \text{nodes}(N) \subseteq m \), such that \( N^+ \neq 0 \).

In the initial round, let \( \forall \) play \( a \in \text{At}\mathfrak{A} \). \( \exists \) plays a network \( N \) with \( N(0, \ldots, n-1) = a \). Then \( N^+ = a \neq 0 \). Recall that here \( \forall \) is offered only one (cylindrifier) move. At a later stage, suppose \( \forall \) plays the cylindrifier move, which we denote by \( (N, (f_0, \ldots, f_{n-2}), k, b, l) \). He picks a previously played network \( N, f_i \in \text{nodes}(N), l < n, k \notin \{ f_i : i < n-2 \} \), such that \( b \leq c_i N(f_0, \ldots, f_{i-1}, x, f_{i+1}, \ldots, f_{n-2}) \) and \( N^+ \neq 0 \). Let \( \bar{a} = (f_0 \ldots f_{i-1}, k, f_{i+1}, \ldots, f_{n-2}) \). Then by second part of (3) we have that \( c_i N^+ \cdot s_b \neq 0 \) and so by first part of (2), there is a network \( M \) such that \( M^+ \cdot c_i N^+ \cdot s_b \neq 0 \). Hence \( M(f_0, \ldots, f_{i-1}, k, f_{i+2}, \ldots, f_{n-2}) = b, \text{nodes}(M) = \text{nodes}(N) \cup \{ k \} \), and \( M^+ \neq 0 \), so this property is maintained. \( \Box \)
\( \mathsf{At} \), hence \( \mathsf{At} \mathfrak{A} \subseteq \mathsf{At} \mathsf{Nr}_n \mathsf{CA}_\omega \) and \( \mathfrak{A} = \mathsf{Cl} \mathsf{At} \mathfrak{A} \subseteq \mathsf{Cl} \mathsf{At} \mathsf{Nr}_n \mathsf{CA}_\omega \). The required now follows from the chain of inclusions: \( \mathsf{RCA}_n = \mathsf{SPFCs}_n \subseteq \mathsf{SPFCmAt}(\mathsf{Nr}_n \mathsf{CA}_\omega) \subseteq \mathsf{SPFCmAt}(\mathsf{El} \mathsf{Nr}_n \mathsf{CA}_\omega) \subseteq \mathsf{SPFCmAt} \mathsf{K} \subseteq \mathsf{SPFCmLCAS}_n \subseteq \mathsf{RCA}_n \), where \( \mathsf{K} \) is given above.

**Theorem 2.5.** Let \( \kappa \) be an infinite cardinal. Then there exists an atomless \( \mathsf{C} \in \mathsf{CA}_\omega \) such that for all \( 2 < n < \omega \), \( \mathsf{Nr}_n \mathsf{C} \) is atomic, with \( |\mathsf{At}(\mathsf{Nr}_n \mathsf{C})| = 2^\kappa \), \( \mathsf{Nr}_n \mathsf{C} \in \mathsf{LCA}_n \), but \( \mathsf{Nr}_n \mathsf{C} \) is not completely representable.

**Proof.** We use the following uncountable version of Ramsey’s theorem due to Erdos and Rado: If \( r \geq 2 \) is finite, \( k \) an infinite cardinal, then \( \exp_r(k)^+ \rightarrow (k^+)^r+1 \) where \( \exp_0(k) = k \) and inductively \( \exp_{r+1}(k) = 2^{\exp_r(k)} \). The above partition symbol describes the following statement. If \( f \) is a coloring of the \( r+1 \) element subsets of a set of cardinality \( \exp_r(k)^+ \) in \( k \) many colors, then there is a homogeneous set of cardinality \( k^+ \) (a set, all whose \( r+1 \) element subsets get the same \( f \)-value). Let \( \kappa \) be the given cardinal. We use a simplified more basic version of a rainbow construction where only the two predominant colours, namely, the reds and blues are available. The algebra \( \mathsf{C} \) will be constructed from a relation algebra possessing an \( \omega \)-dimensional cylindric basis. To define the relation algebra we specify its atoms and the forbidden triples of atoms. The atoms are \( \mathsf{ld}_i, g^i_0 : i < 2^\kappa \) and \( r_j : 1 \leq j < \kappa \), all symmetric. The forbidden triples of atoms are all permutations of \( (\mathsf{ld}_i, g_0^i, g_0^{i'}) \) for \( 1 \leq j < \kappa \) and \( (g_0^i, g_0^{i'}, g_0^{i''}) \) for \( i, i', i'' < 2^\kappa \). Write \( g_0^i \) for \( \{ g_0^i : i < 2^\kappa \} \) and \( r_+ \) for \( \{ r_j : 1 \leq j < \kappa \} \). Call this atom structure \( \alpha \). Consider the term algebra \( \mathfrak{A} \) defined to be the subalgebra of the complex algebra of this atom structure generated by the atoms. We claim that \( \mathfrak{A} \), as a relation algebra, has no complete representation, hence any algebra sharing this atom structure is not completely representable, too. Indeed, it is easy to show that if \( \mathfrak{A} \) and \( \mathfrak{B} \) are atomic relation algebras sharing the same atom structure, so that \( \mathsf{At} \mathfrak{A} = \mathsf{At} \mathfrak{B} \), then \( \mathfrak{A} \) is completely representable \( \iff \mathfrak{B} \) is completely representable. Assume for contradiction that \( \mathfrak{A} \) has a complete representation \( \mathcal{M} \). Let \( x, y \) be points in the representation with \( \mathcal{M} \models r_1(x, y) \). For each \( i < 2^\kappa \), there is a point \( z_i \in \mathcal{M} \) such that \( \mathcal{M} \models g_0^i(x, z_i) \land r_1(z_i, y) \). Let \( Z = \{ z_i : i < 2^\kappa \} \). Within \( Z \), each edge is labelled by one of the \( \kappa \) atoms in \( r_+ \). The Erdos-Rado theorem forces the existence of three points \( z^1, z^2, z^3 \) in \( Z \) such that 
\[ \mathcal{M} \models r_j(z^1, z^2) \land r_j(z^2, z^3) \land r_j(z^3, z_1), \]
for some single \( j < \kappa \). This contradicts the definition of composition in \( \mathfrak{A} \) (since we avoided monochromatic triangles).

Let \( S \) be the set of all atomic \( \mathfrak{A} \)-networks \( N \) with nodes \( \omega \) such that \( \{ r_i : 1 \leq i < \kappa : r_i \) is the label of an edge in \( N \} \) is finite. Then it is straightforward to show \( S \) is an amalgamation class, that is for all \( M, N \in S \) if \( M \equiv_{ij} N \) then there is \( L \in S \) with \( M \equiv_i L \equiv_j N \), witness [8, Definition 12.8] for notation. Now let \( X \) be the set of finite \( \mathfrak{A} \)-networks \( N \) with nodes \( \subseteq \kappa \) such that:

1. each edge of \( N \) is either (a) an atom of \( \mathfrak{A} \) or (b) a cofinite subset of \( r_+ = \{ r_j : 1 \leq j < \kappa \} \) or (c) a cofinite subset of \( g_0 = \{ g_0^i : i < 2^\kappa \} \) and
2. $N$ is ‘triangle-closed’, i.e. for all $l, m, n \in \text{nodes}(N)$ we have $N(l, n) \leq N(l, m); N(m, n)$. That means if an edge $(l, m)$ is labelled by $\text{Id}$ then $N(l, n) = N(m, n)$ and if $N(l, m), N(m, n) \leq g_0$ then $N(l, n) \cdot g_0 = 0$ and if $N(l, m) = N(m, n) = r_j$ (some $1 \leq j < \omega$) then $N(l, n) \cdot r_j = 0$.

For $N \in X$ let $\widehat{N} \in \mathcal{Ca}(S)$ be defined by

$$\{L \in S : L(m, n) \leq N(m, n) \text{ for } m, n \in \text{nodes}(N)\}.$$  

For $i \in \omega$, let $N \upharpoonright_{-i}$ be the subgraph of $N$ obtained by deleting the node $i$. Then if $N \in X$, $i < \omega$ then $c_i N = N \upharpoonright_{-i}$. The inclusion $c_i N \subseteq (N \upharpoonright_{-i})$ is clear. Conversely, let $L \in (\widehat{N} \upharpoonright_{-i})$. We seek $M \equiv_i L$ with $M \in \widehat{N}$. This will prove that $L \in c_i N$, as required. Since $L \in S$ the set $T = \{r_i \notin L\}$ is infinite. Let $T$ be the disjoint union of two infinite sets $Y \cup Y'$, say. To define the $\omega$-network $M$ we must define the labels of all edges involving the node $i$ (other labels are given by $M \equiv_i L$). We define these labels by enumerating the edges and labeling them one at a time. So let $j \neq i < \kappa$. Suppose $j \in \text{nodes}(N)$. We must choose $M(i, j) \leq N(i, j)$. If $N(i, j)$ is an atom then of course $M(i, j) = N(i, j)$. Since $N$ is finite, this defines only finitely many labels of $M$. If $N(i, j)$ is a cofinite subset of $g_0$ then we let $M(i, j)$ be an arbitrary atom in $N(i, j)$. And if $N(i, j)$ is a cofinite subset of $r_+$ then let $M(i, j)$ be an element of $N(i, j) \cap Y$ which has not been used as the label of any edge of $M$ which has already been chosen (possible, since at each stage only finitely many have been chosen so far). If $j \notin \text{nodes}(N)$ then we can let $M(i, j) = r_k \in Y$ some $1 \leq k < \kappa$ such that no edge of $M$ has already been labelled by $r_k$. It is not hard to check that each triangle of $M$ is consistent (we have avoided all monochromatic triangles) and clearly $M \equiv_i \widehat{N}$. The labeling avoided all but finitely many elements of $Y'$, so $M \in S$. So $(N \upharpoonright_{-i}) \subseteq c_i N$.

Now let $\widehat{X} = \{\widehat{N} : N \in X\} \subseteq \mathcal{Ca}(S)$. Then we claim that the subalgebra of $\mathcal{Ca}(S)$ generated by $\widehat{X}$ is simply obtained from $\widehat{X}$ by closing under finite unions. Clearly all these finite unions are generated by $\widehat{X}$. We must show that the set of finite unions of $\widehat{X}$ is closed under all cylindric operations. Closure under unions is given. For $\widehat{N} \in X$ we have $\widehat{N} = \bigcup_{m, n \in \text{nodes}(N)} N_{mn}$ where $N_{mn}$ is a network with nodes $\{m, n\}$ and labeling $N_{mn}(m, n) = -N(m, n)$. $N_{mn}$ may not belong to $X$ but it is equivalent to a union of at most finitely many members of $\widehat{X}$. The diagonal $d_{ij} \in \mathcal{Ca}(S)$ is equal to $\widehat{N}$ where $N$ is a network with nodes $\{i, j\}$ and labeling $N(i, j) = \text{Id}$. Closure under cylindrification is given. Let $\mathcal{C}$ be the subalgebra of $\mathcal{Ca}(S)$ generated by $\widehat{X}$. Then $\mathfrak{A} = \mathfrak{Ra}(\mathcal{C})$. To see why, each element of $\mathfrak{A}$ is a union of a finite number of atoms, possibly a co–finite subset of $g_0$ and possibly a co–finite subset of $r_+$. Clearly $\mathfrak{A} \subseteq \mathfrak{Ra}(\mathcal{C})$.

Conversely, each element $z \in \mathfrak{Ra}(\mathcal{C})$ is a finite union $\bigcup_{N \in F} \widehat{N}$, for some finite subset $F$ of $X$, satisfying $c_i z = z$, for $i > 1$. Let $i_0, \ldots, i_k$ be an enumeration
of all the nodes, other than 0 and 1, that occur as nodes of networks in \( F \). Then, \( c_{i_0} \ldots c_{i_k} z = \bigcup_{N \in F} c_{i_0} \ldots c_{i_k} \hat{N} = \bigcup_{N \in F} (\hat{N}|_{\{0,1\}}) \in \mathfrak{A} \). So \( \text{Ra}(\mathfrak{C}) \subseteq \mathfrak{A} \).

\( \mathfrak{A} \) is relation algebra reduct of \( \mathfrak{C} \in \text{CA}_\omega \) but has no complete representation. Let \( n > 2 \). Let \( \mathfrak{B} = N_r n \mathfrak{C} \). Then \( \mathfrak{B} \in N_r n \text{CA}_\omega \), is atomic, but has no complete representation for plainly a complete representation of \( \mathfrak{B} \) induces one of \( \mathfrak{A} \). In fact, because \( \mathfrak{B} \) is generated by its two dimensional elements, and its dimension is at least three, its \( \text{Df} \) reduct is not completely representable.

It remains to show that the \( \omega \)-dilation \( \mathfrak{C} \) is atomless. For any \( N \in X \), we can add an extra node extending \( N \) to \( M \) such that \( 0 \subseteq M' \subseteq N' \), so that \( N' \) cannot be an atom in \( \mathfrak{C} \).

\[ \square \]

### 2.1 Complete and other notions of representability

In the previous construction used in Proposition 2.5, \( \mathfrak{A} \) also satisfies the Lyndon conditions by [5, Theorem 33] but is not completely representable. Thus:

**Corollary 2.6.** The class CRRA is not elementary.

But we can go further:

**Theorem 2.7.** The class CRRA is not closed under \( \equiv_{\infty, \omega} \).

**Proof.** Take \( \mathfrak{A} \) to be a symmetric, atomic relation algebra with atoms

\[ \text{Id}, r(i), y(i), b(i) : i < \omega. \]

Non-identity atoms have colours, \( r \) is red, \( b \) is blue, and \( y \) is yellow. All atoms are self-converse. Composition of atoms is defined by listing the forbidden triples. The forbidden triples are (Peircean transforms) or permutations of \((\text{Id}, x, y)\) for \( x \neq y \), and

\[ (r(i), r(i), r(j)), (y(i), y(i), y(j)), (b(i), b(i), b(j)) \quad i \leq j < \omega \]

\( \mathfrak{A} \) is the complex algebra over this atom structure. Let \( \alpha \) be an ordinal. \( \mathfrak{A}^\alpha \) is obtained from \( \mathfrak{A} \) by splitting the atom \( r(0) \) into \( \alpha \) parts \( r^k(0) : k < \alpha \) and then taking the full complex algebra. In more detail, we put red atoms \( r^k(0) \) for \( k < \alpha \). In the altered algebra the forbidden triples are \((y(i), y(i), y(j)), (b(i), b(i), b(j))\), \( i \leq j < \omega \), \((r(i), r(i), r(j)), \quad 0 < i \leq j < \omega \), \((r^k(0), r^l(0), r^m(0))) \quad k, l, m < \alpha \). Now let \( \mathfrak{B} = \mathfrak{A}^\omega \) and \( \mathfrak{A} = \mathfrak{A}^n \) with \( n \geq 2^{\aleph_0} \).

For an ordinal \( \alpha \), \( \mathfrak{A}^\alpha \) is as defined in the previous remark. In \( \mathfrak{A}^\alpha \), we use the following abbreviations: \( r(0) = \sum_{k < \alpha} r^k(0) \) \( r = \sum_{i < \omega} r(i) \) \( y = \sum_{i < \omega} y(i) \) \( b = \sum_{i < \omega} b(i) \). These suprema exist because they are taken in the complex algebras which are complete. The index of \( r(i), y(i) \) and \( b(i) \) is \( i \) and the index of \( r^k(0) \) is also \( 0 \). Now let \( \mathfrak{B} = \mathfrak{A}^\omega \) and \( \mathfrak{A} = \mathfrak{A}^n \) with \( n \geq 2^{\aleph_0} \). We claim that \( \mathfrak{B} \in \text{RaCA}_\omega \) and \( \mathfrak{A} \equiv \mathfrak{B} \). For the first required, we show that \( \mathfrak{B} \) has a
cylindric bases by exhibiting a winning strategy for $\exists$ in the the cylindric-basis game, which is a simpler version of the hyperbasis game [8, Definition 12.26]. At some stage of the game, let the play so far be $N_0, N_1, \ldots, N_{t-1}$ for some $t < \omega$. We say that an edge $(m, n)$ of an atomic network $N$ is a diversity edge if $N(m, n) \cdot Id = 0$. Each diversity edge of each atomic network in the play has an owner — either $\exists$ or $\forall$, which we will allocate as we define $\exists$’s strategy. If an edge $(m, n)$ belongs to player $p$ then so does the reverse edge $(n, m)$ and we will only specify one of them. Since our algebra is symmetric, so the label of the reverse edge is equal to the label of the edge, so again need to specify only one. For the next round $\exists$ must define $N_t$ in response to $\forall$’s move. If there is an already played network $N_i$ (some $i < t$) and a finitary map $\sigma : \omega \rightarrow \omega$ such that $N_i \sigma$ ‘answers’ his move, then she lets $N_i = N_i \sigma$. From now on we assume that there is no such $N_i$ and $\sigma$. We consider the three types of $\forall$ can make. If he plays an atom move by picking an atom $a$, $\exists$ plays an atomic network $N$ with $N(0, 1) = a$ and for all $x \in \omega \setminus \{1\}, N(0, x) = Id$.

If $\forall$ plays a triangle move by picking a previously played $N_x$ (some $x < t$), nodes $i, j, k$ with $k \notin \{i, j\}$ and atoms $a, b$ with $a; b \geq N_x(i, j)$, we know that $a, b \neq 1^t$, as we are assuming the $\exists$ cannot play an embedding move (if $a = Id$, consider $N_x$ and the map $[k/i]$). $\exists$ must play a network $N_i \equiv_k N_x$ such that $N_i(i, k) = a, N_i(k, j) = b$. These edges, $(i, k)$ and $(k, j)$, belong to $\forall$ in $N_t$. All diversity edges not involving $k$ have the same owner in $N_t$ as they did in $N_x$. And all edges $(l, k)$ for $k \notin \{i, j\}$ belong to $\exists$ in $N_x$. To label these edges $\exists$ chooses a colour $c$ different than the colours of $a, b$ (we have three colours so this is possible). Then, one at a time, she labels each edge $(l, k)$ by an atom with colour $c$ and a non-zero index which has not yet been used to label any edge of any network played in the game. She does this one edge at a time, each with a new index. There are infinitely many indices to choose, so this can be done.

Finally, $\forall$ can play an amalgamation move by picking $M, N \in \{N_s : s < t\}$, nodes $i, j$ such that $M \equiv_{ij} N$. If there is $N_s$ (some $s < t$) and a map $\sigma : \text{nodes}(N_s) \rightarrow \text{nodes}(M) \cup \text{nodes}(N)$ such that $M \equiv N_s \sigma \equiv_j N$ then $\exists$ lets $N_i = N_s \sigma$. Ownership of edges is inherited from $N_s$. If there is no such $N_s$ and $\sigma$ then there are two cases. If there are three nodes $x, y, z$ in the ‘amalgam’ such that $M(j, x)$ and $N(x, i)$ are both red and of the same index, $M(j, y), N(y, i)$ are both yellow and of the same index and $M(j, z), N(z, i)$ are both blue and of the same index, then the new edge $(i, j)$ belongs to $\forall$ in $N_t$. It will be labelled by either $r^0(0), b(0)$ or $y(0)$ and it is easy to show that at least one of these will be a consistent choice. Otherwise, if there is no such $x, y, z$ then the new edge $(i, j)$ belongs to $\exists$ in $N_t$. She chooses a colour $c$ such that there is no $x$ with $M(j, x)$ and $N(x, i)$ both having colour $c$ and the same index. And she chooses a non-zero index for $N_t(i, j)$ which is new to the game (as with triangle moves). If $k \neq k' \in M \cap N$ then $(j, k)$ has the same owner in $N_t$ as it does in $M$, $(k, i)$ has the same owner in $N_t$.
as it does in \( N \) and \((k, k')\) belongs to \( \exists \) in \( N_t \) if it belongs to \( \exists \) in either \( M \) or \( N \), otherwise it belongs to \( \forall \) in \( N_t \). Now the only way \( \exists \) could lose, is if \( \forall \) played an amalgamation move \((M, N, i, j)\) such that there are \( x, y, z \in M \cap N \) such that \( M(j, x) = r^k(0), \ N(x, i) = r^{k'}(0), \ M(j, y) = N(y, i) = b(0) \) and \( M(j, z) = N(z, i) = y(0) \). But according to \( \exists \)'s strategy, she never chooses atoms with index 0, so all these edges must have been chosen by \( \forall \). This contradiction proves the required.

Now, let \( H \) be an \( \omega \)-dimensional cylindric basis for \( \mathcal{B} \). Then \( \mathfrak{Ca}H \in \mathcal{CA}_\omega \). Consider the cylindric algebra \( C = Sg\mathfrak{Ca}H \mathcal{B} \), the subalgebra of \( \mathfrak{Ca}H \) generated by \( \mathcal{B} \). In principal, new two dimensional elements that were not originally in \( \mathcal{B} \), can be created in \( C \) using the spare dimensions in \( \mathfrak{Ca}(H) \). But next we exclude this possibility. We show that \( \mathcal{B} \) exhausts the 2–dimensional elements of \( \mathfrak{RaC} \), more concisely, we show that \( \mathcal{B} = \mathfrak{RaC} \). For this purpose we want to find out what are the elements of \( \mathfrak{Ca}H \) that are generated by \( \mathcal{B} \). Let \( M \) be a (not necessarily atomic) finite network over \( \mathcal{B} \) whose nodes are a finite subset of \( \omega \).

- Define (using the same notation in the proof of Theorem 2.5) \( \widehat{M} = \{ N \in H : N \leq M \} \in \mathfrak{Ca}H \). \( N \leq M \) means that for all \( i, j \in M \) we have \( N(i, j) \leq M(i, j) \).

- A **block** is an element of the form \( \widehat{M} \) for some finite network \( M \) such that

  1. \( M \) is triangle-closed, i.e. for all \( i, j, k \in M \) we have \( M(i, k) \leq M(i, j); M(j, k) \)

  2. If \( x \) is the label of an irreflexive edge of \( M \) then \( x = \text{id} \) or \( x \leq r \) or \( x \leq y \) or \( x \leq b \) (we say \( x \) is ‘monochromatic’), and \(|\{i : x \cdot (r(i) + y(i) + b(i)) \neq 0\}| \) is either 0, 1 or infinite (we say that the number of indices of \( x \) is either 0, 1 or infinite).

We prove:

  1. For any block \( \widehat{M} \) and \( i < \omega \) we have

\[
c_i\widehat{M} = (M|_{\text{dom}(M) \setminus \{i\}})^\sim
\]

  2. The domain of \( C \) consists of finite sums of blocks.

\( c_i\widehat{M} \subseteq (M|_{\text{dom}(M) \setminus \{i\}})^\sim \) is obvious. If \( i \notin M \) the equality is trivial. Let \( N \in (M|_{\text{dom}(M) \setminus \{i\}})^\sim \), i.e. \( N \leq M|_{\text{dom}(M) \setminus \{i\}} \). We must show that \( N \in c_i\widehat{M} \) and for this we must find \( L \equiv_i N \) with \( L \in \widehat{M} \). \( L \equiv_i N \) determines every edge of \( L \) except those involving \( i \). For each \( j \in M \), if the number of indices in \( M(i, j) \) is just one, say \( M(i, j) = r(k) \), then let \( L(i, j) \) be an arbitrary atom below \( r(k) \). There should be no inconsistencies in the labelling so far defined.
for \( L \), by triangle-closure for \( M \). For all the other edges \((i, j)\) if \( j \in M \) there are infinitely many indices in \( M(i, j) \) and if \( j \notin M \) then we have an unrestricted choice of atoms for the label. These edges are labelled one at a time and each label is given an atom with a new index, thus avoiding any inconsistencies. This defines \( L \equiv_i N \) with \( L \in \tilde{M} \). For the second part, we already have seen that the set of finite sums of blocks is closed under cylindrification. We’ll show that this set is closed under all the cylindric operations and includes \( \mathcal{B} \). For any \( x \in \mathcal{B} \) and \( i, j < \omega \), let \( N^i_j \) be the \( \mathcal{B} \)-network with two nodes \( \{i, j\} \) and labelling \( N^i_j(i, i) = N^i_j(j, j) = \mathbf{1}d \), and \( N^i_j(i, j) = x \), \( N^i_j(j, i) = \bar{x} \).

Clearly \( N^i_j \) is triangle closed. And \( \hat{N}^{01}_{x,i,j} = x \). For any \( x \in \mathcal{B} \), we have \( x = x \cdot \mathbf{1}d + x \cdot r + x \cdot y + x \cdot b \), so \( x = \hat{N}^{01}_{x, \mathbf{1}d} + \hat{N}^{01}_{x, r} + \hat{N}^{01}_{x, y} + \hat{N}^{01}_{x, b} \) and the labels of these four networks are monochromatic. The first network defines a block and for each of the last three, if the number if indices is infinite then it is a block. If the number of indices is finite then it is a finite union of blocks. So every element of \( \mathcal{B} \) is a finite union of blocks.

For the diagonal elements, \( d_{ij} = \hat{N}^{ij}_{\mathbf{1}d} \). Closure under sums is obvious. For negation, take a block \( \tilde{M} \). Then \( \tilde{M} = \sum_{i,j \in M} \hat{N}^{ij}_{-N(i,j)} \). As before we can replace \( \hat{N}^{ij}_{-N(i,j)} \) by a finite union of blocks. Thus the set of finite sums of blocks includes \( \mathcal{B} \) and the diagonals and is closed under all the cylindric operations. Since every block is clearly generated from \( \mathcal{B} \) using substitutions and intersection only. It remains to show that \( \mathcal{B} = \text{RaC} \). Take a block \( \tilde{M} \in \text{RaC} \). Then \( \alpha \tilde{M} = \tilde{M} \) for \( 2 \leq i < \omega \). By the first part of the lemma, \( \tilde{M} = \tilde{M} \upharpoonright \{0, 1\} \in \mathcal{B} \).

We finally show that \( \exists \) has a winning strategy in an Ehrenfeucht–Fraïssé-game over \((\mathcal{A}, \mathcal{B})\) concluding that \( \mathcal{A} \equiv_{\omega} \mathcal{B} \). At any stage of the game, if \( \forall \) places a pebble on one of \( \mathcal{A} \) or \( \mathcal{B} \), \( \exists \) must place a matching pebble, on the other algebra. Let \( \tilde{a} = \langle a_0, a_1, \ldots, a_{n-1} \rangle \) be the position of the pebbles played so far (by either player) on \( \mathcal{A} \) and let \( \tilde{b} = \langle b_0, \ldots, b_{n-1} \rangle \) be the the position of the pebbles played on \( \mathcal{B} \). \( \exists \) maintains the following properties throughout the game.

- For any atom \( x \) (of either algebra) with \( x \cdot r(0) = 0 \) then \( x \in a_i \iff x \in b_i \).

- \( \tilde{a} \) induces a finite partition of \( r(0) \) in \( \mathcal{A} \) of \( 2^n \) (possibly empty) parts \( p_i : i < 2^n \) and \( \tilde{b} \) induces a partition of \( r(0) \) in \( \mathcal{B} \) of parts \( q_i : i < 2^n \). \( p_i \) is finite iff \( q_i \) is finite and, in this case, \( |p_i| = |q_i| \).

Now we show that \( \text{CrRa} \) is not closed under \( \equiv_{\omega, \omega} \). Since \( \mathcal{B} \in \text{RaCA}_\omega \) has countably many atoms, then \( \mathcal{B} \) is completely representable [5, Theorem 29]. For this purpose, we show that \( \mathcal{A} \) is not completely representable. We work with the term algebra, \( \text{ImAtA} \), since the latter is completely representable.
the complex algebra is. Let \( r = \{ r(i) : 1 \leq i < \omega \} \cup \{ r^k(0) : k < 2^{\aleph_0} \} \), \( y = \{ y(i) : i \in \omega \} \), \( b^+ = \{ b(i) : i \in \omega \} \). It is not hard to check every element of \( \text{ImAt}\mathcal{A} \subseteq \varphi(\text{At}\mathcal{A}) \) has the form \( F \cup R_0 \cup B_0 \cup Y_0 \), where \( F \) is a finite set of atoms, \( R_0 \) is either empty or a co-finite subset of \( r \), \( B_0 \) is either empty or a co-finite subset of \( b \), and \( Y_0 \) is either empty or a co-finite subset of \( y \). Using an argument similar to that used in the proof of Theorem 2.5, we show that the existence of a complete representation necessarily forces a monochromatic triangle, that we avoided at the start when defining \( \mathcal{A} \). Let \( x, y \) be points in the representation with \( M \models y(0)(x, y) \). For each \( i < 2^{\aleph_0} \), there is a point \( z_i \in M \) such that \( M \models \text{red}(x, z_i) \land y(0)(z_i, y) \) (some red \( \text{red} \in r \)). Let \( Z = \{ z_i : i < 2^{\aleph_0} \} \). Within \( Z \) each edge is labelled by one of the \( \omega \) atoms in \( y^+ \) or \( b^+ \). The Erdos–Rado theorem forces the existence of three points \( z^1, z^2, z^3 \in Z \) such that \( M \models y(j)(z^1, z^2) \land y(j)(z^2, z^3) \land y(j)(z^3, z_1) \), for some single \( j < \omega \) or three points \( z^1, z^2, z^3 \in Z \) such that \( M \models b(l)(z^1, z^2) \land b(l)(z^2, z^3) \land b(l)(z^3, z_1) \), for some single \( l < \omega \). This contradicts the definition of composition in \( \mathcal{A} \) (since we avoided monochromatic triangles). We have proved that CRRA is not closed under \( \equiv_{\infty, \omega} \), since \( \mathcal{A} \equiv_{\infty, \omega} \mathcal{B}, \mathcal{A} \) is not completely representable, but \( \mathcal{B} \) is completely representable.

\( \square \)

**Theorem 2.8.** Any class \( K \), such that \( \text{RaCA}_\omega \subseteq K \subseteq \text{RaCA}_5 \) is not elementary.

**Proof.** Using the notation in the proof of the last theorem, where we proved that \( \mathcal{B} \in \text{RaCA}_\omega \) and \( \mathcal{A} \equiv \mathcal{B} \); it therefore suffices to show that that \( \mathcal{A} \not\in \text{RaCA}_5 \).

Let \( \kappa = 2^{\aleph_0} \). We use \( \sum \) to denote suprema which exists in \( \mathcal{A} \). Notation, cf. [7] 13.30. For \( \bar{a} \in \rho^n \) we write \( s_\bar{a} \) for an arbitrary string of substitutions \( w \) such that \( \bar{w} = \bar{a} \). In more detail. Let \( n \geq 3 \) and \( i, j < n \). We define a string of substitutions \( s_{ij} \):

\[
\begin{align*}
s_{ij} &= s_0^1s_1^j, \text{ if } j \neq 0 \\
\text{s}_{ij} &= s_0^1s_1^0 \text{ iff } j = 0, i \neq 1 \\
\text{s}_{ij} &= s_0^2s_1^0 \text{ iff } j = 0, i = 1
\end{align*}
\]

We use that if \( \mathcal{C} \in \text{CA}_n \), \( i, j, k < n, \ k \neq i, j \). Then \( s_{ij}(r; s) = c_k(s_{ik}r, s_{kj}s) \), for all \( r, s \in \text{RaC} \). Suppose for contradiction that \( \mathcal{A} = \text{RaC} \) for some \( \mathcal{C} \in \text{CA}_5 \).

Then \( c_0^0(0) = r^0(0) \leq r^k(0); y(0) = c_2(s_1^0r^k(0) \cdot s_2^0 y(0)) \), for each \( k < \kappa \). Therefore

\[
x_k = r^0(0) \cdot s_2^1r^k(0) \cdot s_0^0 y(0) \neq 0.
\]

Then by a tedious computation one shows that \( x_0 \leq \sum_{k<\omega} c_3(s_3^0 s_2 s_1^0 b(i)) \).

This holds for all \( 0 < k < \omega_1 \). So,

\[
x_0 \leq \prod_{0<k<\kappa} \sum_{i<\omega} c_3(s_3^0 s_2 s_1^0 b(i)) \prod_{k<\omega_1} c_4(s_3^0 s_2 s_1^0 b(g(k)))
\]
So there is a function $g: \kappa \setminus \{0\} \to \omega$ such that

$$x_0 \cdot \prod_{0 < k < \omega_1} c_3(s_3^2x_k \cdot s_2^0s_3^1b(g(k))) \neq 0$$

Pick $i < \omega$ such that $X = g^{-1}(i)$ is uncountable, we need an uncountable number of superscripts $k$ at this point only. Then

$$\xi = x_0 \cdot \prod_{k \in X} c_3(s_3^2x_k \cdot s_2^0s_3^1b(i)) \neq 0$$

Let

$$z_k = s_3^2x_k \cdot s_2^0s_3^1b(i) \cdot \xi$$

for each $k \in X$. Let $S_0 = \{z_k : k \in X\}$. $S_0$ has the following properties. There is an index $i < \omega$ such that for all $z, x \in S_0$,

1. $c_4z = z$
2. $c_3z = c_3x$
3. (a) $\exists k < \omega_1$ such that $z \leq s_3^1r^k(0)$
   (b) $\forall k < \omega_1$ if $z, x \leq s_3^1r^k(0)$ then $z = x$
4. $z \leq s_3^2r^0(0) \cdot s_2^0y(0) \cdot s_3^0y(0) \cdot s_3^0s_3^1b(i)$
5. $S_0$ is infinite.

Suppose there is an infinite set $S$ and an index $i < \omega$ with the properties listed above. We show how to construct another infinite set $S'$ and a new index $i' < i$ with the same properties. Iterating this construction $i + 1$ times will then lead to a contradiction since the index cannot be less than 0. Fix $z \in S$. For each $x \in S \setminus \{z\}$ and $j < i$, let

$$\tau_j^x = c_4(s_4^3x \cdot s_2^0s_4^1b(j)).$$

Then by another tedious computation we have: (***) for any $x \in S \setminus \{z\}$,

$$z \leq \sum_{j < i} \tau_j^x$$

We now construct $S'$ and $i'$ from $S$ and $i$ with the required properties. By (***)

$$z \leq \prod_{x \in S \setminus z} \sum_{j < i} \tau_j^x = \sum_{g : S \setminus \{z\} \to i} \prod_{x \in S \setminus \{z\}} \tau_g^x(\cdot)$$
Since \( z \neq 0 \) there is \( g : S \setminus \{z\} \rightarrow i \) (= \{0, 1, \ldots, i - 1\}) such that \( z \cdot \prod_{x \in S \setminus \{z\}} \tau^g(x) \neq 0 \). Pick \( i' < i \) such that \( X = g^{-1}(i') \) is infinite. Then

\[
z \cdot \prod_{x \in X} \tau^g_x \neq 0
\]

Define

\[
\xi = z \cdot \prod_{x \in X} \tau^g_x \neq 0
\]

\[
x' = s_3^4 x \cdot s_3^1 s_4 b(i') \cdot \xi
\]

\[
x'' = s_3^4 s_2 c_2 x'
\]

\[
S' = \{x'' : x \in X\}
\]

We check each of the properties. Property 1 is obvious. By property 3 for \( S \), if \( x \in X \) then there is \( k < \omega \) and \( x \leq s_4^1 s_2^k (0) \). So

\[
x'' \leq s_3^4 s_2^3 c_2 s_3^1 s_4^1 r^k (0)
\]

\[
= s_3^4 s_2^3 c_2 s_3^1 s_4^1 r^k (0)
\]

\[
= s_3^4 s_2^3 s_3^1 c_2 r^k (0)
\]

\[
= s_3^4 s_2^3 s_3^1 r^{k'} (0)
\]

\[
= s_3^1 s_4^1 r^k (0)
\]

\[
= s_3^1 r^k (0)
\]

This gives property 3 for \( S' \) and shows that \( S' \) is infinite (property 5). For property 2 we first prove that if \( x \in S \) then \( c_4 x' = \xi \). First note that \( c_4 \xi = \xi \), so

\[
c_4 x' = c_4 (s_3^4 x \cdot s_3^1 s_4 b(i') \cdot \xi)
\]

\[
= c_4 (s_3^4 x \cdot s_3^1 s_4 b(i')) \cdot \xi
\]

\[
= \tau^g_x \cdot \xi
\]

\[
= \xi
\]

Hence,

\[
c_3 x'' = c_3 s_3^1 s_3^1 s_4^1 c_2 x'
\]

\[
= c_4 s_3^1 s_3^1 s_4^1 c_2 x'
\]

\[
= c_4 s_3^1 s_2^1 c_2 x'
\]

\[
= s_3 s_2^2 c_2 x'
\]

\[
= s_3^1 c_2 \xi
\]
which gives property 2. Finally, for property 4, we must prove that \( x'' \leq s_3^3s_2^3c_2s_{41}y(0) \). For an \( x'' \leq s_3^3s_2^3c_2s_{41}y(0) \) gives property 2. Finally, for property 4, we must prove that \( x'' \leq s_3^3s_2^3c_2s_{41}y(0) \). Therefore

\[
\begin{align*}
x'' & \leq s_3^3s_2^3c_2s_{41}y(0) \\
& = s_3^4s_3^3c_2s_{41}y(0) \\
& = s_3^4s_3^3s_2^3c_2y(0) \\
& = s_3^4s_3^3s_2^3y(0) \\
& = s_0^0s_1^3y(0) \\
& = s_0^0y(0)
\end{align*}
\]

Similarly, we can show that \( x'' \leq s_3^3s_2^3c_2s_{41}y(0) \). And \( x' \leq s_3^3s_2^3c_2s_{41}y(0) \) gives \( x'' \leq s_3^3s_2^3c_2s_{41}b(i') \) gives \( x'' \leq s_3^3s_2^3c_2s_{41}b(i') \). This proves property 4 and we are done.

\[\square\]

3 Complete representations and non-elementary classes

In the construction used in Proposition 2.5, both \( \mathcal{R} \) and \( \mathcal{N}_{rl} \mathcal{C} \) satisfy the Lyndon conditions but are not completely representable. Thus:

**Corollary 3.1.** [7] Let \( 2 < n < \omega \). Then the classes CRRA and \( \text{CRCA}_n \) are not elementary.

We next strengthen the last theorem. We first define a game \( \mathcal{H} \) that involves certain hypernetworks. A \( \lambda \)-neat hypernetwork is roughly a network endowed with hyperedges of length \( \neq n \) allowed to get arbitrarily long but are of finite length, and such hyperedges get their labels from a non-empty set of labels \( \Lambda \); such that all so-called short hyperedges are constantly labelled by \( \lambda \in \Lambda \). The board of the game consists of \( \lambda \)-neat hypernetworks:

**Definition 3.2.** For an \( n \)-dimensional atomic network \( N \) on an atomic \( \text{CA}_n \) and for \( x, y \in \text{nodes}(N) \), set \( x \sim y \) if there exists \( \bar{z} \) such that \( N(x, y, \bar{z}) \leq d_{01} \). Define the equivalence relation \( \sim \) over the set of all finite sequences over \( \text{nodes}(N) \) by \( \bar{x} \sim \bar{y} \) iff \( |\bar{x}| = |\bar{y}| \) and \( x_i \sim y_i \) for all \( i < |\bar{x}| \). (It can be easily checked that this indeed an equivalence relation). A hypernetwork \( N = (N^a, N^\lambda) \) over an atomic \( \text{CA}_n \) consists of an \( n \)-dimensional network \( N^a \) together with a labelling function for hyperlabels \( N^\lambda : \text{nodes}(N) \to \Lambda \) (some arbitrary set of hyperlabels \( \Lambda \)) such that for \( \bar{x}, \bar{y} \in \text{nodes}(N) \) if \( \bar{x} \sim \bar{y} \) then \( N^\lambda(\bar{x}) = N^\lambda(\bar{y}) \). If \( |\bar{x}| = k \in \mathbb{N} \) and \( N^\lambda(\bar{x}) = \lambda \), then we say that \( \lambda \) is a \( k \)-ary hyperlabel. \( \bar{x} \) is referred to as a \( k \)-ary hyperedge, or simply a hyperedge. A hyperedge \( \bar{x} \in \text{nodes}(N) \) is short, if there are \( y_0, \ldots, y_{n-1} \) that are nodes in \( N \), such that \( N(x_i, y_0, \bar{z}) \leq d_{01} \) or \( \ldots N(x_i, y_{n-1}, \bar{z}) \leq d_{01} \) for all
Let $b \in M$ except that we allow a clause for infinitary disjunctions. In more detail, for $U$ by hyperlabels done in the obvious way using the fact that the $M$ such that $M \models \text{nodes}(M) \cap \text{nodes}(N) = N \models \text{nodes}(M) \cap \text{nodes}(N)$, and $\text{nodes}(M) \cap \text{nodes}(N) \neq \emptyset$. This move is denoted $(M,N)$. To make a legal response, $\exists$ must play a $\lambda_0$–neat hypernetwork $L$ extending $M$ and $N$, where $\text{nodes}(L) = \text{nodes}(M) \cup \text{nodes}(N)$.

**Theorem 3.3.** Let $\alpha$ be a countable atom structure. If $\exists$ has a winning strategy in $H_\omega(\alpha)$, then there exists a complete $\mathcal{D} \in \text{RCA}_\omega$ such that $\exists \mathcal{M}_\alpha \equiv \exists \mathfrak{N}_n \mathcal{D}$ and $\alpha \equiv \exists \mathfrak{N}_n \mathcal{D}$. In particular, $\exists \mathcal{M}_\alpha \in \mathfrak{N}_n \mathcal{CA}_\omega$ and $\alpha \in \exists \mathfrak{N}_n \mathcal{CA}_\omega$.

**Proof.** Fix some $a \in \alpha$. The game $H_\omega$ is designed so that using $\exists$'s winning strategy in the game $H_\omega(\alpha)$ one can define a nested sequence $M_0 \subseteq M_1, \ldots$ of $\lambda$–neat hypernetworks where $M_0$ is $\exists$'s response to the initial $\forall$-move $a$, such that: If $M_r$ is in the sequence and $M_r(\bar{x}) \leq c_i a$ for an atom $a$ and some $i < n$, then there is $s \geq r$ and $d \in \text{nodes}(M_s)$ such that $M_s(\bar{y}) = a$, $\bar{y}_i = d$ and $\bar{y} \equiv_i \bar{x}$. In addition, if $M_r$ is in the sequence and $\theta$ is any partial isomorphism of $M_r$, then there is $s \geq r$ and a partial isomorphism $\theta^+$ of $M_s$ extending $\theta$ such that $\text{rng}(\theta^+) \supseteq \text{nodes}(M_s)$ (This can be done using $\exists$'s responses to amalgamation moves). Now let $\mathcal{M}_a$ be the limit of this sequence, that is $\mathcal{M}_a = \bigcup M_i$, the labelling of $n - 1$ tuples of nodes by atoms, and hyperedges by hyperlabels done in the obvious way using the fact that the $M_i$s are nested. Let $L$ be the signature with one $n$-ary relation for each $b \in \alpha$, and one $k$-ary predicate symbol for each $k$-ary hyperlabel $\lambda$. Now we work in $L_{\infty,\omega}$. For fixed $f_a \in n \text{nodes}(\mathcal{M}_a)$, let $\mathfrak{U}_a = \{ f \in n \text{nodes}(\mathcal{M}_a) : \{ i < \omega : g(i) \neq f_a(i) \} \text{ is finite} \}$. We make $\mathfrak{U}_a$ into the base of an $L$ relativized structure $\mathcal{M}_a$ like in [5, Theorem 29] except that we allow a clause for infinitary disjunctions. In more detail, for $b \in \alpha$, $l_0, \ldots , l_{n-1}, i_0, \ldots, i_{k-1} < \omega$, $k$-ary hyperlabels $\lambda$, and all $L$-formulas $\phi, \phi_i, \psi$, and $f \in \mathfrak{U}_a$:

\[
\begin{align*}
\mathcal{M}_a, f \models b(x_0, \ldots , x_{n-1}) & \iff \mathcal{M}_a(f(l_0), \ldots , f(l_{n-1})) = b, \\
\mathcal{M}_a, f \models \lambda(x_0, \ldots , x_{k-1}) & \iff \mathcal{M}_a(f(i_0), \ldots , f(i_{k-1})) = \lambda, \\
\mathcal{M}_a, f \models \neg \phi & \iff \mathcal{M}_a, f \not\models \phi, \\
\mathcal{M}_a, f \models \bigvee_{i \in I} \phi_i & \iff (\exists i \in I)(\mathcal{M}_a, f \models \phi_i), \\
\mathcal{M}_a, f \models \exists x_i \phi & \iff \mathcal{M}_a, f[i/m] \models \phi, \text{ some } m \in \text{nodes}(\mathcal{M}_a).
\end{align*}
\]

For any such $L$-formula $\phi$, write $\phi^{\mathcal{M}_a}$ for $\{ f \in \mathfrak{U}_a : \mathcal{M}_a, f \models \phi \}$. Let $D_a = \ldots$
\{ \phi^{M_a} : \phi \text{ is an } L\text{-formula} \} \text{ and } D_a \text{ be the weak set algebra with universe } D_a. \text{ Let } D = P_{a \in \alpha} D_a. \text{ Then } D \text{ is a generalized complete weak set algebra [4, Definition 3.1.2 (iv)]}. \text{ Now we show that } \alpha \cong \text{AtNr}_\alpha D \text{ and } \mathcal{C}_\alpha \cong \text{Nr}_\alpha D. \text{ Let } x \in D. \text{ Then } x = (x_a : a \in \alpha), \text{ where } x_a \in D_a. \text{ For } b \in \alpha \text{ let } \pi_b : D \rightarrow D_b \text{ be the projection map defined by } \pi_b(x_a : a \in \alpha) = x_b. \text{ Conversely, let } \iota_a : D_a \rightarrow D \text{ be the embedding defined by } \iota_a(y) = (x_b : b \in \alpha), \text{ where } x_a = y \text{ and } x_b = 0 \text{ for } b \neq a. \text{ Suppose } x \in \text{Nr}_\alpha D \setminus \{0\}. \text{ Since } x \neq 0, \text{ then it has a non-zero component } \pi_a(x) \in D_a, \text{ for some } a \in \alpha. \text{ Assume that } \emptyset \neq \phi(x_{i_0}, \ldots, x_{i_k-1})^{D_a} = \pi_a(x), \text{ for some } L\text{-formula } \phi(x_{i_0}, \ldots, x_{i_k-1}). \text{ We have } \phi(x_{i_0}, \ldots, x_{i_k-1})^{D_a} \in \text{Nr}_\alpha D_a. \text{ Pick } f \in \phi(x_{i_0}, \ldots, x_{i_k-1})^{D_a} \text{ and assume that } \mathcal{M}_a, f \models b(x_0, \ldots, x_n) \text{ for some } b \in \alpha. \text{ We show that } b(x_0, x_1, \ldots, x_{n-1})^{D_a} \subseteq \phi(x_{i_0}, \ldots, x_{i_k-1})^{D_a}. \text{ Take any } g \in b(x_0, x_1, \ldots, x_{n-1})^{D_a}, \text{ so that } \mathcal{M}_a, g \models b(x_0, \ldots, x_{n-1}). \text{ The map } \{(f(i), g(i)) : i < n\} \text{ is a partial isomorphism of } \mathcal{M}_a. \text{ Here that short hyperedges are constantly labelled by } \lambda \text{ is used. This map extends to a finite partial isomorphism } \theta \text{ of } \mathcal{M}_a \text{ whose domain includes } f(i_0), \ldots, f(i_{k-1}). \text{ Let } g' \in \mathcal{M}_a \text{ be defined by }

\begin{align*}
g'(i) = \begin{cases} 
\theta(i) & \text{if } i \in \text{dom}(\theta) \\
g(i) & \text{otherwise}
\end{cases}
\end{align*}

We have \mathcal{M}_a, g' \models \phi(x_{i_0}, \ldots, x_{i_k-1}). \text{ But } g'(0) = \theta(0) = g(0) \text{ and similarly } g'(n - 1) = g(n - 1), \text{ so } g \text{ is identical to } g' \text{ over } n \text{ and it differs from } g' \text{ on only a finite set. Since } \phi(x_{i_0}, \ldots, x_{i_k-1})^{D_a} \in \text{Nr}_\alpha D_a, \text{ we get that } \mathcal{M}_a, g \models \phi(x_{i_0}, \ldots, x_{i_k}), \text{ so } g \in \phi(x_{i_0}, \ldots, x_{i_k-1})^{D_a} \text{ (this can be proved by induction on quantifier depth of formulas). This proves that }

b(x_0, x_1 \ldots x_{n-1})^{D_a} \subseteq \phi(x_{i_0}, \ldots, x_{i_k})^{D_a} = \pi_a(x),

\text{ and so }

\iota_a(b(x_0, x_1 \ldots x_{n-1})^{D_a}) \leq \iota_a(\phi(x_{i_0}, \ldots, x_{i_k-1})^{D_a}) \leq x \in D_a \setminus \{0\}.

Now every non-zero element } x \text{ of } \text{Nr}_\alpha D_a \text{ is above a non-zero element of the following form } \iota_a(b(x_0, x_1, \ldots, x_{n-1})^{D_a}) \text{ (some } a, b \in \alpha) \text{ and these are the atoms of } \text{Nr}_\alpha D_a. \text{ The map } \phi(x_{i_0}, \ldots, x_{i_k-1})^{D_a} \subseteq \text{AtNr}_\alpha CA. \text{ Let } X \subseteq \text{Nr}_\alpha D. \text{ Then by completeness of } D, \text{ we get that } d = \sum X \text{ exists. Assume that } i \notin n, \text{ then } c_i d = c_i \sum X = \sum_{x \in X} c_i x = \sum X = d, \text{ because the } c_i s \text{ are completely additive and } c_i x = x, \text{ for all } i \notin n, \text{ since } x \in \text{Nr}_\alpha D. \text{ We conclude that } d \in \text{Nr}_\alpha D, \text{ hence } d \text{ is an upper bound of } X \text{ in } \text{Nr}_\alpha D. \text{ Since } d = \sum x \in X \text{ there can be no } b \in \text{Nr}_\alpha D \subseteq D \text{ with } b < d \text{ such that } b \text{ is an upper bound of } X \text{ for else it will be an upper bound of } X \text{ in } D. \text{ Thus } \sum_{x \in X} X = d \text{ We have shown that } \text{Nr}_\alpha D \text{ is complete. Making the legitimate identification } \text{Nr}_\alpha D \subseteq D \text{ } \mathcal{C}_\alpha \text{ by density, we get that } \text{Nr}_\alpha D = \mathcal{C}_\alpha \text{ (since } \text{Nr}_\alpha D \text{ is complete), hence } \mathcal{C}_\alpha \subseteq \text{Nr}_\alpha CA. \text{ } \square
If $B$ is a Boolean algebra and $b \in B$, then $\mathcal{M}_b B$ denotes the Boolean algebra with domain $\{ x \in B : x \leq b \}$, top element $b$, and other Boolean operations those of $B$ relativized to $b$.

**Lemma 3.4.** In the following $\mathfrak{A}$ and $\mathfrak{D}$ are Boolean algebras.

1. If $\mathfrak{A}$ is atomic and $0 \neq a \in \mathfrak{A}$, then $\mathcal{M}_a \mathfrak{A}$ is also atomic. If $\mathfrak{A} \subseteq_d \mathfrak{D}$, and $a \in A$, then $\mathcal{M}_a \mathfrak{A} \subseteq_d \mathcal{M}_a \mathfrak{D}$.

2. If $\mathfrak{A} \subseteq_d \mathfrak{D}$ then $\mathfrak{A} \subseteq \mathfrak{D}$. In particular, for any class $K$ of BAOs, $K \subseteq S_d K \subseteq S_{\omega} K$. If furthermore $\mathfrak{A}$ and $\mathfrak{D}$ are atomic, then $\text{At} \mathfrak{D} \subseteq \text{At} \mathfrak{A}$.

**Proof.** (1): Let $b \in \mathcal{M}_a \mathfrak{D}$ be non–zero. Then $b \leq a$ and $b$ is non-zero in $\mathfrak{D}$. By atomicity of $\mathfrak{D}$ there is an atom $c$ of $\mathfrak{D}$ such that $c \leq b$. So $c \leq b \leq a$, thus $c \in \mathcal{M}_a \mathfrak{D}$. Also $c$ is an atom in $\mathcal{M}_a \mathfrak{D}$ because if not, then it will not be an atom in $\mathfrak{D}$. The second part is similar.

(2): Assume that $\sum_{a}^{\mathfrak{A}} S = 1$ and for contradiction that there exists $b' \in \mathfrak{D}$, $b' < 1$ such that $s \leq b'$ for all $s \in S$. Let $b = 1 - b'$ then $b \neq 0$, hence by assumption (density) there exists a non-zero $a \in \mathfrak{A}$ such that $a \leq b$, i.e. $a \leq (1 - b')$. If $a \cdot s = 0$ for some $s \in S$, then $a$ is not less than $b'$ which is impossible. So $a \cdot s = 0$ for every $s \in S$, implying that $a = 0$, contradiction. Now we prove the second part. Assume that $\mathfrak{A} \subseteq_d \mathfrak{D}$ and $\mathfrak{D}$ is atomic. Let $b \in \mathfrak{D}$ be an atom. We show that $b \in \text{At} \mathfrak{A}$. By density there is a non–zero $a' \in \mathfrak{A}$, such that $a' \leq b$ in $\mathfrak{D}$. Since $\mathfrak{A}$ is atomic, there is an atom $a \in \mathfrak{A}$ such that $a \leq a' \leq b$. But $b$ is an atom of $\mathfrak{D}$, and $a$ is non–zero in $\mathfrak{D}$, too, so it must be the case that $a = b \in \text{At} \mathfrak{A}$. Thus $\text{At} \mathfrak{A} \subseteq \text{At} \mathfrak{A}$ and we are done. 

**Theorem 3.5.** Let $2 < n < \omega$. Then any class $K$ such that $\text{Nr}_n \text{CA}_\omega \cap \text{CRCA}_n \subseteq K \subseteq S_n \text{Nr}_n \text{CA}_{n+3}$, $K$ is not elementary.

**Proof.** The proof is divided into two parts:

1. **Any class between $S_n \text{Nr}_n \text{CA}_\omega \cap \text{CRCA}_n$ and $S_n \text{Nr}_n \text{CA}_{n+3}$ is not elementary:** We use the construction in [23, Theorem 5.12]. The algebra $\mathcal{C}_{Z,\omega}(\text{RCA}_n)$ based on $\mathbb{Z}$ (greens) and $\mathbb{N}$ (reds) denotes the rainbow-like algebra used in op.cit. It was shown in [23] that $\exists$ has a winning strategy in $G_k(\text{At} \mathcal{C}_{Z,\omega})$ for all $k \in \omega$. With some more effort it can be shown that $\exists$ has a winning strategy in $H_k(\text{At} \mathcal{C}_{Z,\omega})$ for all $k \in \omega$ dealing with the new amalgamation moves without hyperedges. It remains therefore to describe $\exists$’s strategy in dealing with labelling hyperedges in $\lambda$–neat hypernetworks, where $\lambda$ is a constant label kept on short hyperedges. In a play, $\exists$ is required to play $\lambda$–neat hypernetworks, so she has no choice about the the short edges, these are labelled by $\lambda$. In response to a cylindrifier move by $\forall$ extending the current hypernetwork providing a new node $k$, and a previously played coloured hypernetwork $M$ all long hyperedges not incident with $k$ necessarily keep the hyperlabel they had in $M$. All long hyperedges incident with $k$ in $M$ are given
unique hyperlabels not occurring as the hyperlabel of any other hyperedge in $M$. In response to an amalgamation move, which involves two hypernetworks required to be amalgamated, say $(M, N)$ all long hyperedges whose range is contained in $\text{nodes}(M)$ have hyperlabel determined by $M$, and those whose range is contained in $\text{nodes}(N)$ have hyperlabels determined by $N$. If $\bar{x}$ is a long hyperedge of $\exists s$'s response $L$ where $\text{rng}(\bar{x}) \not\subseteq \text{nodes}(M)$, $\text{nodes}(N)$ then $\bar{x}$ is given a new hyperlabel, not used in any previously played hypernetwork and not used within $L$ as the label of any hyperedge other than $\bar{x}$. This completes her strategy for labelling hyperedges. We first show that $\exists$ has a winning strategy in $G_k(\text{AtC}_{\mathbb{Z}, \mathbb{N}})$ where $0 < k < \omega$ is the number of rounds: Let $0 < k < \omega$. We proceed inductively. Let $M_0, M_1, \ldots, M_r$, $r < k$ be the coloured graphs at the start of a play of $G_k$ just before round $r + 1$. Assume inductively, that $\exists$ computes a partial function $\rho_s : \mathbb{Z} \to \mathbb{N}$, for $s \leq r$ : Let $0 < k < \omega$. We proceed inductively. Let $M_0, M_1, \ldots, M_r$, $r < k$ be the coloured graphs at the start of a play of $G_k$ just before round $r + 1$. Assume inductively, that $\exists$ computes a partial function $\rho_s : \mathbb{Z} \to \mathbb{N}$, for $s \leq r$ :

(i) $\rho_0 \subseteq \ldots \rho_t \subseteq \ldots \subseteq \rho_s$ is (strict) order preserving; if $i < j \in \text{dom}(\rho_s)$ then $\rho_s(i) - \rho_s(j) \geq 3^{k-r}$, where $k - r$ is the number of rounds remaining in the game, and

$$\text{dom}(\rho_s) = \{i \in \mathbb{Z} : \exists t \leq s, M_t \text{ contains an } i\text{-cone as a subgraph}\},$$

(ii) for $u, v, x_0 \in \text{nodes}(M_s)$, if $M_s(u, v) = r_{\mu, k}$, $\mu, k \in \mathbb{N}$, $M_s(x_0, u) = g_0^i$, $M_s(x_0, v) = g_0^j$, where $i, j \in \mathbb{Z}$ are tints of two cones, with base $F$ such that $x_0$ is the first element in $F$ under the induced linear order, then $\rho_s(i) = \mu$ and $\rho_s(j) = k$.

For the base of the induction $\exists$ takes $M_0 = \rho_0 = \emptyset$. Assume that $M_r$, $r < k$ (the number of rounds) is the current coloured graph and that $\exists$ has constructed $\rho_r : \mathbb{Z} \to \mathbb{N}$ to be a finite order preserving partial map such conditions (i) and (ii) hold. We show that (i) and (ii) can be maintained in a further round. We check the most difficult case. Assume that $\beta \in \text{nodes}(M_r)$, $\delta \not\in \text{nodes}(M_r)$ is chosen by $\forall v$ in his cylindrifier move, such that $\beta$ and $\delta$ are appreces of two cones having same base and green tints $p \neq q \in \mathbb{Z}$. Now $\exists$ adds $q$ to $\text{dom}(\rho_s)$ forming $\rho_{r+1}$ by defining the value $\rho_{r+1}(p) \in \mathbb{N}$ in such a way to preserve the (natural) order on $\text{dom}(\rho_r) \cup \{q\}$, that is maintaining property (i) Inductively, $\rho_r$ is order preserving and 'widely spaced' meaning that the gap between its elements is at least $3^{k-r}$, so this can be maintained in a further round. Now $\exists$ has to define a (complete) coloured graph $M_{r+1}$ such that $\text{nodes}(M_{r+1}) = \text{nodes}(M_r) \cup \{\delta\}$. In particular, she has to find a suitable red label for the edge $(\beta, \delta)$. Having $\rho_{r+1}$ at hand she proceeds as follows. Now that $p, q \in \text{dom}(\rho_{r+1})$, she lets $\mu = \rho_{r+1}(p)$, $b = \rho_{r+1}(q)$. The red label she chooses for the edge $(\beta, \delta)$ is: (*).
$M_{r+1}(\beta,\delta) = r_{u,b}$. This way she maintains property (ii) for $\rho_{r+1}$. Next we show that this is a winning strategy for $\exists$.

We check consistency of newly created triangles proving that $M_{r+1}$ is a coloured graph completing the induction. Since $\rho_{r+1}$ is chosen to preserve order, no new forbidden triple (involving two greens and one red) will be created. Now we check red triangles only of the form $(\beta,y,\delta)$ in $M_{r+1}$ ($y \in \text{nodes}(M_r)$). We can assume that $y$ is the apex of a cone with base $F$ in $M_r$ and green tint $t$, say, and that $\beta$ is the apex of the $p$–cone having the same base. Then inductively by condition (ii), taking $x_0$ to be the first element of $F$, and taking the nodes $\beta,y$, and the tints $p,t$, for $u,v,i,j$, respectively, we have by observing that $\beta,y \in \text{nodes}(M_r)$, $\beta \in \text{dom}(\rho_r)$ and $\rho_r \subseteq \rho_{r+1}$, the following: $M_{r+1}(\beta,y) = M_r(\beta,y) = r_{p_r(p),\rho_r(t)} = r_{\rho_{r+1}(p),\rho_{r+1}(t)}$. By her strategy, we have $M_{r+1}(y,\delta) = r_{\rho_{r+1}(t),\rho_{r+1}(q)}$ and we know by (*) that $M_{r+1}(\beta,\delta) = r_{\rho_{r+1}(p),\rho_{r+1}(q)}$. The triple $(r_{\rho_{r+1}(p),\rho_{r+1}(t)}, r_{\rho_{r+1}(t),\rho_{r+1}(q)}, r_{\rho_{r+1}(p),\rho_{r+1}(q)})$ of reds is consistent and we are done with this case. All other edge labelling and colouring $n - 1$ tuples in $M_{r+1}$ by yellow shades are exactly like in [7]. But we can go further.

**We show that $\exists$ has a winning strategy in the stronger game $H_k(\text{AtC})$ for all $k \in \omega$: $\exists$’s strategy dealing with $\lambda$–neat hypernetworks, where $\lambda$ is a constant label kept on short hyperedges is already dealt with. Now we change the board of play but only formally. We play on $\lambda$–neat hypergraphs.**

Given a rainbow algebra $\mathfrak{A}$, there is a one to one correspondence between coloured graphs on $\text{At\mathfrak{A}}$ and networks on $\text{At\mathfrak{A}}$ [10, Half of p. 76] denote this correspondence, expressed by a bijection from coloured graphs to networks by (*): $\Gamma \mapsto N_\Gamma$, $\text{nodes}(\Gamma) = \text{nodes}(N_\Gamma)$.

Now the game $H$ can be re-formulated to be played on $\lambda$–neat hypergraphs on a rainbow algebra $\mathfrak{A}$; these are of the form $(\Delta, N^h)$, where $\Delta$ is a coloured graph on $\text{At\mathfrak{A}}$, $\lambda$ is a hyperlabel, and $N^h$ is as before, $N^h : \omega^{\text{nodes}(\Delta)} \to \Lambda$, such that for $\bar{x}, \bar{y} \in \text{nodes}(\Delta)$, if $\bar{x} \sim \bar{y} \Rightarrow N^h(\bar{x}) = N^h(\bar{y})$. Here $\bar{x} \sim \bar{y}$, making the obvious translation, is the equivalence relation defined by: $x \sim y \iff |x| = |y|$ and $N_{\Delta}(x_i, y_i, \bar{z}) \leq d_{01}$ for all $i < |x|$ and some $\bar{z} \in n^{-2}\text{nodes}(\Delta)$. All notions earlier defined for hypernetworks, in particular, $\lambda$–neat ones, translate to $\lambda$–neat hypergraphs, using (*), like short hyperedges, long hypedges, $\lambda$–neat hypergraphs, etc. The game is played now on $\lambda$–neat hypergraphs on which the constant label $\lambda$ is kept on the short hyperedges in $\omega^{\text{nodes}(\Delta)}$. We have already dealt with the ‘graph part’ of the game. We turn to the remaining amalgamation moves. We need some notation and terminology. Every edge of any hypergraph (edge of its graph part) has an owner $\forall$ or $\exists$, namely, the one who coloured this edge. We call such edges $\forall$ edges or $\exists$ edges. Each long hypedge $\bar{x}$ in $N^h$ of a hypergraph $N$ occurring in the play has an envelope $v_N(\bar{x})$ to be defined shortly.

In the initial round, $\forall$ plays $a \in \alpha$ and $\exists$ plays $N_0$ then all edges of $N_0$ belongs
to $\forall$. There are no long hyperedges in $N_0$. If $\forall$ plays a cylindrifier move requiring a new node $k$ and $\exists$ responds with $M$ then the owner in $M$ of an edge not incident with $k$ is the same as it was in $N$ and the envelope in $M$ of a long hyperedge not incident with $k$ is the same as that it was in $N$. All edges $(l, k)$ for $l \in \text{nodes}(N) \sim \{k\}$ belong to $\exists$ in $M$. If $\bar{x}$ is any long hyperedge of $M$ with $k \in \text{rng}((\bar{x}))$, then $v_M(\bar{x}) = \text{nodes}(M)$.

If $\forall$ plays the amalgamation move $(M, N)$ (of two $\lambda$–neat hypergraphs) and $\exists$ responds with $L$ then for $m \neq n \in \text{nodes}(L)$ the owner in $L$ of a edge $(m, n)$ is $\forall$ if it belongs to $\forall$ in either $M$ or $N$, in all other cases it belongs to $\exists$ in $L$. If $\bar{x}$ is a long hyperedge of $L$ then $v_L(\bar{x}) = v_M(\bar{x})$ if $\text{rng}(\bar{x}) \subseteq \text{nodes}(M)$, $v_L(\bar{x}) = v_N(\bar{x})$ and $v_L(\bar{x}) = \text{nodes}(M)$ otherwise. If in a later move, $\forall$ plays the transformation move $(N, \theta)$ and $\exists$ responds with $N\theta$, then owners and envelopes are inherited in the obvious way. This completes the definition of owners and envelopes. The next claim, basically, reduces amalgamation moves to cylindrifier moves. By induction on the number of rounds one can show:

Claim: Let $M, N$ occur in a play of $H_m$, $0 < m \in \omega$. in which $\exists$ uses the above labelling for hyperedges. Let $\bar{x}$ be a long hyperedge of $M$ and let $\bar{y}$ be a long hyperedge of $N$. Then for any hyperedge $\bar{x}'$ with $\text{rng}(\bar{x}') \subseteq v_M(\bar{x})$, if $M(\bar{x}') = M(\bar{x})$ then $\bar{x}' = \bar{x}$. If $\bar{x}$ is a long hyperedge of $M$ and $\bar{y}$ is a long hyperedge of $N$, and $M(\bar{x}) = N(\bar{y})$, then there is a local isomorphism $\theta : v_M(\bar{x}) \rightarrow v_N(\bar{y})$ such that $\theta(x_i) = y_i$ for all $i < |x|$. For any $x \in \text{nodes}(M) \sim v_M(\bar{x})$ and $S \subseteq v_M(\bar{x})$, if $(x, s)$ belong to $\forall$ in $M$ for all $s \in S$, then $|S| \leq 2$.

Next, we proceed inductively with the inductive hypothesis exactly as before, except that now each $N_r$ is a $\lambda$–neat hypergraph. All what remains is the amalgamation move. With the above claim at hand, this turns out an easy task to implement guided by $\exists$’s winning strategy in the graph part.

We consider an amalgamation move at round $0 < r$, $(N_s, N_t)$ chosen by $\forall$ in round $r + 1$, $\exists$ has to deliver an amalgam $N_{r+1}$. $\exists$ lets $\text{nodes}(N_{r+1}) = \text{nodes}(N_s) \cup \text{nodes}(N_t)$, then she, for a start, has to choose a colour for each edge $(i, j)$ where $i \in \text{nodes}(N_s) \sim \text{nodes}(N_t)$ and $j \in \text{nodes}(N_t) \sim \text{nodes}(N_s)$. Let $\bar{x}$ enumerate $\text{nodes}(N_s) \cap \text{nodes}(N_t)$. If $\bar{x}$ is short, then there are at most two nodes in the intersection and this case is identical to the cylindrifier move. If not, that is if $\bar{x}$ is long in $N_s$, then by the claim there is a partial isomorphism $\theta : v_{N_s}(\bar{x}) \rightarrow v_{N_t}(\bar{x})$ fixing $\bar{x}$. We can assume that $v_{N_s}(\bar{x}) = \text{nodes}(N_s) \cap \text{nodes}(N_t) = \text{rng}(\bar{x}) = v_{N_t}(\bar{x})$. It remains to label the edges $(i, j) \in N_{r+1}$ where $i \in \text{nodes}(N_s) \sim \text{nodes}(N_t)$ and $j \in \text{nodes}(N_t) \sim \text{nodes}(N_s)$. Her strategy is now again similar to the cylindrifier move. If $i$ and $j$ are tints of the same cone she chooses a red using $\rho_{r+1}$ (constructed inductively like in the above proof), if not she chooses a white. She never chooses a green. Concerning $n − 1$ tuples she needs to label $n − 1$ hyperedges by shades of yellow. For each tuple $\bar{a} = a_0, \ldots a_{n−2} \in N_{r+1}$, with no edge $(a_i, a_j)$ coloured green (we have already
labelled edges), then \( \exists \) colours \( \bar{a} \) by \( y_S \), where

\[
S = \{ i \in \mathbb{Z} : \text{there is an } i \text{ cone in } N_{r+1} \text{ with base } \bar{a} \}.
\]

We have shown that \( \exists \) has a winning strategy in \( H_k(\text{At} \mathfrak{C}) \) for each finite \( k \).

Using ultrapowers and an elementary chain argument as in [10, Theorem 3.3.5], one gets a countable and atomic \( \mathfrak{B} \in \text{CA}_n \) such \( \exists \) has a winning strategy in \( H_\omega(\text{At} \mathfrak{B}) \), \( \mathfrak{B} \equiv \mathfrak{C}_{Z,N} \) and by Lemma 3.7 \( \text{cmAt} \mathfrak{B} \in \text{Nr}_n \text{CA}_\omega \) and \( \text{At} \mathfrak{B} \in \text{At} \text{Nr}_n \text{CA}_\omega \) (however, we do not guarantee that \( \mathfrak{B} \) itself is in \( \text{Nr}_n \text{CA}_\omega \)). Since \( \mathfrak{B} \subseteq_d \text{cmAt} \mathfrak{B} \), \( \mathfrak{B} \in S_d \text{Nr}_n \text{CA}_\omega \), so \( \mathfrak{B} \in S_d \text{Nr}_n \text{CA}_\omega \). Being countable, it follows by [20, Theorem 5.3.6] that \( \mathfrak{B} \in \text{CRCA}_n \). We show that \( \forall \) has a winning strategy in \( G^{n+3}(\text{At} \mathfrak{C}_{Z,N}) \). For the reader’s convenience we include this short part of the proof. In the initial round \( \forall \) plays a graph \( M \) with nodes \( 0, 1, \ldots, n-1 \) such that \( M(i, j) = w_0 \) for \( i < j < n-1 \) and \( M(i, n-1) = g_i \) \( (i = 1, \ldots, n-2) \), \( M(0, n-1) = g_0^0 \) and \( M(0, 1, \ldots, n-2) = y_Z \). This is a \( 0 \) cone. In the following move \( \forall \) chooses the base of the cone \( 0, \ldots, n-2 \) and demands a node \( n \) with \( M_2(i, n) = g_i \) \( (i = 1, \ldots, n-2) \), and \( M_2(0, n) = g_0^{-1} \). \( \exists \) must choose a label for the edge \((n+1, n)\) of \( M_2 \). It must be a red atom \( r_{mk} \), \( m, k \in \mathbb{N} \). Since \( -1 < 0 \), then by the ‘order preserving’ condition we have \( m < k \). In the next move \( \forall \) plays the face \( 0, \ldots, n-2 \) and demands a node \( n+1 \), with \( M_3(i, n) = g_i \) \( (i = 1, \ldots, n-2) \), such that \( M_3(0, n+2) = g_0^{-2} \). Then \( M_3(n+1, n) \) and \( M_3(n+1, n-1) \) both being red, the indices must match. \( M_3(n+1, n) = r_{lk} \)

and \( M_3(n+1, r-1) = r_{km} \) with \( l < m \in \mathbb{N} \). In the next round \( \forall \) plays \( 0, \ldots, n-2 \) and re-uses the node \( 2 \) such that \( M_4(0, 2) = g_0^3 \). This time we have \( M_4(n, n-1) = r_{jl} \) for some \( j < l \). Continuing in this manner leads to a decreasing sequence in \( \mathbb{N} \). We have proved the required. By Lemma 2.3, \( \mathfrak{C}_{Z,N} \notin S_d \text{Nr}_n \text{CA}_{n+3} \). Let \( \mathfrak{K} \) be a class between \( S_d \text{Nr}_n \text{CA}_{n+3} \cap \text{CRCA}_n \) and \( S_d \text{Nr}_n \text{CA}_{n+3} \). Then \( \mathfrak{K} \) is not elementary, because \( \mathfrak{C}_{Z,N} \notin S_d \text{Nr}_n \text{CA}_{n+3} \). \( \mathfrak{B} \in S_d \text{Nr}_n \text{CA}_{n} \cap \text{CRCA}_n (\subseteq \mathfrak{K}) \), and \( \mathfrak{C}_{Z,N} \equiv \mathfrak{B} \). We are not there yet, for \( \mathfrak{B} \) might still be outside \( \text{Nr}_n \text{CA}_\omega \) (like the \( \mathfrak{E} \) used in item (1) of Theoremrefsquare).

2. Proving the required; removing the \( S_d \): To get the required we resort to an auxiliary construction. We slightly modify the construction in [20, Lemma 5.1.3, Theorem 5.1.4]. Using the same notation, the algebras \( \mathfrak{A} \) and \( \mathfrak{B} \) constructed in op.cit satisfy \( \mathfrak{A} \in \text{Nr}_n \text{CA}_\omega \), \( \mathfrak{B} \notin \text{Nr}_n \text{CA}_{n+1} \) and \( \mathfrak{A} \equiv \mathfrak{B} \). As they stand, \( \mathfrak{A} \) and \( \mathfrak{B} \) are not atomic, but it can be fixed that they are atomic, giving the same result with the rest of the proof unaltered. This is done by interpreting the uncountably many tenary relations in the signature of \( M \) defined in [20, Lemma 5.1.3], which is the base of \( \mathfrak{A} \) and \( \mathfrak{B} \) to be disjoint in \( M \), not just distinct. The construction is presented this way in [18], where (the equivalent of) \( M \) is built in a more basic step-by-step fashion. We work with \( 2 < n < \omega \) instead of only \( n = 3 \). The proof presented in op.cit lift verbatim to any such \( n \). Let \( u \in n^n \). Write \( 1_u \) for \( \chi^M_u \) (denoted by \( 1_u \) (for \( n = 3 \)) in [20, Theorem 5.1.4].) We denote by \( \mathfrak{A}_u \) the Boolean algebra \( \mathfrak{A}_u \mathfrak{A} = \{ x \in \mathfrak{A} : x \leq 1_u \} \) and
similarly for \( \mathfrak{B} \), writing \( \mathfrak{B}_u \) short hand for the Boolean algebra \( \mathfrak{N}_{1, u} \mathfrak{B} = \{ x \in \mathfrak{B} : x \leq 1 \} \). Then exactly like in [20], it can be proved that \( \mathfrak{A} \equiv \mathfrak{B} \). Using that \( M \) has quantifier elimination we get, using the same argument in \textit{op.cit} that \( \mathfrak{A} \in \text{Nr}_n \text{CA}_\omega \). The property that \( \mathfrak{B} \not\in \text{Nr}_n \text{CA}_{n+1} \) is also still maintained. To see why, consider the substitution operator \( _n s(0, 1) \) (using one spare dimension) as defined in the proof of [20, Theorem 5.1.4]. Assume for contradiction that \( \mathfrak{B} = \text{Nr}_n \mathfrak{C} \), with \( \mathfrak{C} \in \text{CA}_{n+1} \). Let \( u = (1, 0, 2, \ldots n - 1) \). Then \( \mathfrak{A}_u = \mathfrak{B}_u \) and so \( |\mathfrak{B}_u| > \omega \). The term \( _n s(0, 1) \) acts like a substitution operator corresponding to the transposition \( [0, 1] \); it ‘swaps’ the first two co-ordinates. Now one can show that \( _n s(0, 1)^{\mathfrak{C}} \mathfrak{B}_u \subseteq \mathfrak{B}^{[0, 1]_u} = \mathfrak{B}_{1d} \), so \( |_n s(0, 1)^{\mathfrak{C}} \mathfrak{B}_u| \) is countable because \( \mathfrak{B}_{1d} \) was forced by construction to be countable. But \( _n s(0, 1) \) is a Boolean automorphism with inverse \( _n s(1, 0) \), so that \( |\mathfrak{B}_u| = |_n s(0, 1)^{\mathfrak{C}} \mathfrak{B}_u| > \omega \), contradiction.

It can be proved exactly like in [20] that the property \( \mathfrak{A} \equiv \mathfrak{B} \) is also still maintained after making the atoms disjoint. In fact, this change offers more for it can be proved that \( \mathfrak{A} \equiv_{\omega} \mathfrak{B} \). We show that \( \exists \) has a winning strategy in an Ehrenfeucht–Fraïssé-game over \((\mathfrak{A}, \mathfrak{B})\) concluding that \( \mathfrak{A} \equiv_{\infty} \mathfrak{B} \). At any stage of the game, if \( \forall \) places a pebble on one of \( \mathfrak{A} \) or \( \mathfrak{B} \), \( \exists \) must place a matching pebble, on the other algebra. Let \( \bar{a} = (a_0, a_1, \ldots, a_{n-1}) \) be the position of the pebbles played so far (by either player) on \( \mathfrak{A} \) and let \( \bar{b} = (b_0, \ldots, b_{n-1}) \) be the the position of the pebbles played on \( \mathfrak{B} \). \( \exists \) maintains the following properties throughout the game: For any atom \( x \) (of either algebra) with \( x \cdot 1_{1d} = 0 \) then \( x \in a_i \iff x \in b_i \) and \( \bar{a} \) induces a finite partition of \( 1_{1d} \) in \( \mathfrak{A} \) of \( 2^n \) (possibly empty) parts \( p_i : i < 2^n \) and \( \bar{b} \) induces a partition of \( 1_{1d} \) in \( \mathfrak{B} \) of parts \( q_i : i < 2^n \). Furthermore, \( p_i \) is finite \( \iff q_i \) is finite and, in this case, \( |p_i| = |q_i| \). That such properties can be maintained is fairly easy to show. Now because \( \mathfrak{A} \in \text{Nr}_n \text{CA}_\omega \cap \text{CRCA}_n \), it suffices to show (since \( \mathfrak{B} \) is atomic) that \( \mathfrak{B} \) is in fact outside \( S_d \text{Nr}_n \text{CA}_{n+1} \cap \text{At} \). Take \( \kappa \) the signature of \( M \); more specifically, the number of \( n \)-ary relation symbols to be \( 2^{2^\omega} \), and assume for contradiction that \( \mathfrak{B} \in S_d \text{Nr}_n \text{CA}_{n+1} \cap \text{At} \). Then \( \mathfrak{B} \subseteq_d \text{Nr}_n \mathfrak{D} \) for some \( \mathfrak{D} \in \text{CA}_{n+1} \) and \( \text{Nr}_n \mathfrak{D} \) is atomic. For brevity, let \( \mathfrak{C} = \text{Nr}_n \mathfrak{D} \). Then by item (1) of Lemma 3.4 \( \mathfrak{N}_{1d} \mathfrak{B} \subseteq_d \mathfrak{N}_{1d} \mathfrak{C} \). Since \( \mathfrak{C} \) is atomic, then by item (1) of the same Lemma \( \mathfrak{N}_{1d} \mathfrak{C} \) is also atomic. Using the same reasoning as above, we get that \( |\mathfrak{N}_{1d} \mathfrak{B}| > 2^{\omega} \) (since \( \mathfrak{C} \in \text{Nr}_n \text{CA}_{n+1} \)). By the choice of \( \kappa \), we get that \( |\text{At} \mathfrak{N}_{1d} \mathfrak{C}| > \omega \). By density, we get from item (2) of Lemma 3.4, that \( \text{At} \mathfrak{N}_{1d} \mathfrak{C} \subseteq \text{At} \mathfrak{N}_{1d} \mathfrak{B} \). Hence \( |\text{At} \mathfrak{N}_{1d} \mathfrak{B}| \geq |\text{At} \mathfrak{N}_{1d} \mathfrak{C}| > \omega \). But by the construction of \( \mathfrak{B} \), \( |\mathfrak{N}_{1d} \mathfrak{B}| = |\text{At} \mathfrak{N}_{1d} \mathfrak{B}| = \omega \), which is a contradiction and we are done. Since \( \text{El} \text{(Nr}_n \text{CA}_\omega \cap \text{CRCA}_n) \not\subseteq S_d \text{Nr}_n \text{CA}_\omega \cap \text{CRCA}_n \), there can be no elementary class between \( \text{Nr}_n \text{CA}_\omega \cap \text{CRCA}_n \) and \( S_d \text{Nr}_n \text{CA}_\omega \cap \text{CRCA}_n \) and \( S_c \text{Nr}_n \text{CA}_{n+3} \), we are done.
Fix finite \( k > 2 \). Then \( V_k = \text{Str}(\text{SN}_{n+k} \text{CA}_{n+k}) \) is not elementary \( \implies \)
\( V_k \) is not-atom canonical. But the converse implication does not hold because (arguing contrapositively) in the case of atom–canonicity, we get that \( \text{Str}(\text{SN}_{n+k} \text{CA}_{n+k}) = \text{At}(\text{SN}_{n+k} \text{CA}_{n+k}) \), and the last class is elementary \([8, \text{Theorem 2.84}]\). In particular, we do not know whether \( \text{Str}(\text{SN}_{n+k} \text{CA}_{n+k}) \), for a particular finite \( k \geq 3 \), is elementary or not. Nevertheless, it is easy to show that there has to be a finite \( k < \omega \) such that \( V_j \) is not elementary for all \( j \geq k \):

**Theorem 3.6.** 1. There is a finite \( k \geq 2 \), such that for all \( m \geq n+k \) the class of frames \( \text{Str}(\text{SN}_{n+k} \text{CA}_m) = \{ \mathcal{F} : \mathcal{E}m\mathcal{F} \in \text{SN}_{n+k} \text{CA}_m \} \) is not elementary. An entirely analogous result holds for RAs,

2. Let \( \mathcal{O} \in \{ S_k, S_d, I \} \) and \( k \geq 3 \). Then the class of frames \( \mathcal{K}_k = \{ \mathcal{F} : \mathcal{E}m\mathcal{F} \in \text{ON}_{n+k} \} \) is not elementary.

*Proof.* (1): We show that \( \text{Str}(\text{SN}_{n+k} \text{CA}_m) \) is not elementary for some finite \( m \geq n+2 \). By \([1] \) \( m \) cannot be equal to \( n+1 \). Let \( (\mathcal{A}_i : i \in \omega) \) be a sequence of (strongly) representable \( \text{CA}_m \)s with \( \mathcal{E}m\mathcal{A}_i = \mathcal{A}_i \), and \( \mathcal{A} = \Pi_{i \in \omega} \mathcal{A}_i \) is not strongly representable with respect to any non-principal ultrafilter \( U \) on \( \omega \). Such algebras exist \([10] \). Hence \( \mathcal{E}m\mathcal{A} \notin \text{SN}_{n+k} \text{CA}_\omega = \bigcap_{i \in \omega} \text{SN}_{n+k} \text{CA}_{n+i} \), so \( \mathcal{E}m\mathcal{A} \notin \text{SN}_{n+k} \text{CA}_l \) for all \( l > m \), for some \( m \in \omega, m \geq n+2 \). But for each such \( l, \mathcal{A}_i \in \text{SN}_{n+k} \text{CA}_l (\subseteq \text{RCA}_n) \), so \( (\mathcal{A}_i : i \in \omega) \) is a sequence of algebras such that \( \mathcal{E}m\mathcal{A}(\mathcal{A}_i) \in \text{SN}_{n+k} \text{CA}_l \) (\( i \in I \)), but \( \mathcal{E}m(\text{At}(\Pi_{i \in \omega} \mathcal{A}_i)) = \mathcal{E}m(\mathcal{A}) \notin \text{SN}_{n+k} \text{CA}_l \), for all \( l \geq m \).

(2): We use the same construction (and notation) in the last item of Theorem 3.5. It suffices to show that the class of algebras \( \mathcal{K}_k = \{ \mathcal{A} \in \text{CA}_n \cap \text{At} : \mathcal{E}m\mathcal{A} \in \text{ON}_{n+k} \text{CA}_k \} \) is not elementary. \( \exists \) has a winning strategy in \( \mathcal{H}_n(\alpha) \) for some countable atom structure \( \alpha \), \( \exists \alpha \subseteq d \mathcal{E}m\alpha \in \text{NR}_{n+k} \text{CA}_\omega \), and \( \exists \alpha \in \text{CRCA}_n \). Since \( \text{e}_{z,n} \notin \text{SN}_{n+k} \text{CA}_{n+3} \), then \( \text{e}_{z,n} = \mathcal{E}m\text{At} \mathcal{E}_{z,n} \notin \mathcal{K}_k \), \( \mathcal{E}_{z,n} \equiv \exists \alpha \) and \( \exists \alpha \in \mathcal{K}_k \) because \( \mathcal{E}m\alpha \in \text{NR}_{n+k} \text{CA}_\omega \subseteq \text{SN}_{n+k} \text{CA}_\omega \subseteq \text{SN}_{n+k} \text{CA}_\omega \). We have shown that \( \mathcal{E}_{z,n} \in \text{ElK}_k \sim \mathcal{K}_k \); proving the required.

To obtain the RA analogue of item (2) of Theorem 3.6, we need to strengthen [5, Theorem 39]. We prove more by allowing infinite conjunctions in constructing a certain model (denoted by \( \mathcal{M}_a \)) as clarified below. The \( k \) rounded game \( H_k (k \leq \omega) \) is defined for relation algebras in [5, Definition 28]. \( \text{Gws}_\beta \) denotes the class of generalized weak set algebras of dimension \( \beta \) in the sense of [4, Definition 3.1.2].

**Theorem 3.7.** Let \( \alpha \) be a countable atom structure. If \( \exists \) has a winning strategy in \( \mathcal{H}_n(\alpha) \), then there exists a complete \( \mathcal{D} \in \text{RCA}_\omega \) such that \( \mathcal{E}m\alpha \cong \text{RaD} \) and \( \alpha \cong \text{AtRaD} \). In particular, \( \mathcal{E}m\alpha \in \text{RaCA}_\omega \), \( \alpha \in \text{AtRaCA}_\omega \) and \( \alpha \) is completely representable.
Proof. Fix some \( a \in \alpha \). As shown in [5], the game \( H_\omega \) is designed so that using \( \exists \)'s winning strategy in the game \( H_\omega(\alpha) \) one can define a nested sequence \( M_0 \subseteq M_1, \ldots \) of \( \lambda \)-neat networks where \( M_0 \) is \( \exists \)'s response to the initial \( \forall \)-move \( a \) such that: If \( M_r \) is in the sequence and \( M_r(x, y) \leq a; b \) for an atoms \( a \) and \( b \) then there is \( s \geq r \) and a witness \( z \in \text{nodes}(M_s) \) such that \( M_s(x, z) = a \) and \( M_s(z, y) = b \). In addition, if \( M_r \) is in the sequence and \( \theta \) is any partial isomorphism of \( M_r \), then there is \( s \geq r \) and a partial isomorphism \( \theta^+ \) of \( M_s \) extending \( \theta \) such that \( \text{rng}(\theta^+) \supseteq \text{nodes}(M_s) \). Now let \( M_a \) be the limit of this sequence as defined in [5]. Let \( L \) be the signature with one binary relation for each \( b \in \alpha \), and one \( k \)-ary predicate symbol for each \( k \)-ary hyperlabel \( \lambda \). We work in \( L_{\infty,\omega} \). For fixed \( f_a \in \text{"nodes}(M_a) \), let \( \mathcal{U}_a = \{ f \in \text{"nodes}(M_a) : \{ i < \omega : g(i) \neq f_a(i) \} \text{ finite} \} \). One makes \( \mathcal{U}_a \) into the base of an \( L \) relativized structure \( M_a \) like in [5, Theorem 29] except that we allow a clause for infinitary disjunctions. We are now working with (weak) set algebras whose semantics are induced by \( L_{\infty,\omega} \) formulas in the signature \( L \), instead of first order ones. For any such \( L \)-formula \( \phi \), write \( \phi^{M_a} \) for \( \{ f \in \mathcal{U}_a : M_a, f \models \phi \} \). Let \( D_a = \{ \phi^{M_a} : \phi \text{ is an } L \text{-formula} \} \) and \( \mathfrak{D}_a \) be the weak set algebra with universe \( D_a \). Let \( \mathfrak{D} = \mathcal{P}_{a \in \alpha} \mathfrak{D}_a \). Then \( \mathfrak{D} \in \text{Gws}_\omega \) and furthermore \( \mathfrak{D} \) is complete. Suprema exists in \( \mathfrak{D} \) because we chose to work with \( L_{\infty,\omega} \) while forming the dilations \( \mathfrak{D}_a \) \( (a \in \alpha) \). Each \( \mathfrak{D}_a \) is complete, hence so is their product \( \mathfrak{D} \). Now \( \alpha \cong \text{AtRaD} \) as proved in [5, Theorem 39]. We show that \( \text{Cm}^{\alpha} \cong \text{RaD} \). Since \( \mathfrak{D} \) is complete, then \( \text{RaD} \) is complete. Making the legitimate identification \( \text{RaD} \subseteq \text{Cm}^{\alpha} \), by density we get that \( \text{RaD} \cong \text{Cm}^{\alpha} \) because \( \text{RaD} \) is complete, so \( \text{Cm}^{\alpha} \in \text{RaCA}_\omega \). Now \( \text{Cm}^{\alpha} \in \text{S}_d \text{RaCA}_\omega (\subseteq \text{S}_e \text{RaCA}_\omega) \) and \( \alpha \) is countable, so \( \alpha \) is completely representable.

Corollary 3.8. Let \( O \in \{ \text{S}_e, \text{S}_d, I \} \) and \( m \geq 5 \). Then the class of frames \( L_m = \{ \mathfrak{F} : \text{Cm}^{\mathfrak{F}} \in \text{ORaCA}_m \} \) is not elementary. Furthermore, any class \( K \) such that \( \text{S}_d \text{RaCA}_\omega \cap \text{CRRA} \subseteq K \subseteq \text{S}_e \text{RaCA}_\omega \), \( K \) is not elementary.

Proof. Let \( L_m = \{ \mathfrak{A} \in \text{RA}_n \cap \text{At} : \text{CmAtA} \in \text{ORaCA}_m \} \). Take the relation algebra atom structure \( \beta \) based on \( N \) and \( Z \) as defined in [5], for which \( \exists \) has a winning strategy in \( H_k(\text{AtA}_Z, N) \), where \( \text{A}_Z, N = \text{Cm} \beta \) for all \( k < \omega \), and \( \forall \) has a winning strategy in \( F^\beta \) with \( F^\gamma \) as in [5, Definition 28]. By [5, Lemma 2.6], we get that \( \text{A}_Z, N \notin \text{S}_d \text{RaCA}_5 \). The usual argument of taking ultrapowers followed by an elementary chain argument, one gets a countable atom structure \( \alpha \), such that \( \text{Cm}^{\alpha} \equiv \text{A}_Z, N \) and \( \exists \) has a winning strategy in \( H(\alpha) \). By Theorem 3.7, there is \( \mathcal{D} \in \text{CA}_\omega \), such that \( \alpha \equiv \beta, \alpha \equiv \text{AtD} \) and \( \text{Cm}^{\alpha} \subseteq \text{RaD} \), so that \( \text{Cm}^{\alpha} \in \text{S}_d \text{RaCA}_\omega \). Since \( \text{A}_Z, N \notin \text{S}_e \text{RaCA}_5 \), then \( \text{A}_Z, N = \text{CmAtC}_Z, N \notin L_m \). \( \text{A}_Z, N \equiv \text{Cm}^{\alpha} \) and \( \text{Cm}^{\alpha} \in L_m \) because \( \text{Cm}^{\alpha} \in \text{RaCA}_\omega \). For the second required one uses the same elementary equivalent algebras \( \text{Cm}^{\alpha} \in \text{S}_d \text{RaCA}_\omega \) and \( \text{A}_Z, N \notin \text{S}_e \text{RaCA}_5 \). \( \square \)

In the next table we summarize the results hitherto obtained on first order definability:
The result in row seven is stronger than the result proved in [6] because $S_d \leq S_c$. Neither of the results in the second and third row is superfluous for the two classes $S_cNr_nCA_{n+3}$ and $S_dNr_nCA_{n+1}$ are mutually distinct: The algebra $B$ in the last item of Theorem 3.5 is completely representable, so $B \in S_dNr_nCA_{n+3} \sim S_dNr_nCA_{n+1}$. Conversely, in [13, §2], a finite algebra $A \in Nr_nCA_{n+3} \sim SNr_nCA_{n+2}(\subseteq S_dNr_nCA_{n+1} \sim S_cNr_nCA_{n+3})$ is constructed. From the no’s in the sixth and last row we cannot answer (negatively) the only question mark in the table. This would be a far too hasty decision (taken mistakenly in [5] and corrected in [6]). The implication $AtA \in AtRaCA_n$ and $\exists mAtA \in RaCA_n \implies A \in RaCA_\omega$ may not be valid. Indeed, the algebra $B$ used in item (1) of Theorem 2.4 confirms our doubts, since $AtB \in AtNr_nCA_\omega$, $\exists mAtB \in Nr_nCA_\omega$, but $B \notin Nr_nCA_\omega$.

Unless otherwise explicitly indicated, fix $2 < n < \omega$: The last example motivates:

**Definition 3.9.** 1. A CA$_n$ atom structure $A$ is weakly representable if there is an $A \in RCA_n$ such that $AtA = A$. The CA$_n$ atom structure $A$ is strongly representable if every $A \in RCA_n$ such that $AtA = A \implies A \in RCA_n$.

2. The class $K$ is gripped by its atom structures or simply gripped, if for $A \in CA_n$, whenever $AtA \in AtK$, then $A \in K$. An $\omega$-rounded game $H$ grips $K$, if whenever $A \in CA_n$ is atomic with countably many atoms and $\exists$ has a winning strategy in $H(A)$, then $A \in K$. The game $H$ weakly grips $K$, if whenever $A \in CA_n$ is atomic with countably many atoms and $\exists$ has a winning strategy in $H(AtA)$, then $AtA \in AtK$. The game $H$ densely grips $K$, if whenever $A \in CA_n$ is atomic with countably many atoms and $\exists$ has a winning strategy in $H(AtA)$, then $AtA \in AtK$ and $\exists mAtA \in K$.

Let $2 < n < m \leq \omega$. It is is known that a CA$_n$ atom structure $A$ is weakly representable $\iff \exists mAt$ (the term algebra which is the subalgebra of the complex algebra $\exists mAt$ generated by the atoms) is representable; it is strongly representable $\iff \exists mAt$ is representable. [23, 12]. The classes RCA$_n$ and $Nr_nCA_m$ are not gripped, by [12] and first item of Theorem 2.4. In [12] a weakly representable CA$_n$ atom structure that is not strongly representable is constructed, showing that RCA$_n$ is not closed under Dedekind-MacNeille completions, because $\exists mAt$ is the Dedekind-MacNeille completion of $\exists mAt$.
in the case of completely additive varieties of BAOs. But we can go even further:

**Theorem 3.10.** For $m \geq m + 3$, the variety $\mathbf{SNr}_n \mathbf{CA}_m$ is not gripped.

**Proof.** It clearly suffices to construct an atom structure $\mathbf{At}$ such that $\mathbf{ImAt} \in \mathbf{RCA}_n$ but $\mathbf{ImAt} \notin \mathbf{SNr}_n \mathbf{CA}_{n+3}$. We divide the proof into four parts:

1. **Blowing up and blurring $\mathfrak{A}_{n+1,n}$ forming a weakly representable atom structure $\mathbf{At}$:** Take the finite rainbow $\mathbf{CA}_n$, $\mathfrak{A}_{n+1,n}$ where the reds $\mathbf{R}$ is the complete irreflexive graph $n$, and the greens are $\{g_i : 1 \leq i \leq n - 1\} \cup \{g_0^i : 1 \leq i \leq n + 1\}$, endowed with the quasi-polyadic equality operations. Denote the finite atom structure of $\mathfrak{A}_{n+1,n}$ by $\mathbf{At}_f$; so that $\mathbf{At}_f = \mathbf{At}(\mathfrak{A}_{n+1,n})$. One then replaces the red colours of the finite rainbow algebra of $\mathfrak{A}_{n+1,n}$ each by infinitely many reds (getting their superscripts from $\omega$), obtaining this way a weakly representable atom structure $\mathbf{At}$. The resulting atom structure after ‘splitting the reds’, namely, $\mathbf{At}$, is like the weakly (but not strongly) representable atom structure of the atomic, countable and simple algebra $\mathfrak{A}$ as defined in [12, Definition 4.1]; the sole difference is that we have $n + 1$ greens and not $\omega$—many as is the case in [12]. We denote the resulting term cylindric algebra $\mathbf{ImAt}$ by $\mathbf{Bb}(\mathfrak{A}_{n+1,n}, r, \omega)$ short hand for blowing up and blurring $\mathfrak{A}_{n+1,n}$ by splitting each red graph (atom) into $\omega$ many. It can be shown exactly like in [12] that $\exists$ can win the rainbow $\omega$-rounded game and build an $n$-homogeneous model $\mathbf{M}$ by using a shade of red $\rho$ outside the rainbow signature, when she is forced a red; [12, Proposition 2.6, Lemma 2.7]. The $n$-homogeniuty entails that any subgraph (substructure) of $\mathbf{M}$ of size $\leq n$, is independent of its location in $\mathbf{M}$; it is uniquely determined by its isomorphism type. One proves like in op.cit that $\mathbf{Bb}(\mathfrak{A}_{n+1,n}, r, \omega)$ is representable as a set algebra having top element "$\mathbf{M}$. We give more details. In the present context, after the splitting ‘the finitely many red colours’ replacing each such red colour $r_{kl}$, $k < l < n$ by $\omega$ many $r^i_{kl}$, $i \in \omega$, the rainbow signature for the resulting rainbow theory as defined in [8, Definition 3.6.9] call this theory $T_{ra}$, consists of $g_i : 1 \leq i < n - 1$, $g_0^i : 1 \leq i \leq n + 1$, $w_i : i < n - 1$, $r_{kl}^i : k < l < n$, $t \in \omega$, binary relations, and $n - 1$ ary relations $y_S$, $S \subseteq \omega n + k - 2$ or $S = n + 1$. The set algebra $\mathbf{Bb}(\mathfrak{A}_{n+1,n}, r, \omega)$ of dimension $n$ has base an $n$-homogeneous model $\mathbf{M}$ of another theory $T$ whose signature expands that of $T_{ra}$ by an additional binary relation (a shade of red) $\rho$. In this new signature $T$ is obtained from $T_{ra}$ by some axioms (consistency conditions) extending $T_{ra}$. Such axioms (consistency conditions) specify consistent triples involving $\rho$. We call the models of $T$ extended coloured graphs. In particular, $\mathbf{M}$ is an extended coloured graph. To build $\mathbf{M}$, the class of coloured graphs is considered in the signature $L \cup \{\rho\}$ like in usual rainbow constructions as given above with the two additional forbidden triples $(r, \rho, \rho)$ and $(r, r^*, \rho)$, where $r, r^*$ are any reds. This model $\mathbf{M}$ is constructed as a countable limit of finite models of $T$ using a game played between $\exists$ and $\forall$. Here, unlike the extended $L_{\omega_1,\omega}$ theory dealt
with in [12], $T$ is a \textit{first order one} because the number of greens used are finite. In the rainbow game [7] §4.3, $\forall$ challenges $\exists$ with \textit{cones} having green \textit{tints} $(g_i)$, and $\exists$ wins if she can respond to such moves. This is the only way that $\forall$ can force a win. $\exists$ has to respond by labelling \textit{apexes} of two successive cones, having the \textit{same base} played by $\forall$. By the rules of the game, she has to use a red label. She resorts to $\rho$ whenever she is forced a red while using the rainbow reds will lead to an inconsistent triangle of reds; [12, Proposition 2.6, Lemma 2.7]. The indicative term ‘blow up and blur’ was introduced in [3]. The ‘blowing up’ refers to ‘splitting each red atom of $\mathfrak{A}_{n+1,n}$ into infinitely many subatoms called its copies obtaining the new weakly representable atom structure $\bar{A}t$. The term ‘blur’ refers to the fact that the algebraic structure of $\mathfrak{A}_{n+1,n}$ is ‘disorganized’ in $T_{\bar{A}}$; it is completely distorted by redefining the operations to the point that the original operations of $\mathfrak{A}_{n+1,n}$ is blurred; it is no longer there. Nevertheless, the algebraic structure of $\mathfrak{A}_{n+1,n}$ reappears in $\mathfrak{CmAt}$, the Dedekind-MacNeille completion of $T_{\bar{A}}$, in the sense that $\mathfrak{A}_{n+1,n}$ embeds into $\mathfrak{CmAt}$ as will be shown in a while. The completion (supremum exists) of $\mathfrak{CmAt}$ plays a key role, because the embedding is done by mapping every atom to the suprema of its copies. This suprema may, and indeed in some of the cases does not exist in $T_{\bar{A}}$.

2. Representing $T_{\bar{A}}$ (and its completion) as (generalized) set algebras: From now on, forget about $\rho$; having done its task as a colour to (weakly) represent $\bar{A}t$, it will play no further role. Having $M$ at hand, one constructs two atomic $n$–dimensional set algebras based on $M$, sharing the same atom structure and having the same top element. The atoms of each will be the set of coloured graphs, seeing as how, quoting Hodkinson [12] such coloured graphs are ‘literally indivisible’. Now $L_n$ and $L^n_{\infty,\omega}$ are taken in the rainbow signature (without $\rho$). Continuing like in op.cit, deleting the one available red shade, set $W = \{\bar{a} \in {}^nM : M \models (\bigwedge_{i<j<n} \neg \rho(x_i, x_j))(\bar{a})\}$, and for $\phi \in L^n_{\infty,\omega}$, let $\phi^W = \{s \in W : M \models \phi[s]\}$. Here $W$ is the set of all $n$–ary assignments in ${}^nM$, that have no edge labelled by $\rho$. Let $\mathfrak{A}$ be the relativized set algebra with domain $\{\varphi^W : \varphi \text{ a first-order } L_n \text{ – formula} \}$ and unit $W$, endowed with the usual concrete quasipolyadic operations read off the connectives. Classical semantics for $L_n$ \textit{rainbow formulas} and their semantics by relativizing to $W$ coincide [12, Proposition 3.13] \textit{but not with respect to} $L^n_{\infty,\omega}$ \textit{rainbow formulas}. This depends essentially on [12, Lemma 3.10], which is the heart and soul of the proof in [12], and for what matters this proof. The referred to lemma says that any permutation $\chi$ of $\omega \cup \{\rho\}$, $\Theta^\chi$ as defined in [12, Definitions 3.9, 3.10] is an $n$ back–and–forth system induced by any permutation of $\omega \cup \{\rho\}$. Let $\chi$ be such a permutation. The system $\Theta^\chi$ consists of isomorphisms between coloured graphs such that superscripts of reds are ‘re-shuffled along $\chi$', in such a way that rainbow red labels are permuted, $\rho$ is replaced by a red rainbow label, and all other colours are preserved. One uses
such $n$-back-and-forth systems mapping a tuple $\bar{b} \in {}^nM \setminus W$ to a tuple $\bar{c} \in W$ preserving any formula in $L_{ra}$ containing the non-red symbols that are ‘moved’ by the system, so if $\bar{b} \in {}^nM$ refutes the $L_n$ rainbow formula $\phi$, then there is a $\bar{c}$ in $W$ refuting $\phi$. Hence the set algebra $\mathfrak{A}$ is isomorphic to a quasi-polyadic equality set algebra of dimension $n$ having top element $^nM$, so $\mathfrak{A}$ is simple, in fact its $\mathfrak{D}$f reduct is simple. Let $\mathfrak{E} = \{ \phi^W : \phi \in L_{\infty,\omega}^n \}$ [12, Definition 4.1] with the operations defined like on $\mathfrak{A}$ the usual way. $\mathfrak{CmAt}$ is a complete $\text{QEA}_n$ and, so like in [12, Lemma 5.3] we have an isomorphism from $\mathfrak{CmAt}$ to $\mathfrak{E}$ defined via $X \mapsto \bigcup X$. Since $\mathfrak{AtA} = \mathfrak{At} \mathfrak{Im(AtA)} = \mathfrak{At}$ and $\mathfrak{ImAtA} \subseteq \mathfrak{A}$, hence $\mathfrak{ImAtA}$ is representable. The atoms of $\mathfrak{A}$, $\mathfrak{ImAtA}$ and $\mathfrak{CmAtA} = \mathfrak{CmAt}$ are the coloured graphs whose edges are not labelled by $\rho$. These atoms are uniquely determined by the interpretation in $M$ of so-called $\text{MCA}$ formulas in the rainbow signature of $\mathfrak{At}$ as in [12, Definition 4.3].

3. Embedding $\mathfrak{A}_{n+1,n}$ into $\mathfrak{CmAt(}\mathfrak{Gb(At}_{n+1,n} r, \omega))$: Let $\text{CRG}_f$ be the class of coloured graphs on $\mathfrak{At}_f$ and $\text{CRG}$ be the class of coloured graph on $\mathfrak{At}$. We can (and will) assume that $\text{CRG}_f \subseteq \text{CRG}$. Write $M_a$ for the atom that is the (equivalence class of the) surjection $a : n \to M$, $M \in \text{CGR}$. Here we identify $a$ with $[a]$; no harm will ensue. We define the (equivalence) relation $\sim$ on $\mathfrak{At}$ by $M_b \sim N_a$, $(M, N \in \text{CGR})$:

- $a(i) = a(j) \iff b(i) = b(j)$,
- $M_a(a(i), a(j)) = r^l \iff N_b(b(i), b(j)) = r^k$, for some $l, k \in \omega$,
- $M_a(a(i), a(j)) = N_b(b(i), b(j))$, if they are not red,
- $M_a(a(k_0), \ldots, a(k_{n-2})) = N_b(b(k_0), \ldots, b(k_{n-2}))$, whenever defined.

We say that $M_a$ is a copy of $N_b$ if $M_a \sim N_b$ (by symmetry $N_b$ is a copy of $M_a$.) Indeed, the relation ‘copy of’ is an equivalence relation on $\mathfrak{At}$. An atom $M_a$ is called a red atom, if $M_a$ has at least one red edge. Any red atom has $\omega$ many copies, that are cylindrically equivalent, in the sense that, if $N_a \sim M_b$ with one (equivalently both) red, with $a : n \to N$ and $b : n \to M$, then we can assume that $\text{nodes}(N) = \text{nodes}(M)$ and that for all $i < n$, $a \restriction n \sim \{i\} = b \restriction n \sim \{i\}$. Any red atom has $\omega$ many copies that are cylindrically equivalent, in the sense that, if $N_a \sim M_b$ with one (equivalently both) red, with $a : n \to N$ and $b : n \to M$, then we can assume that $\text{nodes}(N) = \text{nodes}(M)$ and that for all $i < n$, $a \restriction n \sim \{i\} = b \restriction n \sim \{i\}$. In $\mathfrak{CmAt}$, we write $M_a$ for $\{M_a\}$ and we denote suprema taken in $\mathfrak{CmAt}$, possibly finite, by $\sum$. Define the map $\Theta$ from $\mathfrak{A}_{n+1,n} = \mathfrak{CmAt}_{f}$ to $\mathfrak{CmAt}_{f}$, by specifying first its values on $\mathfrak{At}_f$, via $M_a \mapsto \sum_j M_a^{(j)}$ where $M_a^{(j)}(j)$ is a copy of $M_a$. So each atom maps to the suprema of its copies. This map is well-defined because $\mathfrak{CmAt}$ is complete. We check that $\Theta$ is an injective homomorphism. Injectivity follows from $M_a \leq \Theta(M_a)$, hence $\Theta(x) \neq 0$ for every atom $x \in \mathfrak{At}(\mathfrak{A}_{n+1,n})$. We check preservation of all the $\text{QEA}_n$ operations. The Boolean join is obvious.
• For complementation: It suffices to check preservation of complementation ‘at atoms’ of $\mathbf{At}_f$. So let $M_a \in \mathbf{At}_f$ with $a : n \to M, M \in \text{CGR}_f \subseteq \text{CGR}$. Then:

$$
\Theta(\sim M_a) = \Theta(\bigcup_{[b] \neq [a]} M_b) = \bigcup_{[b] \neq [a]} \Theta(M_b) = \bigcup_{[b] \neq [a]} \sum_j M^{(j)}_b
$$

$$
= \bigcup_{[b] \neq [a]} \sim \sum_j [(\sim M_a)^{(j)}] = \bigcup_{[b] \neq [a]} \sim \sum_j M^{(j)}_b
$$

$$
= \bigwedge_{[b] \neq [a]} \bigcup_j M^{(j)}_b = \bigwedge_{[b] \neq [a]} (\sim M_a)^{(j)} = \sim (\sum M^{(j)}_a) = \sim \Theta(a)
$$

• Diagonal elements. Let $l < k < n$. Then:

$$
M_x \leq \Theta(d^{\text{emAt}_f}_{lk}) \iff M_x \leq \bigcup_{a_l = a_k} M^{(j)}_a
$$

$$
\iff M_x \leq \bigcup_{a_l = a_k} \sum_j M^{(j)}_a
$$

$$
\iff M_x = M^{(j)}_a \text{ for some } a : n \to M \text{ such that } a(l) = a(k)
$$

$$
\iff M_x \in d^{\text{emAt}_f}_{lk}.
$$

• Cylindrifiers. Let $i < n$. By additivity of cylindrifiers, we restrict our attention to atoms $M_a \in \mathbf{At}_f$ with $a : n \to M$, and $M \in \text{CGR}_f \subseteq \text{CGR}$. Then:

$$
\Theta(c^{\text{emAt}_f}_i M_a) = f(\bigcup_{[c] = [a]} M_c) = \bigcup_{[c] = [a]} \Theta(M_c)
$$

$$
= \bigcup_{[c] = [a]} \sum_j M^{(j)}_c = \bigcup_{[c] = [a]} \sum_j M^{(j)}_c = \sum_j c^{\text{emAt}_f}_i M^{(j)}_a
$$

$$
= c^{\text{emAt}_f}_i (\sum_j M^{(j)}_a) = c^{\text{emAt}_f}_i \Theta(M_a).
$$

4.: $\forall$ has a winning strategy in $G^{n+3}\mathbf{At}(\exists_{n+1,n})$; and the required result: It is straightforward to show that $\forall$ has winning strategy first in the Ehrenfeucht–Fraïssé forth private game played between $\exists$ and $\forall$ on the complete irreflexive graphs $n + 1$ and $n$ in $n + 1$ rounds $\text{EF}^{n+1}_{n+1}(n + 1, n)$ [10, Definition 16.2] since $n + 1$ is ‘longer’ than $n$. Here $r$ is the number of rounds and $p$ is the number of pairs of pebbles on board. Using (any) $p > n$ many pairs of pebbles available on the board $\forall$ can win this game in $n + 1$ many rounds. In each round $0, 1 \ldots n$, $\exists$ places a new pebble on a new element of $n + 1$. The edge relation in $n$ is irreflexive so to avoid losing $\exists$ must respond by
placing the other pebble of the pair on an unused element of \( n \). After \( n \) rounds there will be no such element, so she loses in the next round. \( \forall \) lifts his winning strategy from the private Ehrenfeucht–Fraïssé forth game \( \text{EF}_{n+1}^n(n, n+1) \) to the graph game on \( \text{At}_f = \text{At}(\mathfrak{A}_{n+1,n}) \) [7, pp. 841] forcing a win using \( n+3 \) nodes. He bombards \( \exists \) with cones having common base and distinct green tints until \( \exists \) is forced to play an inconsistent red triangle (where indices of reds do not match). By Lemma 2.3, \( \mathfrak{A}_{n+1,n} \not\in \text{SNr}_n \mathfrak{C}_n \). Since \( \mathfrak{A}_{n+1,n} \) is finite, then \( \mathfrak{A}_{n+1,n} \not\in \text{SNr}_n \mathfrak{C}_n \), because \( \mathfrak{A}_{n+1} = \mathfrak{A}_{n+1,n} \) (its canonical extension) and \( \mathfrak{D} \in \text{SNr}_n \mathfrak{C}_n \) \( \rightarrow \mathfrak{A}^+ \in \text{SNr}_n \mathfrak{C}_n^+ \). But \( \mathfrak{A}_{n+1,n} \) embeds into \( \text{cmAt}\mathfrak{A} \), hence \( \text{cmAt}\mathfrak{A} \) is outside the variety \( \text{SNr}_n \mathfrak{C}_n^+ \), as well. By Lemma 2.3, the required follows.

\[ \square \]

On the other hand, the class \( \text{SNr}_n \mathfrak{C}_n^m \) is gripped. For any \( n < \omega \), the class \( \text{CRCA}_n \), and its elementary closure, namely, the class of algebras satisfying the Lyndon conditions as defined in [10] is gripped. The usual atomic game \( G \) weakly grips, densely grips and grips \( \text{CRCA}_n \).

**Theorem 3.11.** The game \( \mathbf{H}_\omega \) densely grips \( \text{Nr}_n \mathfrak{C}_\omega \), but does not grip \( \text{Nr}_n \mathfrak{C}_n^\omega \).

**Proof.** The first part follows from Theorem 3.7. For the second more tricky part. Take \( \mathfrak{E} \in \mathfrak{C}_n \) to be the algebra in the first item of Theorem 2.4. We know that \( \text{At}\mathfrak{E} \in \text{At}\text{Nr}_n \mathfrak{C}_n \), and \( \text{cmAt}\mathfrak{E} \in \mathfrak{C}_n \), but \( \mathfrak{E} \not\in \text{Nr}_n \mathfrak{C}_n^1 \). We show that \( \exists \) has a winning strategy in \( \mathbf{H}_\omega(\text{At}\mathfrak{E}) \). \( \exists \)'s strategy dealing with \( \lambda \)-neat hypernetworks, where \( \lambda \) is a constant label kept on short hyperedges is exactly like in the proof of Theorem 3.7. The rest of her winning strategy is to play \( \lambda \)-neat hypernetworks \( (N^a, N^h) \) with \( \text{nodes}(N_a) \subseteq \omega \) such that \( (N^a)^+ \neq 0 \) (recall that \( (N^a)^+ \) is as defined in the proof of lemma 2.3). In the initial round, let \( \forall \) play \( a \in \text{At} \). \( \exists \) plays a network \( N \) with \( N^a(0, 1, \ldots n - 1) = a \). Then \( (N^a)^+ = a \neq 0 \). The response to the cylindrifier move is exactly like in the first part of Lemma 2.3 because \( \mathfrak{E} \) is completely representable so \( \mathfrak{E} \in \text{SNr}_n \mathfrak{C}_\omega \) [20, Theorem 5.3.6]. For transformation moves: if \( \forall \) plays \((M, \theta), \) then it is easy to see that we have \( (M^a \theta)^+ \neq 0 \), so this response is maintained in the next round. For each \( J \subseteq \omega \) with \( |J| = n \), say, for \( \mathfrak{A} \in \mathfrak{C}_\omega \), let \( \text{Nr}_j \mathfrak{A} = \{ x \in \mathfrak{A} : c_i x = x, \forall i \in \omega \sim J \} \). We have \( \text{At}\mathfrak{E} \in \text{At}\text{Nr}_n \mathfrak{C}_n \), so assume that \( \text{At}\mathfrak{E} = \text{At}\text{Nr}_n \mathfrak{D} \) with \( \mathfrak{D} \in \mathfrak{C}_n \). Then for all \( y \in \text{Nr}_j \mathfrak{D} \), where \( J = \{ i_0, i_1, \ldots, i_{n-1} \} \), \( i_0, \ldots, i_{n-1} \in \omega \), the following holds for \( a \in \text{At}\mathfrak{E} \): \( s_{i_0, \ldots, i_{n-1}} a \cdot y \not= 0 \Rightarrow s_{i_0, \ldots, i_{n-1}} a \leq y \). Now we are ready to describe \( \exists \)'s strategy in response to amalgamation moves. For better readability, we write \( \tilde{i} \) for \( \{ i_0, i_1, \ldots, i_{n-1} \} \), if it occurs as a set, and we write \( s_i \) short for \( s_{i_0} s_{i_1} \ldots s_{i_{n-1}} \). Also we only deal with the network part of the game. Now suppose that \( \forall \) plays the amalgamation move \( (M, N) \) where \( \text{nodes}(M) \cap \text{nodes}(N) = \{ \tilde{i} \} \), then \( M(\tilde{i}) = N(\tilde{i}) \). Let \( \mu = \text{nodes}(M) \sim \tilde{i} \) and \( v = \text{nodes}(N) \sim \tilde{i} \). Then \( c_{(v)} M^+ = M^+ \) and \( c_{(\mu)} N^+ = M^+ \). Hence
using (*), we have: $c(u)M^+ = s_i M(\bar{i}) = s_i N(\bar{i}) = c(v)N^+$ so $c(v)M^+ = M^+ \leq c(u)M^+ = c(u)N^+$ and $M^+ \cdot N^+ = x \neq 0$. So there is $L$ with $\text{nodes}(L) = \text{nodes}(M) \cup \text{nodes}(N)$, and $L^+ \cdot x \neq 0$, thus $L^+ \cdot M^+ \neq 0$ and consequently $L|_{\text{nodes}(M)} = M|_{\text{nodes}(M)}$, hence $M \subseteq L$ and similarly $N \subseteq L$, so that $L$ is the required amalgam.

\[\square\]

**Example 3.12.** Fix $2 < n < \omega$. The game $H$ weakly and densely grips $\text{Nr}_n \text{CA}_\omega$ but $H$ does not grip $\text{Nr}_n \text{CA}_\omega$. We devise an $\omega$–rounded non-atomic game $G$ gripping $\text{Nr}_n \text{CA}_\omega$. By non-atomic, we mean that arbitrary elements of the algebra not necessarily atoms are allowed during the play. By the example in item 1 of Theorem 2.4, $G$ is strictly stronger than $H$. That is to say, $\exists$ has a winning strategy in $G \implies \exists$ has a winning strategy in $H$, but the converse implication is false as, using the notation in *op.cit*, $\exists$ has a winning strategy in $H(\text{At}\mathcal{C})$ but does not have a winning strategy in $G(\mathcal{C})$.

The game $G$ is played on both $\lambda$–neat hypernetworks as defined for $H$, and complete labelled graphs (possibly by non–atoms) with no consistency conditions. The play at a certain point, like in $H$, will be a $\lambda$–neat hypernetwork, call its network part $X$, and we write $X(\bar{x})$ for the atom the edge $\bar{x}$. By network part we mean forgetting $k$–hypedges getting non–atomic labels. An $n$–matrix is a finite complete graph with nodes including $0, \ldots, n - 1$ with all edges labelled by arbitrary elements of $\mathcal{B}$. No consistency properties are assumed. $\forall$ can play an arbitrary $n$–matrix $N$, $\exists$ must replace $N(0, \ldots, n - 1)$, by some element $a \in \mathcal{B}$; this is a non-atomic move. The final move is that $\forall$ can pick a previously played $n$–matrix $N$, and pick any tuple $\bar{x} = (x_0, \ldots, x_{n-1})$ whose atomic label is below $N(0, \ldots, n - 1)$. $\exists$ must respond by extending $X$ to $X'$ such that there is an embedding $\theta$ of $N$ into $X'$ such that $\theta(0) = x_0, \ldots, \theta(n - 1) = x_{n-1}$ and for all $i_0, \ldots, i_{n-1} \subseteq N$, we have

$$X(\theta(i_0) \ldots, \theta(i_{n-1})) \leq N(i_0, \ldots, i_{n-1}).$$

This ensures that in the limit, the constraints in $N$ really define the element $a$. Assume that $\mathcal{B} \in \text{CA}_n$ is atomic and has countably many atoms. If $\exists$ has a winning strategy in $G(\mathcal{B})$, then the extra move involving non–atoms labelling matrices, ensures that that every $n$–dimensional element generated by $\mathcal{B}$ in a dilation $\mathcal{D} \in \text{RCA}_\omega$ having base $M$, constructed from a winning strategy in $G$ as the limit of the $\lambda$–neat hypernetworks played during the game (and further assuming without loss that $\forall$ plays every possible move) is already an element of $\mathcal{B}$. For $k < \omega$, let $G_k$ be the game $G$ truncated to $k$ rounds, and let $G^a_k$ and $G^r_k$ be the relation algebra analogue of the game obtained by adding the non–atomic move replacing $n$–matrices by $2$–matrices, to the game $H$ as defined for relation algebras [5, Definition 28].

Using the argument in the proof of the Theorem 3.7 replacing $H$ by $G$
we get: Assume that $2 < n < m < \omega$. If there exists a countable atom structure $\alpha$ such that $\exists$ has a winning strategy in $G_k(\mathcal{C}^m\alpha)$ for all $k \in \omega$ and $\forall$ has a winning strategy in $F^m$, then any class $K$, such that $\mathcal{N}_n \mathcal{C}_\omega \subseteq K \subseteq \mathcal{S}_\omega \mathcal{N}_n \mathcal{C}_m$, is not elementary. We have already proved the last result. The relation algebra case is more interesting. Undefined notation can be found in [5]; detailed citation is given in the proof.

**Theorem 3.13.** Assume that $2 < m < \omega$. If there exists a countable atom structure $\alpha$ such that $\exists$ has a winning strategy in $G_k(\mathcal{C}^m\alpha)$ for all $k \in \omega$ and $\forall$ has a winning strategy in $F^m$, then any class $K$, such that $\mathcal{R}_{\alpha} \mathcal{C}_\omega \subseteq K \subseteq \mathcal{S}_\omega \mathcal{R}_{\alpha} \mathcal{C}_5$, is not elementary.

**Proof.** The analogous result can be obtained for relation algebras for $2 < m < \omega$ obtained by replacing $\mathcal{N}_n$ by $\mathcal{R}_{\alpha}$. One uses the arguments in [5, Theorem 39, 45], but resorting to the game $G_{\alpha}^k$ in place of $H_k$ ($k < \omega$), as defined for relation algebras [5, Definition 28]. By assumption we have a countable relation algebra atom structure $\alpha$, for which $\exists$ has a winning strategy in $G_{\alpha}^k(\mathcal{C}^m\alpha)$, for all $k < \omega$, and $\forall$ has a winning strategy in $F^m(\alpha)$ with $F^m$ as defined in [5, Definition 28]. By the RA analogue of lemma 2.3 proved in [5, Theorem 33], we get that $\mathcal{C}^m\alpha \not\in \mathcal{S}_\omega \mathcal{R}_{\alpha} \mathcal{C}_5$. The usual argument of taking an ultrapower of $\mathcal{C}^m\alpha$, followed by a downward elementary chain argument, one gets a countable $\mathcal{B} \in \mathcal{R}_{\alpha}$, such that $\mathcal{B} \equiv \mathcal{C}^m\alpha$ and $\exists$ has a winning strategy in $G^m(\mathcal{B})$, so $\mathcal{B} \in \mathcal{R}_{\alpha} \mathcal{C}_\omega$ because $G^m$ grips $\mathcal{R}_{\alpha} \mathcal{C}_\omega$. Hence for any $K$, such that $\mathcal{R}_{\alpha} \mathcal{C}_\omega \subseteq K \subseteq \mathcal{S}_\omega \mathcal{R}_{\alpha} \mathcal{C}_5$, we have $\mathcal{C}^m\alpha \not\in K$, $\mathcal{B} \in K$ and $\mathcal{C}^m\alpha \equiv \mathcal{B}$. 

Fix $2 < n < \omega$. Now we investigate the analogue of the result proved in Theorem ?? for several cylindric–like algebras, like Pinter’s substitution algebras ($\mathcal{S}_\alpha$) and quasi-polyadic (equality) algebras, denoted by QA, (QEA). We start with a lemma that generalizes [20, Theorem 5.1.4] to any class of algebras between $\mathcal{S}_\alpha$ and QEA. Such a result was proved in several publications of the author’s dealing separately with different cases; relevant references can be found among the references in the article [20] collected at the end of [2]. These references include (the non exhaustive list) [16, 17, 18, 19]. Following [4] in the next proof $\mathcal{P}_{\alpha}$ denotes the class of polyadic equality set algebras of dimension $\alpha$. For a Boolean algebra with operators $\mathcal{A} \in \mathcal{B}_{\alpha}$, $\mathcal{B} \mathcal{A}$ denotes its Boolean reduct.

**Lemma 3.14.** For any ordinal $\alpha > 1$, and any uncountable cardinal $\kappa \geq |\alpha|$, there exist completely representable algebras $\mathcal{A}, \mathcal{B} \in \mathcal{Q}_{\alpha} \mathcal{A}$, that are weak set algebras, such that $|\mathcal{A}| = |\mathcal{B}| = \kappa$, $\mathcal{A} \in \mathcal{N}_\alpha \mathcal{Q}_{\alpha+1}$, $\mathcal{R}_{\alpha} \mathcal{S}_\alpha \mathcal{B} \not\in \mathcal{N}_\alpha \mathcal{S}_\alpha \mathcal{C}_\alpha$, $\mathcal{A} \equiv_\omega \mathcal{B}$ and $\mathcal{A} \mathcal{B} \equiv_\infty \mathcal{A}$.

**Proof.** Here we consider only finite dimensions which is all we need. Also we count in dimension 2. Fix $1 < n < \omega$. Let $L$ be a signature consisting of the unary relation symbols $P_0, P_1, \ldots, P_{n-1}$ and uncountably many $n$–ary
predicate symbols. $M$ is as in [20, Lemma 5.1.3], but the tenary relations are replaced by $n$-ary ones, and we require that the interpretations of the $n$-ary relations in $M$ are *pairwise disjoint* not only distinct. This can be fixed. In addition to pairwise disjointness of $n$-ary relations, we require their symmetry, that is, permuting the variables does not change their semantics. In fact the construction is presented this way in [16, 17]. For $u \in {}^n n$, let $\chi_u$ be the formula $\bigwedge_{u \in {}^n n} P_u(x_i)$. We assume that the $n$-ary relation symbols are indexed by (an uncountable set) $I$ and that there is a binary operation $+$ on $I$, such that $(I, +)$ is an abelian group, and for distinct $i \neq j \in I$, we have $R_i \circ R_j = R_{i+j}$. For $n \leq k \leq \omega$, let $A_k = \{ \phi^M : \phi \in L_k \}(\subseteq \phi^{(kM)})$, where $\phi$ is taken in the signature $L$, and $\phi^M = \{ s \in {}^k M : M \models \phi[s] \}$.

Let $\mathfrak{A} = \mathfrak{A}_u$, then $\mathfrak{A} \in \mathfrak{P}e_{\omega}$ by the added symmetry condition. Also $A \cong N_{\omega} \mathfrak{A} : \omega$, the isomorphism is given by $\phi^M \mapsto \phi^M$. The map is obviously an injective homomorphism; it is surjective, because $M$ (as stipulated in [20, item (1) of lemma 5.1.3]), has quantifier elimination. For $u \in {}^n n$, let $\mathfrak{A}_u = \{ x \in \mathfrak{A} : x \leq \chi_u^M \}$. Then $\mathfrak{A}_u$ is an uncountable and atomic Boolean algebra (atomicity follows from the new disjointness condition) and $\mathfrak{A}_u \cong \text{Cof}(|I|)$, the finite–cofinite Boolean algebra on $|I|$. Define a map $f : \mathfrak{B} \mathfrak{A} \to \mathfrak{P} u \in {}^n n \mathfrak{A}_u$, by $f(a) = \langle a \cdot \chi_u \rangle_{u \in {}^n n+1}$. Let $\mathcal{P}$ denote the structure for the signature of Boolean algebras expanded by constant symbols $1_u, u \in {}^n n$, $d_{ij}$, and unary relation symbols $s_{[i,j]}$ for each $i, j \in n$. Then for each $i < j < n$, there are quantifier free formulas $\eta_i(x, y)$ and $\eta_{ij}(x, y)$ such that $\mathcal{P} \models \eta_i(f(a), b) \iff b = f(c^a_i)$ and $\mathcal{P} \models \eta_{ij}(f(a), b) \iff b = f(s_{[i,j]} a)$. The one corresponding to cylindrifiers is exactly like the CA case [20, pp.113-114]. For substitutions corresponding to transpositions, it is simply $y = s_{[i,j]} x$. The diagonal elements and the Boolean operations are easy to interpret. Hence, $\mathcal{P}$ is interpretable in $\mathfrak{A}$, and the interpretation is one dimensional and quantifier free. For $v \in {}^n n$, by the Tarski–Skölem downward theorem, let $\mathfrak{B}_v$ be a countable elementary subalgebra of $\mathfrak{A}_u$. (Here we are using the countable signature of $\text{PEA}_n$). Let $S_n(\subseteq {}^n n)$ be the set of permutations in $n$.

Take $u_1 = (0, 1, 0, \ldots, 0)$ and $u_2 = (1, 0, 0, \ldots, 0) \in {}^n n$. Let $v = \tau(u_1, u_2)$ where $\tau(x, y) = c_1(c_0 x \cdot s^1_0 y) \cdot c_1 x \cdot c_0 y$. We call $\tau$ an approximate witness. It is not hard to show that $\tau(u_1, u_2)$ is actually the composition of $u_1$ and $u_2$, so that $\tau(u_1, u_2)$ is the constant zero map; which we denote by $0$; it is also in ${}^n n$.

Clearly for every $i < j < n, s_{[i,j]}^{-n} \{ 0 \} = 0 \notin \{ u_1, u_2 \}$. We can assume without loss that the Boolean reduct of $\mathfrak{A}$ is the following product:

$$\mathfrak{A}_{u_1} \times \mathfrak{A}_{u_2} \times \mathfrak{A}_0 \times \mathfrak{P} u \in V \sim J \mathfrak{A}_u,$$

where $J = \{ u_1, u_2, 0 \}$. Let

$$\mathfrak{B} = (\mathfrak{A}_{u_1} \times \mathfrak{A}_{u_2} \times \mathfrak{B}_0 \times \mathfrak{P} u \in V \sim J \mathfrak{A}_u), 1_u, d_{ij}, s_{[i,j]} x)_{i,j < n},$$
call an contradiction, that Rd by the Feferman–Vaught theorem, we get that B existential.

In this back–forth game, \( \exists \) can maintain these two properties in every round.

Then \( \exists \) has a winning strategy in an Ehrenfeucht–Fraïssé back–forth game over the now atomic \((\mathfrak{A}, \mathfrak{B})\).

At any stage of the game, if \( \forall \) places a pebble on one of \( \mathfrak{A} \) or \( \mathfrak{B} \), \( \exists \) must place a matching pebble on the other algebra. Let \( \bar{a} = \langle a_0, a_1, \ldots, a_{m-1} \rangle \) be the position of the pebbles played so far (by either player) on \( \mathfrak{A} \) and let \( \bar{b} = \langle b_0, \ldots, b_{m-1} \rangle \) be the the position of the pebbles played on \( \mathfrak{B} \). Denote \( \chi_u^M \), by \( 1_u \). Then \( \exists \) has to maintain the following properties throughout the game:

- for any atom \( x \) (of either algebra) with \( x \cdot 1_0 = 0 \), then \( x \in a_i \) if \( x \in b_i \),

- \( \bar{a} \) induces a finite partition of \( 1_0 \) in \( \mathfrak{A} \) of \( 2^m \) (possibly empty) parts \( p_i : i < 2^m \) and the \( \bar{b} \) induces a partition of \( 1_u \) in \( \mathfrak{B} \) of parts \( q_i : i < 2^m \) such that \( p_i \) is finite iff \( q_i \) is finite and, in this case, \( |p_i| = |q_i| \).

It is easy to see that \( \exists \) can maintain these two properties in every round. In this back–forth game, \( \exists \) will always find a matching pebble, because
the pebbles in play are finite. For each \( w \in \mathbb{N} \) the component \( \mathcal{B}_w = \{ x \in \mathcal{B} : x \leq 1_w \} \)\( \subseteq \mathcal{A}_w = \{ x \in \mathcal{A} : x \leq 1_w \} \) contains infinitely many atoms. For any \( w \in V \), \( |\mathcal{A}_w| = |I| \), while for \( u \in V \sim \{0\} \), \( \mathcal{A}_u = \mathcal{A} \mathcal{B}_u \). For \( |\mathcal{A} \mathcal{B}_0| = \omega \), but it is still an infinite set. Therefore \( \mathcal{A} \equiv_\infty \mathcal{B} \). It is clear that the above argument works for any \( \mathcal{C} \) such that \( \mathcal{A} \mathcal{C} = \mathcal{A} \mathcal{B} \), hence \( \mathcal{B} \equiv_\infty \omega \mathcal{C} \).

\[ \square \]

To obtain the required result (generalizing Theorem 3.5 to any class of algebras between \( \mathcal{S}\mathcal{c} \) and \( \mathcal{Q}\mathcal{E}\mathcal{A} \) (to be formulated in a while as Theorem 3.15) the following changes are needed: wn that \( \exists \) has a winning strategy in \( \mathcal{H}_c^q(\mathcal{A} \mathcal{C}) \).

- modifying the game \( \mathcal{H} \) to the \( \mathcal{Q}\mathcal{E}\mathcal{A} \) case, call the new game \( \mathcal{H}^q \). Here one has to modify only the ‘network part’ of \( \lambda \)-neat hypernetworks by adding a symmetry condition \( (s_{i,j})N(\bar{x}) = N(\bar{x} \circ [i, j]) \),

- working with \( \mathcal{C}_{Z,N} \) now expanded with the unary operations \( s_{i,j} \) \((i < j < n)\), call the resulting rainbow algebra \( \mathcal{C}_{Z,N}^q \) \( (\in \mathcal{R}\mathcal{Q}\mathcal{E}\mathcal{A}_n) \),

- generalizing winning strategy’s \((\forall \) given in Theorem 3.5 in \( \mathcal{H}_c^q(\mathcal{A} \mathcal{C}_{Z,N}) \) to the game \( \mathcal{H}_c^q(\mathcal{A} \mathcal{R}\mathcal{D}_s \mathcal{C}_{Z,N}^q) \) and \( \forall \)’s winning strategy in \( \mathcal{H}_c^q(\mathcal{A} \mathcal{C}_{Z,N}^q) \) for each \( k < \omega \); the last is given in the proof of the same theorem. Then one uses the (easy) modification of lemma 3.7 to the \( \mathcal{Q}\mathcal{E}\mathcal{A}_n \) case (in the modification, for example in defining the weak model \( \mathcal{M}_n \) \((a \in \alpha)\) one adds a clause to formulas for satisfiability of the unary connectives \( s_{i,j} \) \((i < j < n)\) interpreted as swapping the \( i \)th and \( j \)th variables). Using lemma 2.3, one shows that \( \mathcal{R}\mathcal{D}_s \mathcal{C}_{Z,N}^q \notin \mathcal{S}_c \mathcal{N}_{n+1} \mathcal{C}_{n+3}^d \). Exactly the \( \mathcal{C}A \) case, we get that any class \( \mathcal{K} \) between \( \mathcal{S}\mathcal{c} \) nd \( \mathcal{Q}\mathcal{E}\mathcal{A} \), and any class between the two classes \( \mathcal{S}_d \mathcal{N}_{n+1} \mathcal{K}_n \cap \mathcal{C} \mathcal{R} \mathcal{K}_n \) and \( \mathcal{S}_c \mathcal{N}_{n+1} \mathcal{K}_n^{ad} \), is not elementary,

- to remove the \( \mathcal{S}_d \), from \( \mathcal{S}_d \mathcal{N}_{n+1} \mathcal{K}_n \) like was done in the \( \mathcal{C}A \) case, one adjoins the (splitting) construction in [17] for classes between \( \mathcal{S}\mathcal{c} \) and \( \mathcal{Q}\mathcal{E}\mathcal{A} \) in place of the modification of the construction in [20, Theorem 5.1.4] (as appeared [18]) addressing only the \( \mathcal{C}A \) case. The construction in [24] shows that this last item is necessary. In \( \textit{opcit} \), an atomic \( \mathcal{C} \in \mathcal{R}\mathcal{Q}\mathcal{E}\mathcal{A}_n \) such that \( \mathcal{C} \mathcal{m} \mathcal{A} \mathcal{C} \in \mathcal{N}_{n+3} \mathcal{Q}\mathcal{A}_n \) and \( \mathcal{A} \mathcal{C} \in \mathcal{A} \mathcal{N}_{n+3} \mathcal{Q}\mathcal{E}\mathcal{A}_n \), but \( \mathcal{R}\mathcal{D}_s \mathcal{A} \notin \mathcal{N}_{n+3} \mathcal{S}_{n+1} \mathcal{C}_{n+3} \), is constructed. In fact, this \( \mathcal{C} \) is simple (has no proper ideals), so that it is a set algebra of dimension \( n \) which is the \( \mathcal{Q}\mathcal{E}\mathcal{A}_n \) generated by the same generators of \( \mathcal{C} \) in the first item of Theorem 2.4 but now in the full \( \textit{quasi}polyadic \ equality \ set \ algebra \ with \ top \ element \ n\mathcal{Q} \). Like in Theorem 3.11, it can be shown that \( \exists \) has a winning strategy in \( \mathcal{H}_c^q(\mathcal{A} \mathcal{C}) \).

So for diagonal free reducts of \( \mathcal{Q}\mathcal{E}\mathcal{A} \) namely, \( \mathcal{S}\mathcal{c} \) and \( \mathcal{Q}\mathcal{A} \), we obtain the weaker result which we formulate only for \( \mathcal{S}\mathcal{c}s \). The Theorem is true for any
class between $\mathcal{Sc}$ and $\mathcal{QA}$. Recall that $\mathcal{CRSc}_n$ denotes the class of completely representable $\mathcal{Sc}_n$s.

**Theorem 3.15.** Any class between $\mathcal{CRSc}_n \cap \mathcal{Nr}_n \mathcal{Sc}_\omega$ and $\mathcal{S}_c \mathcal{Nr}_n \mathcal{Sc}^{ad}_{n+3}$ is not elementary.

Complete additivity appears on the left hand side (giving a smaller class than the class $\mathcal{S}_c \mathcal{Nr}_n \mathcal{Sc}_{n+3}$) due to the intrusion of lemma 2.3. A discrepancy that deserves to be highlighted here is that in the case of non–atom canonicity, proved in Theorem 3.11, though the same lemma 2.3 was also used, additivity did not interfere at all for diagonal free reducts (like $\mathcal{Sc}_s$s) because algebras, more specifically dilations of algebras used, were finite.

**Corollary 3.16.** Let $2 < n < \omega$, $K$ any class between $\mathcal{Sc}$ and $\mathcal{QEA}$, $m > n$, and $k \geq 3$. Then the following classes are not elementary: $\mathcal{CRK}_n \ [7, 14]$, $\mathcal{Nr}_n \mathcal{K}_m$, $\mathcal{Nr}_2 \mathcal{K}_m$, $\mathcal{S}_d \mathcal{Nr}_n \mathcal{K}_{n+k}$ and $\mathcal{S}_c \mathcal{Nr}_n \mathcal{K}_{n+k}$.

**Corollary 3.17.** Let $k \geq 5$. Then the classes $\mathcal{CRRA}$, $\mathcal{RaCA}_k$, $\mathcal{S}_d \mathcal{RaCA}_k$ and $\mathcal{S}_c \mathcal{RaCA}_k$ are not elementary $[7, 5]$. The first two classes are not closed under $\equiv_{\omega}$, but are closed under ultraproducts. Furthermore, $\mathcal{RaCA}_\omega \subseteq \mathcal{S}_d \mathcal{RaCA}_\omega \subsetneq \mathcal{S}_c \mathcal{RaCA}_\omega$.

**Proof.** The first two classes are closed under ultraproducts because they are psuedo-elementary (reducts of elementary classes), cf. [8, Item (2), p. 279], [5, Theorem 21]. Proving the strictness of the last inclusion can be easily distilled from the proof of [5, Theorem 36].

**References**

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Received: December 5, 2019; Published: January 16, 2020