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$\mathfrak{Ra}\mathbf{CA}_n$ IS NOT ELEMENTARY, FOR $n \geq 5$

Abstract

We show that the class $\mathfrak{Ra}\mathbf{CA}_n$ of relation algebras reducts of cylindric algebras of dimension n is not elementary when $n \geq 5$. This answers a long standing open question in algebraic logic posed by Németi and independently by Maddux.

1. Introduction

The notion of neat reducts is a venerable old notion in cylindric algebras that is gaining some momentum. Indeed, the notion of neat reducts has been studied quite intensely lately [1], [2], [3], [4], [6], [13], [12]. [1], [2] and [3] deal with the notion of neat reducts on its own, while [4] and [6] deal with the related notion of *neat embeddings* in connection to amalgamation and representability. In [13] neat reducts are studied in connection to proof theory. Such results are surveyed in [9] and [7].

The investigation of the relationship between cylindric algebras **CA** and relation algebras **RA** goes via neat reducts. Given $\mathfrak{A} \in \mathbf{CA}_n$ with $n > 2$, one can obtain an algebra similar to relation algebras by forming the neat 2-reduct, that is taking the 2 - dimensional elements of $\mathfrak{A}$ and defining converse and composition using a spare dimension. When $n \geq 4$ this algebra, denoted by $\mathfrak{Ra}\mathfrak{A}$ happens to be a relation algebra. For $n = 3$, $\mathfrak{Ra}\mathfrak{A}$ is only a semi-associative relation algebra. The investigation of the relationship between $\mathbf{CA}_n$ and **RA** has always been, and continues to be, a popular rich topic in algebraic logic. Tarski conjectured that the study of **RA** can be reduced to the study of some subclass of $\mathbf{CA}_3$. In this direction Monk [17] discovered a class K properly contained in

$\mathbf{CA}_3$ and proved that $\mathbf{RA} = \mathfrak{Ra}K$, where $\mathfrak{Ra}L$ for a class L denotes the class of relation algebra reducts of algebras in L. The definition of K however, referred explicitly to the operations of $\mathfrak{Ra}\mathbf{CA}_3$, therefore it was expected to try and substitute K with some (more) natural subclass of $\mathbf{CA}_3$ whose operations involve only the cylindric operations. For $\alpha < \beta$ and $L \subseteq \mathbf{CA}_\beta$, let $\mathfrak{Nr}_\alpha L$ denote the class of all neat α - reducts of algebras in L. Henkin and Tarski [11] proved that for $\beta \geq 4$, $\mathfrak{Nr}_3\mathbf{CA}_\beta \subseteq K$. Therefore, it followed that $\mathfrak{Ra}\mathfrak{Nr}_3 CA_\beta \subseteq \mathbf{RA}$. But Monk [17] proved that the other inclusion fails when $\beta \geq 5$. The problem thus arose as to whether $\mathbf{RA} \subseteq \mathfrak{Ra}\mathfrak{Nr}_3\mathbf{CA}_4$ or not. An equivalent form of this is whether $\mathbf{RA} = \mathfrak{Ra}\mathbf{CA}_4$. A partial positive solution was found by Maddux quoted as Theorem 5.3.17 in the monograph [11] which says that $\mathbf{RA} = \mathbf{S}\mathfrak{Ra}\mathbf{CA}_4$ where $\mathbf{S}$ is the operation of forming subalgebras. Answering Monk's original question amounts to deciding whether or not $\mathbf{S}$ can be dropped in the preceding equation. Németi and Simon [15] prove that for $n > 2$, $\mathfrak{Ra}\mathbf{CA}_n$ is not closed under forming subalgebras, hence is not a variety. In other words, the $\mathbf{S}$ cannot be dropped. Independently Maddux [16] proved the same result. It is easy to show that $\mathfrak{Ra}\mathbf{CA}_n$ is closed under products and homomorphic images, and is in fact a pseudo-elementary class (A reduct of an elementary class.) It is therefore closed under ultraproducts. A natural question in this context (posed independently by Maddux and Németi) is whether this class itself is elementary or not, i.e. whether it is closed under ultraroots or equivalently elementary equivalence. In this note we show that $\mathfrak{Ra}\mathbf{CA}_n$ when $n \geq 5$ is not closed under elementary equivalence, hence is not elementary. The cases $n = 3, 4$ remain open. We start by the basic definitions needed:

2. Proof

Our proof, that $\mathfrak{Ra}\mathbf{CA}_n$ is not elementary for $5 \leq n \leq \omega$, is based on the relation algebra $\mathfrak{A}(3, \omega)$ of [14], which in turn is based on a construction used by Monk and Maddux.

Definition 1. $\mathcal{M}$ is a symmetric, atomic relation algebra with atoms $1', \mathsf{r}(i), \mathsf{y}(i), \mathsf{b}(i) : i < \omega$. Non-identity atoms have colours, r is red, b is blue, and y is yellow. All atoms are self-converse. Composition of atoms is defined by listing the forbidden triples .

The set of forbidden triples is the complement of the set of consistent triples. If (a, b, c) is a forbidden triple of atoms, then so are its Peircean transforms $(a, b, c), (b, \check{c}, \breve{a}), (\check{c}, a, \breve{b}), (\breve{b}, \breve{a}, \check{c})(\breve{a}, \check{c}, b), (c, \breve{b}, a)$. The forbidden triples are (Peircean transforms) or permutations of $(1', x, y)$ for $x \neq y$, and

$$(\mathsf{r}(i), \mathsf{r}(i), \mathsf{r}(j)),\ (\mathsf{y}(i), \mathsf{y}(i), \mathsf{y}(j)),\ (\mathsf{b}(i), \mathsf{b}(i), \mathsf{b}(j))\ \ i \leq j < \omega$$

$\mathcal{M}$ is the complex algebra over this atom structure.

Let α be an ordinal. $\mathcal{M}^\alpha$ is obtained from $\mathcal{M}$ by splitting the atom $\mathsf{r}(0)$ into α parts $\mathsf{r}^k(0) : k < \alpha$. and then taking the full complex algebra. In more detail, we put red atoms $r^k(0)$ for $k < \alpha$. In the altered algebra the forbidden triples are

$$(\mathsf{y}(i), \mathsf{y}(i), \mathsf{y}(j)), \qquad i \leq j < \omega$$

$$(\mathsf{b}(i), \mathsf{b}(i), \mathsf{b}(j)), \qquad i \leq j < \omega$$

$$(\mathsf{r}(i), \mathsf{r}(i), \mathsf{r}(j)), \qquad 0 < i \leq j < \omega$$

$$(\mathsf{r}^k(0), \mathsf{r}^l(0), \mathsf{r}(j)), \qquad 0 < j < \omega, k, l < \alpha$$

$$(\mathsf{r}^k(0), \mathsf{r}^l(0), \mathsf{r}^m(0)), \qquad k, l, m < \alpha$$

The *index* of $\mathsf{r}(i), \mathsf{y}(i)$ and $\mathsf{b}(i)$ is i and the index of $\mathsf{r}^k(0)$ is also 0. In $\mathcal{M}^\alpha$, we use the following abbreviations:

$$\begin{aligned} \mathsf{r}(0) &= \sum_{k<\alpha} \mathsf{r}^k(0) \\ \mathsf{r} &= \sum_{i<\omega} \mathsf{r}(i) \\ \mathsf{y} &= \sum_{i<\omega} \mathsf{y}(i) \\ \mathsf{b} &= \sum_{i<\omega} \mathsf{b}(i) \end{aligned}$$

Let $\mathfrak{A} = \mathcal{M}^{\omega_1}$ (ω_1 is the first uncountable ordinal), and $\mathfrak{B} = \mathcal{M}^\omega$.

We will prove the following: (i) $\mathfrak{A} \notin \mathfrak{RaCA}_5$, (ii) $\mathfrak{B} \in \mathfrak{RaCA}_\omega$ and (iii) $\mathfrak{A} \equiv \mathfrak{B}$.

Notation, cf. [14] 13.30. For $\bar{a} \in {}^{<n}n$ we write $\mathsf{s}_{\bar{a}}$ for an arbitrary string of substitutions w such that $\hat{w} = \bar{a}$.

In more detail. Let $n \geq 3$ and $i, j < n$. We define a string of substitutions s_{ij}:

$$\mathsf{s}_{ij} = \mathsf{s}_i^0 \mathsf{s}_j^1, \text{ if } j \neq 0$$

$$\mathsf{s}_{ij} = \mathsf{s}_0^1 \mathsf{s}_i^0 \text{ iff } j = 0, i \neq 1$$

$$\mathsf{s}_{ij} = \mathsf{s}_0^2 \mathsf{s}_1^0 \mathsf{s}_2^1 \text{ iff } j = 0, i = 1$$

We'll need the following lemma [14] 13.31:

LEMMA 1. *Let $\mathfrak{C} \in \mathbf{CA}_n$, $i, j, k < n,$, $k \neq i, j$. Then $\mathsf{s}_{ij}(r; s) = \mathsf{c}_k(\mathsf{s}_{ik} r.\mathsf{s}_{kj} s)$, for all $r, s \in \mathfrak{Ra}\mathfrak{C}$.*

THEOREM 2. $\mathfrak{A} \notin \mathfrak{Ra}\mathbf{CA}_5$.

PROOF: Suppose $\mathfrak{A} = \mathfrak{Ra}\mathfrak{C}$ for some $\mathfrak{C} \in \mathbf{CA}_5$. Then $\mathsf{c}_2 \mathsf{r}^0(0) = \mathsf{r}^0(0) \leq \mathsf{r}^k(0); \mathsf{y}(0) = \mathsf{c}_2(\mathsf{s}_2^1 \mathsf{r}^k(0).\mathsf{s}_2^0 \mathsf{y}(0))$, for each $k < \omega_1$. Therefore

$$x_k = \mathsf{r}^0(0).\mathsf{s}_2^1 \mathsf{r}^k(0).\mathsf{s}_2^0 \mathsf{y}(0) \neq 0.$$

Note that $x_0 \leq \mathsf{r}^k(0); \mathsf{y}(0) = \mathsf{c}_3(\mathsf{s}_3^1 \mathsf{r}^k(0).\mathsf{s}_3^0 \mathsf{y}(0))$ so $\mathsf{s}_3^1 \mathsf{r}^k(0).\mathsf{s}_3^0 \mathsf{y}(0).x_0 \neq 0$. Hence

$$\begin{aligned} x_0 &= \mathsf{c}_3(\mathsf{s}_3^2 x_k.\mathsf{s}_2^1 \mathsf{r}^0(0).\mathsf{s}_2^0 \mathsf{y}(0)) \\ &\leq \mathsf{c}_3(\mathsf{s}_3^2 x_k.\mathsf{s}_{23} 1) \\ &= \mathsf{c}_3(\mathsf{s}_3^2 x_k.\mathsf{s}_{23} \sum_{a \in At\mathcal{A}} a) \\ &= \sum_{a \in At\mathcal{A}} \mathsf{c}_3(\mathsf{s}_3^2 x_k.\mathsf{s}_{23} a) \end{aligned}$$

Next we'll show that $x_0. \sum_{a \in \{1', \mathsf{r}^t(0), \mathsf{y}(i): i < \omega,\ t < \omega_1\}} \mathsf{c}_3(\mathsf{s}_3^2 x_k.\mathsf{s}_{23} a) = 0$. This will show that $x_0 \leq \sum_{i<\omega} \mathsf{c}_3(\mathsf{s}_3^2 x_k.\mathsf{s}_{23} \mathsf{b}(i))$.

For $a \in At\mathfrak{A}$,

$$\begin{aligned} x_0.\mathsf{c}_3(\mathsf{s}_3^2 x_k.\mathsf{s}_{23} a) &\leq \mathsf{c}_3(\mathsf{s}_2^1 \mathsf{r}^0(0).\mathsf{s}_3^1 \mathsf{r}^k(0).\mathsf{s}_{23} a) \\ &\leq \mathsf{c}_3(\mathsf{c}_2(\mathsf{s}_{02} \mathsf{r}^0(0).\mathsf{s}_{23} a(k)).\mathsf{s}_{03} \mathsf{r}^k(0)) \\ &= \mathsf{c}_3 \mathsf{s}_{03}(\mathsf{r}^0(0); a \,.\, \mathsf{r}^k(0)) \end{aligned}$$

So if $a \in \{1', \mathsf{r}^t(0) : t < \omega_1\}$, then $x_0.\mathsf{c}_3(\mathsf{s}_3^2 x_k.\mathsf{s}_{23}a) = 0$. Similarly, because $x_0.\mathsf{c}_3(\mathsf{s}_3^2 x_k.\mathsf{s}_{23}a) \leq \mathsf{s}_2^0\mathsf{y}(0).\mathsf{s}_3^0\mathsf{y}(0).\mathsf{s}_{23}a$, we can deduce that $x_0.\mathsf{c}_3(\mathsf{s}_3^2 x_k.\mathsf{s}_{23}a) \leq \mathsf{c}_3\mathsf{s}_{21}(\mathsf{y}(0)\ .\ a; \mathsf{y}(0))$ and if $a \in \{\mathsf{y}(i) : i < \omega\}$ this is zero. So

$$x_0 \leq \sum_{i<\omega} \mathsf{c}_3(\mathsf{s}_3^2 x_k \mathsf{s}_{23}\mathsf{b}(i))$$

This holds for all $0 < k < \omega_1$. So,

$$\begin{aligned} x_0 &\leq \prod_{0<k<\omega_1} \sum_{i<\omega} \mathsf{c}_3(\mathsf{s}_3^2 x_k.\mathsf{s}_2^0\mathsf{s}_3^1 b(i)) \\ &= \sum_{g:\omega_1\setminus\{0\}\to\omega} \prod_{0<k<\omega_1} \mathsf{c}_3(\mathsf{s}_3^2 x_k.\mathsf{s}_2^0\mathsf{s}_3^1 b(g(k))) \end{aligned}$$

So there is a function $g : \omega_1 \setminus \{0\} \to \omega$ such that

$$x_0\ .\ \prod_{0<k<\omega_1} \mathsf{c}_3(\mathsf{s}_3^2 x_k.\mathsf{s}_2^0\mathsf{s}_3^1\mathsf{b}(g(k))) \neq 0$$

Pick $i < \omega$ such that $X = g^{-1}(i)$ is uncountable, or at least infinite (we need an uncountable number of superscripts k at this point only). Then

$$\xi = x_0\ .\ \prod_{k\in X} \mathsf{c}_3(\mathsf{s}_3^2 x_k.\mathsf{s}_2^0\mathsf{s}_3^1\mathsf{b}(i)) \neq 0$$

Let

$$z_k = \mathsf{s}_3^2 x_k.\mathsf{s}_2^0\mathsf{s}_3^1\mathsf{b}(i).\xi$$

for each $k \in X$. Let $S_0 = \{z_k : k \in X\}$. S_0 has the following properties. There is an index $i < \omega$ such that for all $z, x \in S_0$,

1. $\mathsf{c}_4 z = z$
2. $\mathsf{c}_3 z = \mathsf{c}_3 x$
3. (a) $\exists k < \omega_1$ such that $z \leq \mathsf{s}_3^1\mathsf{r}^k(0)$
   (b) $\forall k < \omega_1$ if $z, x \leq \mathsf{s}_3^1\mathsf{r}^k(0)$ then $z = x$
4. $z \leq \mathsf{s}_2^1\mathsf{r}^0(0).\mathsf{s}_2^0\mathsf{y}(0).\mathsf{s}_3^0\mathsf{y}(0).\mathsf{s}_2^0\mathsf{s}_3^1\mathsf{b}(i)$
5. S_0 is infinite.

Suppose there is an infinite set S and an index $i < \omega$ with the properties listed above. We show how to construct another infinite set S' and a new index $i' < i$ with the same properties. Iterating this construction $i + 1$

times will then lead to a contradiction since the index cannot be less than 0.

Fix $z \in S$. For each $x \in S \setminus \{z\}$ and $j < i$, let

$$\tau_x^j = \mathsf{c}_4(\mathsf{s}_4^3 x.\mathsf{s}_2^0\mathsf{s}_4^1\mathsf{b}(j)).$$

LEMMA 3. *For any $x \in S \setminus \{z\}$,*

$$z \leq \sum_{j<i} \tau_x^j$$

PROOF:

$$z.\mathsf{s}_4^3 x \leq 1 = \mathsf{s}_3^0\mathsf{s}_4^1 1 = \mathsf{s}_3^0\mathsf{s}_4^1 \sum_{a \in At\mathcal{A}} a = \sum_{a \in At\mathcal{A}} \mathsf{s}_3^0\mathsf{s}_4^1 a$$

We'll show that $z.\mathsf{s}_4^3 x.\mathsf{s}_3^0\mathsf{s}_4^1 a = 0$ if $a \in \{1', \mathsf{r}^k(i), \mathsf{y}(i) : i < \omega,\ k < \omega_1\}$.

Let $a \in At\mathcal{A}$ and suppose $w = z.\mathsf{s}_4^3 x.\mathsf{s}_3^0\mathsf{s}_4^1 a \neq 0$. By property 3, $z \leq \mathsf{s}_3^1\mathsf{r}^k(0)$ and $x \leq \mathsf{s}_3^1\mathsf{r}^l(0)$ for some $k \neq l < \omega_1$. So $w \leq \mathsf{s}_4^3 x \leq \mathsf{s}_0^0\mathsf{s}_4^1\mathsf{r}^l(0)$. Therefore

$$w \leq \mathsf{s}_0^0\mathsf{s}_3^1\mathsf{r}^k(0).\mathsf{s}_3^0\mathsf{s}_4^1 a.\mathsf{s}_0^0\mathsf{s}_4^1\mathsf{r}^l(0)$$

By Lemma 1,

$$w \leq \mathsf{s}_0^0\mathsf{s}_4^1(\mathsf{r}^k(0); a\ .\ \mathsf{r}^l(0))$$

and so $\mathsf{r}^k(0); a\ .\ \mathsf{r}^l(0) \neq 0$. This means that $a \notin \{1', \mathsf{r}^m(0) : m < \omega_1\}$.

Similarly, $x, z \leq \mathsf{s}_3^0\mathsf{y}(0)$ so

$$\mathsf{s}_4^3 x \leq \mathsf{s}_4^3\mathsf{s}_3^0\mathsf{y}(0) = \mathsf{s}_4^0\mathsf{s}_0^3\mathsf{y}(0) = \mathsf{s}_4^0\mathsf{y}(0)$$

and

$$\begin{aligned} w &\leq \mathsf{s}_3^0\mathsf{s}_4^1 a.\mathsf{s}_4^0\mathsf{s}_1^1\mathsf{y}(0).\mathsf{s}_3^0\mathsf{s}_1^1\mathsf{y}(0) \\ &\leq \mathsf{s}_3^0\mathsf{s}_1^1(a; \mathsf{y}(0)\ .\ \mathsf{y}(0)) \end{aligned}$$

So $a; \mathsf{y}(0)\ .\ \mathsf{y}(0) \neq 0$ and $a \notin \mathsf{y}(i) : i < \omega$.

Finally, $x, z \leq \mathsf{s}_2^0\mathsf{s}_3^1\mathsf{b}(i)$, by property 4, so

$$\begin{aligned} \mathsf{s}_4^3 x &\leq \mathsf{s}_4^3\mathsf{s}_2^0\mathsf{s}_3^1\mathsf{b}(i) \\ &= \mathsf{s}_2^0\mathsf{s}_4^3\mathsf{s}_3^1\mathsf{b}(i) \\ &= \mathsf{s}_2^0\mathsf{s}_4^1\mathsf{s}_1^3\mathsf{b}(i) \\ &= \mathsf{s}_2^0\mathsf{s}_4^1\mathsf{b}(i) \end{aligned}$$

Hence

$$w \leq \mathsf{s}_2^0\mathsf{s}_4^1\mathsf{b}(i).\mathsf{s}_2^0\mathsf{s}_3^1\mathsf{b}(i).\mathsf{s}_3^0\mathsf{s}_4^1 a \leq \mathsf{s}_2^0\mathsf{s}_4^1(\mathsf{b}(i)\ .\ \mathsf{b}(i); a)$$

and so $\mathsf{b}(i)\ .\ \mathsf{b}(i); a \neq 0$. Therefore $a \notin \{\mathsf{b}(j); i \leq j < \omega\}$.

It follows that $z.\mathsf{s}_4^3 x.\mathsf{s}_3^0\mathsf{s}_4^1 a \neq 0$ implies $a \in \{\mathsf{b}(j) : j < i\}$. Thus $z.\mathsf{s}_4^3 x \leq \sum_{j<i} \mathsf{s}_3^0\mathsf{s}_4^1\mathsf{b}(j)$.

Now to prove the lemma, observe that

$$\begin{aligned}
z &= z.\mathsf{c}_3 z = z.\mathsf{c}_3 x \\
&= z.\mathsf{c}_3\mathsf{s}_3^4 x = z.\mathsf{c}_4\mathsf{s}_4^3 x \\
&= \mathsf{c}_4(z.\mathsf{s}_4^3 x) \\
&= \mathsf{c}_4(z.\mathsf{s}_4^3 x.\sum_{j<i}\mathsf{s}_3^0\mathsf{s}_4^1\mathsf{b}(j) \\
&\leq \mathsf{c}_4(\mathsf{s}_4^3 x.\sum_{j<i}\mathsf{s}_3^0\mathsf{s}_4^1\mathsf{b}(j) \\
&= \mathsf{c}_4\sum_{j<i}(\mathsf{s}_4^3 x.\mathsf{s}_3^0\mathsf{s}_4^1\mathsf{b}(j)) \\
&= \sum_{j<i}\mathsf{c}_4(\mathsf{s}_4^3 x.\mathsf{s}_3^0\mathsf{s}_4^1\mathsf{b}(j)) \\
&= \sum_{j<i}\tau_x^j
\end{aligned}$$

□

We now construct S' and i' from S and i with the required properties. By the lemma,

$$z \leq \prod_{x\in S\setminus z}\sum_{j<i}\tau_x^j = \sum_{g:S\setminus\{z\}\to i}\prod_{x\in S\setminus\{z\}}\tau_x^{g(x)}$$

Since $z \neq 0$ there is $g : S \setminus \{z\} \to i \quad (= \{0, 1, \ldots, i-1\})$ such that $z.\prod_{x\in S\setminus\{z\}}\tau_x^{g(x)} \neq 0$. Pick $i' < i$ such that $X = g^{-1}(i')$ is infinite. Then

$$z.\prod_{x\in X}\tau_x^{i'} \neq 0$$

Define

$$\begin{aligned}
\xi &= z.\prod_{x\in X}\tau_x^{i'} \quad \neq 0 \\
x' &= \mathsf{s}_4^3 x.\mathsf{s}_3^0\mathsf{s}_4^1\mathsf{b}(i').\xi \\
x" &= \mathsf{s}_3^4\mathsf{s}_2^3\mathsf{c}_2 x' \\
S' &= \{x" : x \in X\}
\end{aligned}$$

We check each of the properties. Property 1 is obvious. By property 3 for S, if $x \in X$ then there is $k < \omega_1$ and $x \le \mathsf{s}_3^1\mathsf{r}^k(0)$. So

$$\begin{aligned} x\text{''} &\le \mathsf{s}_3^4\mathsf{s}_2^3\mathsf{c}_2\mathsf{s}_4^3\mathsf{s}_3^1\mathsf{r}^k(0) \\ &= \mathsf{s}_3^4\mathsf{s}_2^3\mathsf{c}_2\mathsf{s}_4^1\mathsf{s}_1^3\mathsf{r}^k(0) \\ &= \mathsf{s}_3^4\mathsf{s}_2^3\mathsf{s}_4^1\mathsf{c}_2\mathsf{r}^k(0) \\ &= \mathsf{s}_3^4\mathsf{s}_4^1\mathsf{s}_2^3\mathsf{r}^k(0) \\ &= \mathsf{s}_3^1\mathsf{s}_1^4\mathsf{r}^k(0) \\ &= \mathsf{s}_3^1\mathsf{r}^k(0) \end{aligned}$$

This gives property 3 for S' and shows that S' is infinite (property 5).

For property 2 we first prove that if $x \in S$ then $\mathsf{c}_4 x' = \xi$. First note that $\mathsf{c}_4\xi = \xi$, so

$$\begin{aligned} \mathsf{c}_4 x' &= \mathsf{c}_4(\mathsf{s}_4^3 x.\mathsf{s}_{34}\mathsf{b}(i').\xi) \\ &= \mathsf{c}_4(\mathsf{s}_4^3 x.\mathsf{s}_{34}\mathsf{b}(i')).\xi \\ &= \tau_x^{i'}.\xi \\ &= \xi \end{aligned}$$

Hence,

$$\begin{aligned} \mathsf{c}_3 x'' &= \mathsf{c}_3\mathsf{s}_3^4\mathsf{s}_2^3\mathsf{c}_2 x' \\ &= \mathsf{c}_4\mathsf{s}_3^4\mathsf{s}_2^3\mathsf{c}_2 x' \\ &= \mathsf{c}_4\mathsf{s}_2^3\mathsf{c}_2 x' \\ &= \mathsf{s}_2^3\mathsf{c}_2\mathsf{c}_4 x' \\ &= \mathsf{s}_2^3\mathsf{c}_2\xi \end{aligned}$$

which gives property 2.

Finally, for property 4, we prove that $x'' \le \mathsf{s}_2^1\mathsf{r}^0(0).\mathsf{s}_2^0\mathsf{y}(0).\mathsf{s}_3^0\mathsf{y}(0).\mathsf{s}_2^0\mathsf{s}_3^1\mathsf{b}(i)$, for each $x'' \in S'$. Property 4 for S says that $x \le \mathsf{s}_3^0\mathsf{y}(0)$ and since $x' \le \mathsf{s}_4^3 x$ we get $x' \le \mathsf{s}_4^0\mathsf{y}(0)$. Therefore

$$\begin{aligned} x'' &\le \mathsf{s}_3^4\mathsf{s}_2^3\mathsf{c}_2\mathsf{s}_{41}\mathsf{y}(0) \\ &= \mathsf{s}_3^4\mathsf{s}_2^3\mathsf{c}_2\mathsf{s}_4^0\mathsf{y}(0) \\ &= \mathsf{s}_3^4\mathsf{s}_2^3\mathsf{s}_4^0\mathsf{c}_2\mathsf{y}(0) \\ &= \mathsf{s}_3^4\mathsf{s}_4^0\mathsf{s}_2^3\mathsf{y}(0) \\ &= \mathsf{s}_3^0\mathsf{s}_0^4\mathsf{y}(0) \\ &= \mathsf{s}_3^0\mathsf{y}(0) \end{aligned}$$

Similarly, we can show that $x'' \leq \mathsf{s}_2^1\mathsf{r}^0(0).\mathsf{s}_2^0\mathsf{y}(0)$. And $x' \leq \mathsf{s}_3^0\mathsf{s}_4^1\mathsf{b}(i')$ gives $x'' \leq \mathsf{s}_3^4\mathsf{s}_2^3\mathsf{c}_2\mathsf{s}_3^0\mathsf{s}_4^1\mathsf{b}(i') \leq \mathsf{s}_2^0\mathsf{s}_3^1\mathsf{b}(i')$. This proves property 4.

□

PROBLEM 1. *If $\mathcal{D}$ is an atomic relation algebra with no n-dimensional hyperbasis, does it follow that $\mathcal{D} \notin \mathfrak{RaCA}_n$? It is known that there is no complete relation algebra embedding of $\mathcal{D}$ into an atomic n-dimensional cylindric algebra, but we must also take the non-atomic cylindric algebras into account.*

For the cylindric-basis game the reader is referred to [14].

LEMMA 4. *$\exists$ can win the cylindric-basis game $G_\omega(\mathfrak{B})$.*

PROOF: In this game, $\forall$ can play *atom moves*, *triangle moves* and *amalgamation moves*. At some stage of the game, let the play so far be

$$N_0, N_1, \ldots, N_{t-1}$$

for some $t < \omega$. We say that an edge (m, n) of an atomic network N is a *diversity edge* if $N(m, n).1' = 0$. Each diversity edge of each atomic network in the play has an owner — either $\exists$ or $\forall$, which we will allocate as we define $\exists$'s strategy. If an edge (m, n) belongs to player p then so does the reverse edge (n, m) and we will only specify one of them. Our algebra is symmetric, so the label of the reverse edge is equal to the label of the edge. Again, we only specify one.

For the next round $\exists$ must define N_t in response to $\forall$'s move. Whatever move he makes, if there is an already played network N_i (some $i < t$) and a finitary map $\sigma : \omega \to \omega$ such that $N_t\sigma$ 'answers' his move, then she lets $N_t = N_i\sigma$. Recall that $N_i\sigma$ has labelling defined by $N_i\sigma(m, n) = N_i(\sigma(m), \sigma(n))$. Lets call this an *embedding move* by $\exists$. For each diversity edge (m, n) of N_t we let (m, n) belong to $\exists$ (respectively $\forall$) in N_t iff $(\sigma(m), \sigma(n))$ (which is a diversity edge of N_i) belongs to $\exists$ ($\forall$) in N_i.

From now on we assume that there is no such N_i and σ. We consider the three types of $\forall$ can make. If he plays an atom move by picking an atom a, $\exists$ plays an atomic network N with $N(0, 1) = a$ and for all $x \in \omega \setminus \{1\}$, $N(0, x) = 1'$. This determines N.

If $\forall$ plays a triangle move by picking a previously played N_x (some $x < t$), nodes i, j, k with $k \notin \{i, j\}$ and atoms a, b with $a; b \geq N_x(i, j)$, we know that $a, b \neq 1'$, as we are assuming the $\exists$ cannot play an embedding move (if $a = 1'$, consider N_x and the map $[k/i]$). $\exists$ must play a network $N_t \equiv_k N_x$ such that $N_t(i,k) = a$, $N_t(k,j) = b$. These edges, (i,k) and (k,j), belong to $\forall$ in N_t. All diversity edges not involving k have the same owner in N_t as they did in N_x. And all edges (l,k) for $k \notin \{i,j\}$ belong to $\exists$ in N_x. To label these edges $\exists$ chooses a colour c different to the colours of a, b. Then, one at a time, she labels each edge (l,k) by an atom with colour c and a non-zero index which has not yet been used to label any edge of any network played in the game. She does this one edge at a time, each with a new index. There are infinitely many indices to choose, so this is OK.

Thirdly, he can play an amalgamation move by picking $M, N \in \{N_s : s < t\}$, nodes i, j such that $M \equiv_{ij} N$. If there is N_s (some $s < t$) and a map $\sigma : nodes(N_s) \to nodes(M) \cup nodes(N)$ such that $M \equiv_i N_s\sigma \equiv_j N$ then $\exists$ lets $N_t = N_s\sigma$. Ownership of edges is inherited from N_s. If there is no such N_s and σ then there are two cases. If there are three nodes x, y, z in the 'amalgam' such that $M(j,x)$ and $N(x,i)$ are both red and of the same index, $M(j,y), N(y,i)$ are both yellow and of the same index and $M(j,z), N(z,i)$ are both blue and of the same index, then the new edge (i,j) belongs to $\forall$ in N_t. It will be labelled by either $\mathsf{r}^0(0), \mathsf{b}(0)$ or $\mathsf{y}(0)$ and it can be shown that at least one of these will be a consistent choice. Otherwise, if there is no such x, y, z then the new edge (i,j) belongs to $\exists$ in N_t. She chooses a colour c such that there is no x with $M(j,x)$ and $N(x,i)$ both having colour c and the same index. And she chooses a non-zero index for $N_t(i,j)$ which is new to the game (as with triangle moves). If $k \neq k' \in M \cap N$ then (j,k) has the same owner in N_t as it does in M, (k,i) has the same owner in N_t as it does in N and (k,k') belongs to $\exists$ in N_t if it belongs to $\exists$ in *either* M or N, otherwise it belongs to $\forall$ in N_t. The following Lemma is easy to check

LEMMA 5. *If $\exists$ plays according to this strategy then if $\forall$ plays an amalgamation move (M,N,i,j) in some round such that (1) j is incident with at least three edges belonging to $\forall$ in M and (2) i is incident with at least three edges belonging to $\forall$ in N, then there is a previously played network L and a map σ such that $M \equiv_i L\sigma \equiv_j N$.*

Now the only way $\exists$ could be defeated is if $\forall$ played an amalgamation move (M,N,i,j) such that there are $x, y, z \in M \cap N$ such that $M(j,x) = \mathsf{r}^k(0)$, $N(x,i) = \mathsf{r}^{k'}(0)$, $M(j,y) = N(y,i) = \mathsf{b}(0)$ and $M(j,z) = N(z,i) =$

$\mathsf{y}(0)$. But according to $\exists$'s strategy, she never chooses atoms with index 0, so all these edges must have been chosen by $\forall$. This contradiction proves the Theorem. $\square$

THEOREM 6. $\mathfrak{B} \in \mathfrak{RaCA}_\omega$.

PROOF: Let $\mathcal{H}$ be an ω-dimensional cylindric basis for $\mathfrak{B}$. $\mathfrak{Ca}\mathcal{H}$ is an ω-dimensional cylindric algebra. Consider the cylindric algebra $\mathfrak{C} = Sg^{\mathfrak{Ca}\mathcal{H}}\mathfrak{B}$, the subalgebra of $\mathfrak{Ca}\mathcal{H}$ generated by $\mathfrak{B}$. We want to find out what elements of $\mathfrak{Ca}\mathcal{H}$ are generated by $\mathfrak{B}$.

DEFINITION 2. Let M be a (not necessarily atomic) finite network over $\mathfrak{B}$ whose nodes are a finite subset of ω.

- Define $\widehat{M} = \{N \in \mathcal{H} : N \leq M\} \in \mathfrak{Ca}\mathcal{H}$. ($N \leq M$ means that for all $i, j \in M$ we have $N(i, j) \leq M(i, j)$.)
- A *block* is an element of the form $\widehat{M}$ for some finite network M such that
  1. M is triangle-closed, i.e. for all $i, j, k \in M$ we have $M(i, k) \leq M(i, j); M(j, k)$
  2. If x is the label of an irreflexive edge of M then $x = 1'$ or $x \leq \mathsf{r}$ or $x \leq \mathsf{y}$ or $x \leq \mathsf{b}$ (we say x is 'monochromatic'), and $|\{i : x.(\mathsf{r}(i) + \mathsf{y}(i) + \mathsf{b}(i)) \neq 0\}|$ is either $0, 1$ or infinite (we say that the number of indices of x is either $0, 1$ or infinite).

LEMMA 7.

1. *For any block* $\widehat{M}$ *and* $i < \omega$ *we have*

$$\mathsf{c}_i\widehat{M} = (M\restriction_{dom(M)\setminus\{i\}})\widehat{\ }$$

2. *The domain of* $\mathfrak{C}$ *consists of finite sums of blocks.*

PROOF: $\mathsf{c}_i\widehat{M} \subseteq (M\restriction_{dom(M)\setminus\{i\}})\widehat{\ }$ is obvious. Also, if $i \notin M$ the equality is trivial. Let $N \in (M\restriction_{dom(M)\setminus\{i\}})\widehat{\ }$, i.e. $N \leq M\restriction_{dom(M)\setminus\{i\}}$. We must show that $N \in \mathsf{c}_i\widehat{M}$ and for this we must find $L \equiv_i N$ with $L \in \widehat{M}$. $L \equiv_i N$ determines every edge of L except those involving i. For each $j \in M$, if the number of indices in $M(i, j)$ is just one, say $M(i, j) = \mathsf{r}(k)$, then let $L(i, j)$ be an arbitrary atom below $\mathsf{r}(k)$. There should be no inconsistencies in the labelling so far defined for L, by triangle-closure for M. For all the other edges (i, j) if $j \in M$ there are infinitely many indices in $M(i, j)$ and if $j \notin M$ then we have an unrestricted choice of atoms for the label. These edges are labelled one at a time and each label is given an atom with a new index, thus avoiding any inconsistencies. This defines $L \equiv_i N$ with $L \in \widehat{M}$ and proves the first part of the lemma.

For the second part, we already have seen that the set of finite sums of blocks is closed under cylindrification. We'll show that this set is closed under all the cylindric operations and includes $\mathfrak{B}$. For any $x \in \mathfrak{B}$ and $i, j < \omega$, let N_x^{ij} be the $\mathfrak{B}$-network with two nodes $\{i, j\}$ and labelling $N_x^{ij}(i,i) = N_x^{ij}(j,j) = 1'$, and $N^{ij}(i,j) = x$, $N_x^{ij}(j,i) = \breve{x}$. Clearly N_x^{ij} is triangle closed. And $\widehat{N_x^{01}} = x$. For any $x \in \mathcal{B}$, we have $x = x.1' + x.\mathsf{r} + x.\mathsf{y} + x.\mathsf{b}$, so $x = \widehat{N_{x.1'}^{01}} + \widehat{N_{x.\mathsf{r}}^{01}} + \widehat{N_{x.\mathsf{y}}^{01}} + \widehat{N_{x.\mathsf{b}}^{01}}$ and the labels of these four networks are monochromatic. The first network defines a block and for each of the last three, if the number if indices is infinite then it is a block. If the number of indices is finite then it is a finite union of blocks. So every element of $\mathfrak{B}$ is a finite union of blocks.

For the diagonals, $\mathsf{d}_{ij} = \widehat{N_{1'}^{ij}}$.

Closure under sums is obvious.

For negation, take a block $\widehat{M}$.

$$-\widehat{M} = \sum_{i,j \in M} \widehat{N_{-N(i,j)}^{ij}}$$

As before we can replace $\widehat{N_{-N(i,j)}^{ij}}$ by a finite union of blocks.

Thus the set of finite sums of blocks includes $\mathfrak{B}$ and the diagonals and is closed under all the cylindric operations. Since every block is clearly generated from $\mathfrak{B}$ using substitutions and intersection only, the lemma follows. □

It remains to show that $\mathfrak{B} = \mathfrak{RaC}$. Take a block $\widehat{M} \in \mathfrak{RaC}$. Then $\mathsf{c}_i\widehat{M} = \widehat{M}$ for $2 \le i < \omega$. By the first part of the lemma, $\widehat{M} = \widehat{M\restriction_{\{0,1\}}} \in \mathfrak{B}$. □

THEOREM 8. $\mathfrak{A} \equiv \mathfrak{B}$.

PROOF: We show that $\exists$ has a winning strategy in an Ehrenfeucht–Fraïssé-game over $(\mathfrak{A}, \mathfrak{B})$. At any stage of the game, if $\forall$ places a pebble on one of $\mathfrak{A}$ or $\mathfrak{B}$, $\exists$ must place a matching pebble on the other algebra. Let $\bar{a} = \langle a_0, a_1, \ldots, a_{n-1}\rangle$ be the position of the pebbles played so far (by either player) on $\mathfrak{A}$ and let $\bar{b} = \langle b_0, \ldots, b_{n-1}\rangle$ be the the position of the pebbles played on $\mathfrak{B}$. $\exists$ maintains the following properties throughout the game.

- For any atom x (of either algebra) with $x.\mathsf{r}(0) = 0$ then $x \in a_i \iff x \in b_i$.
- $\bar{a}$ induces a finite partition of $\mathsf{r}(0)$ in $\mathcal{A}$ of 2^n (possibly empty) parts $p_i : i < 2^n$ and $\bar{b}$ induces a partition of $\mathsf{r}(0)$ in $\mathcal{B}$ of parts $q_i : i < 2^n$. p_i is finite iff q_i is finite and, in this case, $|p_i| = |q_i|$.

It is easy to see that $\exists$ can maintain these properties in every round. Therefore she can win the game. Therefore $\mathfrak{A} \equiv \mathfrak{B}$. $\square$

Hence,

THEOREM 9. *For any* n *with* $5 \leq n \leq \omega$, $\mathfrak{Ra}\mathbf{CA}_n$ *is not elementary.*

PROOF: $\mathfrak{B} \in \mathfrak{Ra}\mathbf{CA}_\omega \subseteq \mathfrak{Ra}\mathbf{CA}_n$ and $\mathfrak{A} \notin \mathfrak{Ra}\mathbf{CA}_5 \supseteq \mathfrak{Ra}\mathbf{CA}_n$ and $\mathfrak{A} \equiv \mathfrak{B}$. So $\mathfrak{Ra}\mathbf{CA}_n$ is not closed under elementary equivalence. $\square$

Let $\mathbf{Lf}_\kappa$ denote the class of locally finite $\mathbf{CA}_\kappa$. An algebra $\mathfrak{D}$ is such if $\Delta x = \{i \in \kappa : \mathsf{c}_i x \neq x\}$ is finite for every $x \in \mathfrak{D}$.

THEOREM 10. *For* $\kappa > \omega$, $\mathfrak{Ra}\mathbf{CA}_\kappa$ *is not elementary.*

PROOF: We prove that $\mathfrak{B} \in \mathfrak{Ra}\mathbf{CA}_\kappa$. Assume that $\mathfrak{B} = \mathfrak{Ra}\mathfrak{C}$, with $\mathfrak{C} \in \mathbf{CA}_\omega$. Let $\mathfrak{C}'$ be the subalgebra of $\mathfrak{C}$ generated by (the set) B. Then $\mathfrak{C}' \in \mathbf{Lf}_\omega$. By $\mathbf{Lf}_\omega = \mathfrak{Nr}_\omega \mathbf{Lf}_\kappa$ [11] we have $\mathfrak{C}' = \mathfrak{Nr}_\omega \mathfrak{D}'$ for some $\mathfrak{D}' \in \mathbf{Lf}_\kappa$.Thus

$$\mathfrak{B} = \mathfrak{Ra}\mathfrak{Nr}_\omega \mathfrak{D}' = \mathfrak{Ra}\mathfrak{D}',$$

and the theorem follows. $\square$

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