Math. Log. Quart. **54**, No. 3, 294 – 306 (2008) / **DOI** 10.1002/malq.200710038

Weakly representable atom structures that are not strongly representable, with an application to first order logic

Tarek Sayed Ahmed[*]

Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt

Received 27 June 2007, revised 26 September 2007, accepted 28 September 2007
Published online 8 May 2008

Key words Algebraic logic, cylindric algebras, relation algebras, atom structures, omitting types.
MSC (2000) 03G15

Let $n > 2$. A weakly representable relation algebra that is not strongly representable is constructed. It is proved that the set of all n by n basic matrices forms a cylindric basis that is also a weakly but not a strongly representable atom structure. This gives an example of a binary generated atomic representable cylindric algebra with no complete representation. An application to first order logic is given.

1 Introduction

The results of this article are known, however, the construction is new. The proofs adopted herein substantially simplify known proofs in [2, 9, 10, 11, 12]. We assume familiarity of the basic notions of cylindric and relation algebras. We follow [7]. In particular, CA_n stands for the class of cylindric algebras of dimension n. RA stands for the class of relation algebras. Any first order relational structure can be turned into a complex algebra. If the structure has the right similarity type, then its complex algebra will have the same similarity type as cylindric algebras, and similarly for relation algebras.

Definition 1.1 Let $\mathcal{B} = (B, T_k, E_{kl})_{k,l<\alpha}$ be any structure with $T_k \subseteq B \times B$ and $E_{kl} \subseteq B$ for all $k, l < \alpha$. *The complex algebra of* $\mathcal{B}$ is

$$\mathfrak{Cm}\mathcal{B} = (\wp(B), \cup, \setminus, T_k^*, E_{kl})_{k,l<\alpha},$$

where $(\wp(B), \cup, \setminus)$ is the Boolean algebra of all subsets of B and

$$T_k^*(X) = \{y \,:\, (\exists x \in X)((x, y) \in T_k)\}.$$

$\mathcal{B}$ is a *cylindric algebra atom structure of dimension* α if $\mathfrak{Cm}\mathcal{B}$ is a CA_α. $\mathcal{B}$ *is strongly representable* if $\mathfrak{Cm}\mathcal{B}$ is a representable cylindric algebra.

Definition 1.2 Let $\mathcal{B} = (B, I, \breve{}, C)$ be any structure such that $C \subseteq B \times B \times B$, $\breve{}$ is a function from B to B, and $I \subseteq B$. *The complex algebra of* $\mathcal{B}$ is

$$\mathfrak{Cm}\mathcal{B} = (\wp(B), \cup, \setminus, I, \breve{}, C^*),$$

where

$$C^*(X, Y) = \{z \,:\, (\exists x \in X)(\exists y \in Y)((x, y, z) \in C)\}$$

and

$$\breve{X} = \{\breve{x} \,:\, x \in X\}.$$

$\mathcal{B}$ is a *relation algebra atom structure* if $\mathfrak{Cm}\mathcal{B}$ is a relation algebra. $\mathcal{B}$ *is strongly representable* if $\mathfrak{Cm}\mathcal{B}$ is a representable relation algebra.

[*] e-mail: rutahmed@gmail.com

The term algebra of $\mathcal{B}$, $\mathfrak{Tm}\mathcal{B}$ for short, is the subalgebra of $\mathfrak{Cm}\mathcal{B}$ generated by the singletons. $\mathcal{B}$ *is weakly representable* if $\mathfrak{Tm}\mathcal{B}$ is representable. Note that if $\mathcal{B}$ is strongly representable, then it is weakly representable. (Here we show that the converse fails.) Special structures are those arising from atomic algebras (hence the name atomic) and those arising from basic matrices defined on a relation algebra.

Definition 1.3 Let $\mathcal{A} = (A, +, -, 0, 1, 1', \breve{}, ;)$ be an atomic relation algebra. Let $\mathrm{At}\mathcal{A}$ denote the set of atoms. Then $\mathcal{A}$ determines an atom structure $(\mathrm{At}\mathcal{A}, \mathrm{Id}, \breve{}, C)$ whose domain is the set of atoms of $\mathcal{A}$, the identity Id is

$$\mathrm{Id} = \{e \in \mathrm{At}A : e \leq 1'\},$$

the conversion is the restriction of $\breve{}$ on the atoms of $\mathcal{A}$, and the ternary relation C is defined by

$$C(a, b, c) \leftrightarrow a; b \geq c \text{ for all atoms } a, b, c.$$

A triple in C is said to be a *consistent triple*.

Atomic cylindric algebras also determine atom structures.

Definition 1.4 Let $\mathcal{A} = (A, +, -, 0, 1, \mathsf{c}_i, \mathsf{d}_{ij})_{i,j<\alpha}$ be an atomic cylindric algebra of dimension α. Let $\mathrm{At}\mathcal{A}$ denote the set of atoms. Then $\mathcal{A}$ determines an atom structure $(\mathrm{At}\mathcal{A}, T_i, D_{ij})_{i,j<\alpha}$ whose domain is the set of atoms of $\mathcal{A}$ and for $i, j < \alpha$, $D_{ij} = \{e \in \mathrm{At}A : e \leq \mathsf{d}_{ij}\}$, and the binary relation T_i is defined by

$$T_i(a, b) \leftrightarrow a \leq \mathsf{c}_i b \text{ for all atoms } a, b.$$

We now turn to defining basic matrices over an atomic relation algebra:

Definition 1.5 Let $\mathcal{A}$ be an atomic relation algebra. Let $n > 2$. M_n is the set of all n by n matrices of atoms in $\mathcal{A}$ which satisfy the following conditions for all $i, j, k < n$:

(1) $m_{kk} \leq 1'$,

(2) $m_{ij} = m_{ji}$,

(3) $m_{ij} \leq m_{jk}; m_{ki}$.

The matrices in M_n are called *basic matrices*. Two matrices m and m' in M_n *agree up to* k if $m_{ij} = m'_{ij}$ whenever $i, j \in n \setminus \{k\}$. For any $i, j < n$, let

$$T_i = \{(m, m') \in M_n \times M_n : m \text{ and } m' \text{ agree up to } i\}, \quad E_{ij} = \{m \in M_n : m_{ij} \leq 1'\}.$$

Then $\mathcal{M}_n = (M_n, T_i, E_{ij})_{i,j<n}$ is a relational structure of cylindric type. Sometimes $\mathfrak{Cm}\mathcal{M}_n$ is a cylindric algebra of dimension n. This occurs when $\mathcal{M}_n$ is a cylindric basis [13].

Our main theorem is:

Theorem 1.6 *Let* $2 < n < \omega$. *Then there is a weakly representable relation atom structure that is not strongly representable with an* n*-dimensional cylindric basis* $\mathcal{M}_n$. *Furthermore,* $\mathcal{M}_n$ *is a weakly representable cylindric algebra atom structure that is not strongly representable.*

Examples of weakly representable atom structures that are not strongly representable are known [10, 11]. Here we give a new construction. A novelty occuring here is that our cylindric atom structure is defined from the relation algebra atom structure. That is, we obtain our desired atom structures in one blow. (This is also done in [2]. But we should add that the construction in [2] is completely different and much more complex.)

Our notation is mostly standard. An ordinal is the set of all smaller ordinals: for $n < \omega$, $n = \{0, 1, \ldots, n-1\}$. Maps are regarded formally as sets of ordered pairs. Thus, if θ is a map, we write $|\theta|$ for the cardinality of the set that is θ. We write $\mathrm{dom}(\theta), \mathrm{rng}(\theta)$ for the domain and range of θ, respectively. We write Id_X for the identity map on X. $\wp(X)$ denotes the power set of X.

We write $\bar{a}, \bar{x}$ for sequences. A sequence (or tuple) $\bar{a}$ of elements of a set X, of length n, is formally an element of the set ${}^nX$. We write a_i for the ith element of this sequence, and $\mathrm{rng}(\bar{a})$ for $\{a_0, \ldots, a_{n-1}\}$. We may write $\bar{a}$ as $(a_0, \ldots, a_{n-1})$. If $\theta : X \longrightarrow Y$ is a map, we write $\theta(\bar{a})$ for the sequence $(\theta(a_0), \ldots, \theta(a_{n-1}))\} \in {}^nY$. If $\bar{a}$ and $\bar{b}$ are n-sequences, we write $(\bar{a} \longmapsto \bar{b})$ for the map $\{(a_i, b_i) : i < n\}$. For $i < n$, we write $\bar{a} =_i \bar{b}$ if $a_j = b_j$ for all $j < n$ with $j \neq i$.

2 The atom structure

Definition 2.1 Fix integers $n \geq 3$ (it will be the dimension of the cylindric algebra) and $N \geq n(n-1)/2$. The graph $\mathfrak{G}$ consists of a countable infinite collection of pairwise disjoint N-cliques (that is, complete graphs with N nodes).

The relation algebra atom structure $\alpha(\mathfrak{G})$ is of the form $(\{1'\} \cup (\mathfrak{G} \times n), R_{1'}, \breve{R}, R_{;})$. The only identity atom is $1'$. All atoms are self-converse, so

$$\breve{R} = \{(a, a) : a \text{ an atom}\}.$$

The colour of an atom $(a', i) \in \mathfrak{G} \times n$ is i. The identity $1'$ has no colour. A triple (a, b, c) of atoms in $\alpha(\mathfrak{G})$ is *consistent* if $R_{;}(a, b, c)$ holds. So the consistent triples are (a, b, c), where

1. one of a, b, c is $1'$ and the other two are equal; or
2. none of a, b, c is $1'$ and they do not all have the same colour; or
3. $a = (a', i)$, $b = (b', i)$, and $c = (c', i)$ for some $i < n$ and $a', b', c' \in \mathfrak{G}$, and there exists at least one graph edge of $\mathfrak{G}$ in $\{a', b', c'\}$.

$\alpha(\mathfrak{G})$ is an instance of atom structures defined in [10], except that we use n colours instead of just three. So $\alpha(\mathfrak{G})$ can be checked to be a relation atom structure.

We need to prove more:

1. The term algebra over $\alpha(\mathfrak{G})$ is representable.
2. The term cylindric algebra over the collection of all $n \times n$ matrices over $\alpha(\mathfrak{G})$ is representable.
3. The complex algebra $\mathfrak{Cm}\alpha(\mathfrak{G})$ is not representable.

3 Labelled graphs

Definition 3.1 A *labelled graph* is an undirected graph Γ such that every edge (unordered pair of distinct nodes) of Γ is labelled by a unique label from $(\mathfrak{G} \cup \{\varrho\}) \times n$, where $\varrho \notin \mathfrak{G}$ is a new element. *The colour of* (ϱ, i) is defined to be i. *The colour of* (a, i) for $a \in \mathfrak{G}$ is i, as before.

Notation 3.2 We will write $\Gamma(x, y)$ for the label of an edge (x, y) in the labelled graph Γ. Note that these may not always be defined: for example, $\Gamma(x, x)$ is not.

If Γ is a labelled graph, and $D \subseteq \Gamma$, we write $\Gamma \restriction D$ for the induced subgraph of Γ on the set D (it inherits the edges and colours of Γ, on its domain D). We write $\Delta \subseteq \Gamma$ if Δ is an induced subgraph of Γ in this sense.

Definition 3.3 Let Γ, Δ be labelled graphs, and $\theta : \Gamma \longrightarrow \Delta$ be a map. θ is said to be a *labelled graph embedding*, or simply an *embedding*, if it is injective and preserves all edges, and all colours, where defined, in both directions. An *isomorphism* is a bijective embedding.

Now we define a class $\mathcal{G}$ of certain labelled graphs.

Definition 3.4 The class $\mathcal{G}$ consists of all complete labelled graphs Γ (possibly the empty graph) such that for all distinct $x, y, z \in \Gamma$, writing $(a, i) = \Gamma(y, x)$, $(b, j) = \Gamma(y, z)$, $(c, l) = \Gamma(x, z)$, we have:

(1) $|\{i, j, l\}| > 1$; or

(2) $a, b, c \in \mathfrak{G}$ and $\{a, b, c\}$ has at least one edge of $\mathfrak{G}$; or

(3) exactly one of a, b, c – say, a – is ϱ, and (b, c) is an edge of $\mathfrak{G}$; or

(4) two or more of a, b, c are ϱ.

Clearly, $\mathcal{G}$ is closed under isomorphism and under induced subgraphs.

3.1 The main construction

Proposition 3.5 *There is a countable labelled graph $M \in \mathcal{G}$ with the following property:*

If $\Delta \subseteq \Delta' \in \mathcal{G}$, $|\Delta'| \leq n$, and $\theta : \Delta \longrightarrow M$ is an embedding, then θ extends to an embedding $\theta' : \Delta' \longrightarrow M$.

P r o o f. Two players, $\forall$ and $\exists$, play a game to build a labelled graph M. They play by choosing a chain

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots$$

of finite graphs in $\mathcal{G}$; the union of the chain will be the graph M.

There are ω rounds. In each round, $\forall$ and $\exists$ do the following. Let $\Gamma \in \mathcal{G}$ be the graph constructed up to this point in the game. $\forall$ chooses $\Delta \in \mathcal{G}$ of size $< n$, an embedding $\theta : \Delta \longrightarrow \Gamma$, and an extension $\Delta \subseteq \Delta^+ \in \mathcal{G}$, where

$$|\Delta^+ \setminus \Delta| \leq 1.$$

These choices, $(\Delta, \theta, \Delta^+)$, constitute his move. $\exists$ must respond with an extension $\Gamma \subseteq \Gamma^+ \in \mathcal{G}$ such that θ extends to an embedding $\theta^+ : \Delta^+ \longrightarrow \Gamma^+$. Her response ends the round.

The starting graph $\Gamma_0 \in \mathcal{G}$ is arbitrary but we will take it to be the empty graph in $\mathcal{G}$.

Lemma 3.6 *$\exists$ never gets stuck – she can always find a suitable extension $\Gamma^+ \in \mathcal{G}$.*

P r o o f. Let $\Gamma \in \mathcal{G}$ be the graph built at some stage, and let $\forall$ choose the graphs $\Delta \subseteq \Delta^+ \in \mathcal{G}$ and the embedding $\theta : \Delta \longrightarrow \Gamma$. Thus, his move is $(\Delta, \theta, \Delta^+)$.

We now describe $\exists$'s response. If Γ is empty, she may simply play Δ^+, and if $\Delta = \Delta^+$, she plays Γ. Otherwise, let $F = \mathrm{rng}(\theta) \subseteq \Gamma$. (So $|F| < n$.) Since Δ and $\Gamma \restriction F$ are isomorphic labelled graphs (via θ), and $\mathcal{G}$ is closed under isomorphism, we may assume with no loss of generality that $\forall$ actually played $(\Gamma \restriction F, \mathrm{Id}_F, \Delta^+)$, where $\Gamma \restriction F \subseteq \Delta^+ \in \mathcal{G}$, $\Delta^+ \setminus F = \{\delta\}$, and $\delta \notin \Gamma$. We may view $\forall$'s move as building a labelled graph $\Gamma^* \supseteq \Gamma$, whose nodes are those of Γ together with δ, and whose edges are the edges of Γ together with edges from δ to every node of F. The labelled graph structure on Γ^* is given by:

1. Γ is an induced subgraph of Γ^* (i. e., $\Gamma \subseteq \Gamma^*$);
2. $\Gamma^* \restriction (F \cup \{\delta\}) = \Delta^+$.

Now $\exists$ has to extend Γ^* to a complete graph on the same nodes and complete the colouring, which yields a graph $\Gamma^+ \in \mathcal{G}$. Thus, she has to define the colour $\Gamma^+(\beta, \delta)$ for all nodes $\beta \in \Gamma \setminus F$, in such a way as to meet the conditions of Definition 3.4. She does this as follows. The set of colours of the labels in $\{\Delta^+(\delta, \varphi) : \varphi \in F\}$ has cardinality at most $n - 1$. Let $i < n$ be a "colour" not in this set. $\exists$ labels (δ, β) by (ϱ, i) for every $\beta \in \Gamma \setminus F$. This completes the definition of Γ^+.

It remains to check that this strategy works – that the conditions from the definition of $\mathcal{G}$ (Definition 3.4) are met. It is sufficient to note that

1. if $\varphi \in F$ and $\beta \in \Gamma \setminus F$, then the labels in Γ^+ on the edges of the triangle (β, δ, φ) are not all of the same colour (by choice of i);
2. if $\beta, \gamma \in \Gamma \setminus F$, then two of the labels in Γ^+ on the edges of the triangle (β, γ, δ) are (ϱ, i).

This proves the lemma. □

Now there are only countably many finite graphs in $\mathcal{G}$ up to isomorphism, and each of the graphs built during the game is finite. Hence $\forall$ may arrange to play every possible $(\Delta, \theta, \Delta^+)$ (up to isomorphism) at some round in the game. Suppose he does this, and let M be the union of the graphs played in the game. We check that M is as required. Certainly, $M \in \mathcal{G}$, since $\mathcal{G}$ is clearly closed under unions of chains. Also, let $\Delta \subseteq \Delta' \in \mathcal{G}$, $|\Delta'| \leq n$, and $\theta : \Delta \longrightarrow M$ be an embedding. We prove that θ extends to Δ', by induction on $d = |\Delta' \setminus \Delta|$. If this is 0, there is nothing to prove.

Assume the result for smaller d. Choose $a \in \Delta' \setminus \Delta$ and let

$$\Delta^+ = \Delta' \restriction (\Delta \cup \{a\}) \in \mathcal{G}.$$

As, $|\Delta| < n$, at some round in the game where the graph built so far was Γ, say, $\forall$ would have played $(\Delta, \theta, \Delta^+)$ (or some isomorphic triple). Hence, if $\exists$ constructed Γ^+ in that round, there exists an embedding $\theta^+ : \Delta^+ \longrightarrow \Gamma^+$ extending θ. As $\Gamma^+ \subseteq M$, θ^+ is also an embedding $\Delta^+ \longrightarrow M$. Since $|\Delta' \setminus \Delta^+| < d$, θ^+ extends inductively to an embedding $\theta' : \Delta' \longrightarrow M$, as required. □

4 Model theory of M

Here we establish the main properties of the graph M of Proposition 3.5. To do so, we will need some (fairly) standard notions from model theory, and we discuss these first.

Let L be a signature without function or constant symbols, and let A be an L-structure.

4.1 Classical semantics

Definition 4.1 Recall the definition of the n-variable infinitary language $L^n_{\infty\omega}$. We use variables $x_0, \ldots, x_{n-1}$. The atomic formulas are $x_i = x_j$ for any $i, j < n$, and $R(\bar{x})$ for any k-ary $R \in L$ and any k-tuple $\bar{x}$ of variables taken from $x_0, \ldots, x_{n-1}$. If φ is an $L^n_{\infty\omega}$-formula, then so are $\neg\varphi$ and $\exists x_i \varphi$ for $i < n$. If Φ is a set of $L^n_{\infty\omega}$-formulas, then $\bigwedge \Phi$ and $\bigvee \Phi$ are also $L^n_{\infty\omega}$-formulas. Of course, we write $\bigwedge\{\varphi, \psi\}$ as $\varphi \wedge \psi$, etc.

The logic $L^n_{\infty\omega}$ is given semantics in A in the usual way, defining $A \models \varphi(\bar{a})$ for an n-tuple $\bar{a}$ of elements of A by induction on the formula φ. Note that not all of $x_0, \ldots, x_{n-1}$ need occur free in φ: so, for example,

$$A \models x_3 = x_2(a_0, \ldots, a_{n-1}) \quad \text{iff} \quad a_3 = a_2.$$

We generally use the notation $A \models \varphi(\bar{a})$ only when $\bar{a}$ is an n-tuple, though if $R \in L$ has arity k, we do write

$$A \models R(a_1, \ldots, a_k)$$

if $(a_1, \ldots, a_k)$ stands in the relation defined by R in A. A similar convention holds for $A \models a = b$.

Definition 4.2 Let L^n denote the first order fragment of $L^n_{\infty\omega}$

Definition 4.3 An *n-back-and-forth system on A* is a set Θ of one-to-one partial maps $A \longrightarrow A$ such that:

1. If $\theta \in \Theta$, then $|\theta| \leq n$.
2. If $\theta' \subseteq \theta \in \Theta$, then $\theta' \in \Theta$.
3. If $\theta \in \Theta$, $|\theta| < n$, and $a \in A$, then there is $\theta' \supseteq \theta$ in Θ with $a \in \mathrm{dom}(\theta')$ (forth).
4. If $\theta \in \Theta$, $|\theta| < n$, and $a \in A$, then there is $\theta' \supseteq \theta$ in Θ with $a \in \mathrm{rng}(\theta')$ (back).

This is independent of the signature of A. We could require that Θ is non-empty, but this will always be so in the applications in any case.

Definition 4.4 Recall that a *partial isomorphism of A* is a partial map $\theta : A \longrightarrow A$ that preserves all quantifier-free L-formulas.

Fact 4.5 *Let Θ be an n-back-and-forth system of partial isomorphisms on A, let $\bar{a}, \bar{b} \in {}^nA$, and suppose that $\theta = (\bar{a} \longmapsto \bar{b})$ is a map in Θ. Then $A \models \varphi(\bar{a})$ iff $A \models \varphi(\bar{b})$, for any formula φ of $L^n_{\infty\omega}$.*

P r o o f. By induction on the structure of φ. If φ is quantifier-free, the result is immediate because θ is a partial isomorphism of A. The Boolean cases are also evident. If the result holds inductively for φ, then consider $\exists x_i \varphi$. If $A \models \exists x_i \varphi(\bar{a})$, then for some $\bar{a}' \in {}^nA$ with $\bar{a}' =_i \bar{a}$, we have $A \models \varphi(\bar{a}')$. Let

$$\theta^- = \theta \restriction \{a_j : j \neq i\}.$$

Then $\theta^- \in \Theta$ and $|\theta^-| < n$. Using the "forth" property of Θ, take $\theta' \in \theta$ extending θ^- and defined on a'_i, and let $\bar{b}' = \theta'(\bar{a}')$. By the inductive hypothesis, $A \models \varphi(\bar{b}')$. Since $\bar{b}' =_i \bar{b}$, we have $A \models \exists x_i \varphi(\bar{b})$. The converse is similar, using the "back" property of Θ. $\square$

4.2 Relativised semantics

Suppose that $W \subseteq {}^nA$ is a given non-empty set. We can relativise quantifiers to W, giving a new semantics $\models_W$ for $L^n_{\infty\omega}$, which has been intensively studied in recent times (see, e. g., [3]). If $\bar{a} \in W$:

1. For atomic φ, $A \models_W \varphi(\bar{a})$ iff $A \models \varphi(\bar{a})$.
2. The Boolean clauses are as expected.
3. For $i < n$, $A \models_W \exists x_i \varphi(\bar{a})$ iff $A \models_W \varphi(\bar{a}')$ for some $\bar{a}' \in W$ with $\bar{a}' =_i \bar{a}$.

Corollary 4.6 *If the set W is $L^n_{\infty\omega}$-definable, Θ is an n-back-and-forth system of partial isomorphisms on A, $\bar{a}, \bar{b} \in W$, and $\bar{a} \longmapsto \bar{b} \in \Theta$, then $A \vDash_W \varphi(\bar{a})$ iff $A \vDash_W \varphi(\bar{b})$ for any formula φ of $L^n_{\infty\omega}$.*

Proof. Assume that W is definable by the $L^n_{\infty\omega}$-formula ψ, so that $W = \{\bar{a} \in {}^nA : A \vDash \psi(a)\}$. We may relativise the quantifiers of $L^n_{\infty\omega}$-formulas to ψ. For each $L^n_{\infty\omega}$-formula φ we obtain a relativised one, φ^ψ, by induction, the main clause in the definition being

$$(\exists x_i \varphi)^\psi = \exists x_i(\psi \wedge \varphi^\psi).$$

Then clearly, $A \vDash_W \varphi(\bar{a})$ iff $A \vDash \varphi^\psi(\bar{a})$, for all $\bar{a} \in W$. The corollary now follows from Fact 4.5. □

4.3 Labelled graphs and model theory

We wish to view the graph M of Proposition 3.5 as a classical structure.

Definition 4.7 Let L^+ be the signature consisting of the binary relation symbols (a, i), for each $a \in \mathfrak{G} \cup \{\varrho\}$ and $i < n$. Let $L = L^+ \setminus \{(\varrho, i) : i < n\}$. From now on, the logics $L^n, L^n_{\infty\omega}$ are taken in this signature.

We may regard any non-empty labelled graph equally as an L^+-structure, in the obvious way. The n-homogeneity built into M by its construction would suggest that the set of all partial isomorphisms of M of cardinality at most n forms an n-back-and-forth system. This is indeed true, but we can go further.

Notation 4.8 Write the set of nodes of $\mathfrak{G}$ as $N \times \omega$, where $(a, i), (b, j)$ are in the same N-clique iff $i = j$. So in (a, i), i is "the clique number" and a is the element of the clique.

Any element of $\alpha(\mathfrak{G}) \setminus \{1'\}$ is an element of $\mathfrak{G} \times n$ and will therefore be of the form $((a, i), j)$, where $a < N$, $i < \omega$, and $j < n$. But we write it as (a, i, j) for simplicity.

Definition 4.9 Let χ be a permutation of the set $\omega \cup \{\varrho\}$, $\Gamma, \Delta \in \mathcal{G}$ have the same size, and let $\theta : \Gamma \longrightarrow \Delta$ be a bijection. We say that θ *is a χ-isomorphism from* Γ *to* Δ if for each distinct $x, y \in \Gamma$:

1. If $\Gamma(x, y) = (a, i, j)$, then

$$\Delta(\theta(x), \theta(y)) = \begin{cases} (a, \chi(i), j) & \text{if } \chi(i) \neq \varrho, \\ (\varrho, j) & \text{otherwise.} \end{cases}$$

2. If $\Gamma(x, y) = (\varrho, j)$, then

$$\Delta(\theta(x), \theta(y)) \in \begin{cases} \{(a, \chi(\varrho), j) : a < N\} & \text{if } \chi(\varrho) \neq \varrho, \\ \{(\varrho, j)\} & \text{otherwise.} \end{cases}$$

Definition 4.10 For any permutation χ of $\omega \cup \{\varrho\}$, Θ^χ is the set of partial one-to-one maps from M to M of size at most n that are χ-isomorphisms on their domains. We write Θ for $\Theta^{\mathrm{Id}_{\omega \cup \{\varrho\}}}$.

Lemma 4.11 *For any permutation χ of $\omega \cup \{\varrho\}$, Θ^χ is an n-back-and-forth system on M.*

Proof. Clearly, Θ^χ is closed under restrictions. We check the "forth" property. Let $\theta \in \Theta^\chi$ have size $t < n$. Enumerate $\mathrm{dom}(\theta), \mathrm{rng}(\theta)$, respectively, as

$$\{a_0, \dots, a_{t-1}\}, \quad \{b_0, \dots, b_{t-1}\},$$

with $\theta(a_i) = b_i$ for $i < t$. Let $a_t \in M$ be arbitrary, let $b_t \notin M$ be a new element, and define a complete labelled graph $\Delta \supseteq M \restriction \{b_0, \dots, b_{t-1}\}$ with nodes $\{b_0, \dots, b_t\}$ as follows.

Choose distinct "nodes" $e_s < N$ for each $s < t$ such that no (e_s, i, j) labels any edge in $M \restriction \{b_0, \dots, b_{t-1}\}$. This is possible because $N \geq n(n-1)/2$, which bounds the number of edges in Δ. We can now define the colour of edges (b_s, b_t) of Δ for $s = 0, \dots, t-1$.

1. If $M(a_s, a_t) = (e, i, j)$, then

$$\Delta(b_s, b_t) = \begin{cases} (e, \chi(i), j) & \text{if } \chi(i) \neq \varrho, \\ (\varrho, j) & \text{otherwise.} \end{cases}$$

2. If $M(a_s, a_t) = (\varrho, j)$, then

$$\Delta(b_s, b_t) = \begin{cases} (e_s, \chi(\varrho), j) & \text{if } \chi(\varrho) \neq \varrho, \\ (\varrho, j) & \text{otherwise.} \end{cases}$$

This completes the definition of Δ. It is easy to check that $\Delta \in \mathcal{G}$. Hence, by Proposition 3.5, there is a graph embedding $\varphi : \Delta \longrightarrow M$ extending the map $\mathrm{Id}_{\{b_0,\dots,b_{t-1}\}}$. Note that $\varphi(b_t) \notin \mathrm{rng}(\theta)$. So the map

$$\theta^+ = \theta \cup \{(a_t, \varphi(b_t))\}$$

is injective, and it is easily seen to be a χ-isomorphism in Θ^χ and defined on a_t.

The converse, "back" property is similarly proved (or by symmetry, using the fact that the inverse of maps in Θ are χ^{-1}-isomorphisms). □

As a special case, we obtain:

Corollary 4.12 *The set* $\Theta = \Theta^{\mathrm{Id}_{\omega\cup\{\varrho\}}}$ *of partial* L^+*-isomorphisms of* M (*partial isomorphisms of* M *regarded as an* L^+*-structure*) *of size at most* n *is an* n*-back-and-forth system on* M.

But we can also derive a connection between classical and relativised semantics in M over the following set W:

Definition 4.13 Let $W = \{\bar{a} \in {}^nM \,:\, M \models (\bigwedge_{i<j<n, l<n} \neg(\varrho, l)(x_i, x_j))(\bar{a})\}$.

W is simply the set of tuples $\bar{a}$ in ${}^nM$ such that the edges between the elements of $\bar{a}$ do not have a label involving ϱ. Their labels are all of the form (a, i, j). Lemma 4.11 allows us to replace ϱ-labels by suitable (a, i, j)-labels within an n-back-and-forth system. Thus, we may arrange that the system maps a tuple $\bar{b} \in {}^nM \setminus W$ to a tuple $\bar{c} \in W$, and by Fact 4.5 this will preserve any formula containing no relation symbols (a, i, j) that are "moved" by the system. The next proposition uses this idea to show that the classical and W-relativised semantics agree.

Proposition 4.14 $M \models_W \varphi(\bar{a})$ *iff* $M \models \varphi(\bar{a})$, *for all* $\bar{a} \in W$ *and all* L^n*-formulas* φ.

Proof. The proof is by induction on φ. If φ is atomic, the result is clear; and the Boolean cases are simple.

Let $i < n$ and consider $\exists x_i \varphi$. If $M \models_W \exists x_i \varphi(\bar{a})$, then there is $\bar{b} \in W$ with $\bar{b} =_i \bar{a}$ and $M \models_W \varphi(\bar{b})$. Inductively, $M \models \varphi(\bar{b})$, so clearly, $M \models_W \exists x_i \varphi(\bar{a})$.

For the (more interesting) converse, suppose that $M \models \exists x_i \varphi(\bar{a})$. Then there exists $\bar{b} \in {}^nM$ such that $\bar{b} =_i \bar{a}$ and $M \models \varphi(\bar{b})$. Take $L_{\varphi,\bar{b}}$ to be any finite subsignature of L containing all the symbols from L that occur in φ or as a label in $M \restriction \mathrm{rng}(\bar{b})$. (Here we use the fact that φ is first order. The result may fail for infinitary formulas with infinite signature.) Choose a permutation χ of $\omega \cup \{\varrho\}$ fixing any i' such that some (a, i', j) occurs in $L_{\varphi,\bar{b}}$, and moving ϱ.

Let $\theta = \mathrm{Id}_{\{a_m \,:\, m \neq i\}}$. Take any distinct $l, m \in n \setminus \{i\}$. If $M(a_l, a_m) = (a, i', j)$, then $M(b_l, b_m) = (a, i', j)$ because $\bar{a} =_i \bar{b}$, so $(a, i', j) \in L_{\varphi,\bar{b}}$ by definition of $L_{\varphi,\bar{b}}$. So $\chi(i') = i'$ by definition of χ. $M(a_l, a_m) \neq (\varrho, j)$ (any j) because $\bar{a} \in W$. It now follows that θ is a χ-isomorphism on its domain, so that $\theta \in \Theta^\chi$.

Extend θ to $\theta' \in \Theta^\chi$ defined on b_i, using the "forth" property of Θ^χ (Lemma 4.11). Let $\bar{c} = \theta'(\bar{b})$. Now by choice of χ, no labels on edges of the subgraph of M with domain $\mathrm{rng}(\bar{c})$ involve ϱ. Hence, $\bar{c} \in W$.

Moreover, each map in Θ^χ is evidently a partial isomorphism of the reduct of M to the signature $L_{\varphi,\bar{b}}$. Now φ is an $L_{\varphi,\bar{b}}$-formula. Hence by Fact 4.5 applied to $L_{\varphi,\bar{b}}$ and Lemma 4.11, we have that

$$M \models \varphi(\bar{a}) \quad \text{iff} \quad M \models \varphi(\bar{c}).$$

So $M \models \varphi(\bar{c})$. Inductively, $M \models_W \varphi(\bar{c})$. Since $\bar{c} =_i \bar{a}$, we have $M \models_W \exists x_i \varphi(\bar{a})$ by definition of the relativised semantics. This completes the induction. □

5 The algebra of L^n-definable subsets of ${}^nM$

We can now extract from the labelled graph M of Proposition 3.5 a relativised set algebra $\mathcal{A}$, which will turn out to be representable (hence a cylindric algebra) and atomic.

First, we recall some relevant facts about cylindric algebras.

5.1 Cylindric algebras

We do not wish to give a comprehensive introduction to these (those who want one may read the standard reference [7]), but we feel we should list those of their features that are relevant here.

Let n be an ordinal (finite, in this paper). Recall that an *n-dimensional cylindric algebra* is an algebra $\mathcal{A}$ in the signature consisting of the Boolean operations $\cdot, -, 0, 1$, constants d_{ij} for $i, j < n$ ("diagonals"), and unary functions c_i for $i < n$ ("cylindrifications"), and satisfying certain equations which can be found in [7] and which we will not go into here. We only need to know that every cylindric algebra is a Boolean algebra with operations, and that the complex algebra of the atom structure of any atomic cylindric algebra is also a cylindric algebra.

We generally write $d_{ij}^{\mathcal{A}}$, $c_i^{\mathcal{A}}$, etc., for the interpretetions of the respective operation in $\mathcal{A}$. An *n-dimensional set algebra* is an algebra of n-ary relations of the form

$$\mathcal{A} = (A, \cap, -, \emptyset, W, d_{ij}^{\mathcal{A}}, c_i^{\mathcal{A}})_{i,j<n},$$

where W is of the form ${}^nU$ for some non-empty set U, $(A, \cap, -, \emptyset, W)$ is a Boolean subalgebra of the Boolean algebra $(\wp(W), \cap, \setminus, \emptyset, W)$, $d_{ij}^{\mathcal{A}} = \{\bar{a} \in W \,:\, a_j = a_j\}$, and for $X \in A$,

$$c_i^{\mathcal{A}} X = \{\bar{a} \in W \,:\, \bar{a} =_i \bar{b} \text{ for some } \bar{b} \in X\}.$$

The set W is called *the unit of* $\mathcal{A}$. Set algebras are automatically cylindric algebras, but not conversely, even up to isomorphism. A *relativised set algebra* (Crs_n) is similar, but has a weaker condition on W: we only require that $W \subseteq {}^nU$ for some set U. Relativised set algebras are not necessarily cylindric algebras.

Let $\mathcal{A}$ be an algebra of the similarity type of cylindric algebras. A *representation of* $\mathcal{A}$ is an algebra embedding h from $\mathcal{A}$ into a direct product of set algebras, and $\mathcal{A}$ is said to be *representable* if there is such a representation. Because the class of cylindric algebras is a variety and so closed under taking products and subalgebras, any representable algebra – for example, a representable relativised set algebra – is a cylindric algebra.

5.2 Definition of $\mathcal{A}$

Θ will continue to denote the set of all partial L^+-isomorphisms of M of size $\leq n$, it is an n-back-and-forth system on M. W remains as in Definition 4.13.

Definition 5.1

1. For an $L^n_{\infty\omega}$-formula φ, we define φ^W to be the set

$$\{\bar{a} \in W \,:\, M \vDash_W \varphi(\bar{a})\}.$$

Here we use the relativised semantics of Section 4.2.

2. We define $\mathcal{A}$ to be the relativised set algebra with domain $\{\varphi^W \,:\, \varphi \text{ a first order } L^n\text{-formula}\}$ and unit W, endowed with the algebraic operations d_{ij}, c_i, etc., in the standard way (see the passage on cylindric algebras above).

Note that $\mathcal{A}$ is indeed closed under the operations and so is a bona fide relativised set algebra. For, reading off from the definitions of the standard operations and the relativised semantics, we see that for all L^n-formulas φ, ψ,

(i) $-^{\mathcal{A}}(\varphi^W) = (\neg\varphi)^W$;

(ii) $\varphi^W \cdot^{\mathcal{A}} \psi^W = (\varphi \wedge \psi)^W$;

(iii) $d_{ij}^{\mathcal{A}} = (x_i = x_j)^W$ for all $i, j < n$;

(iv) $c_i^{\mathcal{A}}(\varphi^W) = (\exists x_i \varphi)^W$ for all $i < n$.

5.3 $\mathcal{A}$ is representable

Proposition 5.2 *$\mathcal{A}$ is representable. Hence, $\mathcal{A}$ is a cylindric algebra.*

P r o o f. Let $\mathcal{S}$ be the set algebra with domain $\wp({}^nM)$ and unit ${}^nM$. By Proposition 4.14, the map $h : \mathcal{A} \longrightarrow S$ given by $h : \varphi^W \longmapsto \{\bar{a} \in {}^nM \,:\, M \vDash \varphi(\bar{a})\}$ can be checked to be well-defined and one-to-one. It clearly respects the cylindric algebra operations. So it is a representation of $\mathcal{A}$. □

5.4 Atoms of $\mathcal{A}$

Here we show that $\mathcal{A}$ is atomic.

Definition 5.3 A formula α of L^n is said to be a *maximal conjunction of atomic formulas* (an MCA) if

(i) $M \models \exists x_0, \ldots, x_{n-1}\alpha$;

(ii) α is of the form

$$\bigwedge_{i \neq j < n} \alpha_{ij}(x_i, x_j),$$

where for each i, j, α_{ij} is either $x_i = x_j$ or $R(x_i, x_j)$ for some binary relation symbol R of L.

The rough idea is that "a formula α being MCA" means that the set defined by α in ${}^nM$ is nonempty, and that if $M \models \alpha(\bar{a})$, then the graph $M \restriction \mathrm{rng}(\bar{a})$ is determined up to isomorphism and has no edge whose label is of the form (ϱ, i). So any two tuples satisfying α are isomorphic and one is mapped to the other by the n-back-and-forth system Θ. By Fact 4.5, no $L^n_{\infty\omega}$-formula can distinguish them. So α defines an atom of $\mathcal{A}$ – it is literally indivisible. Since the MCA-formulas clearly "cover" W, the atoms defined by them are dense in $\mathcal{A}$. So $\mathcal{A}$ is atomic, as required. This, informally, is the content of the next two results.

Lemma 5.4 *Let φ be any $L^n_{\infty\omega}$-formula, and α any MCA-formula. If $\varphi^W \cap \alpha^W \neq \emptyset$, then $\alpha^W \subseteq \varphi^W$.*

P r o o f. Take $\bar{a} \in \varphi^W \cap \alpha^W$. Let $\bar{b} \in \alpha^W$ be arbitrary. It is clear that the map $(\bar{a} \longmapsto \bar{b})$ is in Θ. Also, W is $L^n_{\infty\omega}$-definable in M, since we have

$$W = \{\bar{a} \in {}^nM \,:\, M \models (\bigwedge_{i<j<n}(x_i = x_j \vee \bigvee_{R\in L} R(x_i, x_j)))(\bar{a})\}.$$

By Corollaries 4.6 and 4.12, we have $M \models_W \varphi(\bar{a})$ iff $M \models_W \varphi(\bar{b})$. Since $M \models_W \varphi(\bar{a})$, we have $M \models_W \varphi(\bar{b})$. Since $\bar{b}$ was arbitrary, we see that $\alpha^W \subseteq \varphi^W$. □

Definition 5.5 Let $F = \{\alpha^W \,:\, \alpha \text{ an MCA-}L^n\text{-formula}\} \subseteq \mathcal{A}$.

Evidently, $W = \bigcup F$.

Proposition 5.6 *$\mathcal{A}$ is an atomic algebra, with F as its set of atoms.*

P r o o f. First, we show that any non-empty element φ^W of $\mathcal{A}$ contains an element of F. Take $\bar{a} \in W$ such that $M \models_W \varphi(\bar{a})$. Since $\bar{a} \in W$, there is an MCA-formula α such that $M \models_W \alpha(\bar{a})$. By Lemma 5.4, $\alpha^W \subseteq \varphi^W$. By definition, if α is an MCA-formula, then α^W is non-empty. By Lemma 5.4 again, if φ is an L^n-formula such that $\emptyset \neq \varphi^W \subseteq \alpha^W$, then $\varphi^W = \alpha^W$. It follows that each α^W (for MCA α) is an atom of $\mathcal{A}$. □

Remark 5.7 It follows from the foregoing that the identity map on $\mathcal{A}$ is a complete relativised representation of $\mathcal{A}$ – an isomorphism from $\mathcal{A}$ onto a relativised set algebra that preserves infinite meets and joins where defined.

In any event, $\mathcal{A}$ has an atom structure, which we denote as $\mathrm{At}\mathcal{A}$ as usual. We now show that the atom structure of $\mathcal{A}$ is isomorphic to the atom structure consisting of n by n basic matrices over $\alpha(\mathfrak{G})$.

Proposition 5.8 *The atom structure of $\mathcal{A}$ is isomorphic (as a cylindric algebra atom structure) to the atom structure $\mathcal{M}_n$ of all n-dimensional basic matrices over the relation algebra atom structure $\alpha(\mathfrak{G})$.*

P r o o f. For each $m \in \mathcal{M}_n$, let $\alpha_m = \bigwedge_{i,j<n} \alpha_{ij}$. Here α_{ij} is $x_i = x_j$ if $m_{ij} = 1'$ and $R(x_i, x_j)$ otherwise, where $R = m_{ij} \in L$. Then the map $(m \longmapsto \alpha^W_m)_{m\in\mathcal{M}_n}$ is a well-defined isomorphism of n-dimensional cylindric algebra atom structures. □

We conclude that there is a representable atomic n-dimensional cylindric algebra (namely $\mathcal{A}$) with atom structure $\mathcal{M}_n$. It follows that the subalgebra of $\mathfrak{Cm}\mathcal{M}_n$ generated by the atoms, or the term algebra over $\mathcal{M}_n$, is representable. We recall that RaRCA_n stands for the class of relation algebra reducts of algebras in RCA_n, while RRA stands for the class of representable relation algebras.

Corollary 5.9 *The term relation algebra over the atom structure $\alpha(\mathfrak{G})$ is representable.*

P r o o f. It can be checked that the relation algebra reduct $\mathcal{B}$ of $\mathcal{A}$ is an atomic relation algebra with atom structure isomorphic to $\alpha(\mathfrak{G})$. So $\mathcal{B} \in \mathrm{RaRCA}_n \subseteq \mathrm{RRA}$. Since there exists a representable atomic relation algebra $\mathcal{B}$ with atom structure $\alpha(\mathfrak{G})$, it follows that the term algebra over $\alpha(\mathfrak{G})$ is representable. □

6 The complex algebras

Now we prove that $\mathfrak{Cm}\alpha(\mathfrak{G})$ is not representable. This will imply that the full complex cylindric algebra over $\mathcal{M}_n$ is not representable either. Our result follows from Ramsey's Theorem by noting that $\mathfrak{G}$ has a finite colouring. In more detail:

Definition 6.1 Let $\Gamma = (V, E)$ be an undirected graph. V is the set of vertices and E is an irreflexive, symmetric, binary relation on V. Let C be a non-empty set of colours:

1. $X \subseteq V$ is *independent* if $(x, y) \notin E$ for all $x, y \in X$.
2. A function $f : V \longrightarrow C$ is called a C*-colouring* if $(v, w) \in E$ implies $f(v) \neq f(w)$.

As before, write the set of nodes of $\mathfrak{G}$ as $N \times \omega$, where $(a, i), (b, j)$ are in the same N-clique iff $i = j$. So in the node (a, i), i is "the clique number" and a is the element of the clique.

Any element of $\alpha(\mathfrak{G}) \setminus \{1'\}$ is an element of $\mathfrak{G} \times n$ and will therefore be of the form $((a, i), j)$, where $a < N$, $i < \omega$, and $j < n$. But again we write it as (a, i, j) for simplicity. We can reformulate the definition of α by listing the forbidden triples. These are the complements of consistent ones. We have that $(1', c, d)$ is forbidden if $c \neq d$. Also, the triple $((a, i, k), (a', i', k), (a'', i'', k))$ is forbidden iff (a, i), (a', i'), (a'', i'') are independent in $N \times \omega$. The map $f : N \times \omega \longrightarrow N$ defined by $f(k, i) = k$ is a finite colouring. For $X \subseteq N \times \omega$ and $k < n$ define

$$(X, k) = \{(a, i, k) : (a, i) \in X\}.$$

A non-zero element s of $\mathfrak{Cm}\alpha(\mathfrak{G})$ is *monochromatic* if $s \leq 1'$ or $s \leq (\Gamma, k)$ for some $k < n$.

Theorem 6.2 $\mathfrak{Cm}\alpha(\mathfrak{G})$ *is not representable. Hence the full complex cylindric algebra over the set of* n *by* n *basic matrices is not representable either.*

Proof. Assume for contradiction that $g : \mathfrak{Cm}(\alpha(\mathfrak{G})) \longrightarrow \mathbb{C}$ is an embedding into a proper relation set algebra $\mathbb{C}$ with base set X. $\mathfrak{G}$ has a finite colouring, so partition the nodes of $N \times \omega$ into sets $\{C_j : j < N\}$ such that there are no edges within any C_j. Let

$$J = \{1', (C_j, k) : j < N, k < n\}.$$

Then $\sum J = 1$ in $\mathfrak{Cm}\alpha$. As J is finite, for any $x, y \in X$ there is $P \in J$ with $(x, y) \in h(P)$. Since $\mathfrak{Cm}\alpha(\mathfrak{G})$ is infinite, X is infinite. By Ramsey's Theorem, there are distinct $x_i \in X$ $(i < \omega)$ and $P \in J$ such that $(x_i, x_j) \in h(P)$ for all $i < j < \omega$. It is clear that $P \neq 1'$. Also $(P; P) \cdot P \neq 0$. This follows from the fact that if $x_0, x_1, x_2 \in X$, $a, b, c \in \mathfrak{Cm}\alpha(\mathfrak{G})$, $(x_0, x_1) \in h(a)$, $(x_1, x_2) \in h(b)$, and $(x_0, x_2) \in h(c)$, then $(a; b) \cdot c \neq 0$. Now P is monochromatic, it follows from the definition of $\alpha(\mathfrak{G})$ that $(P; P) \cdot P = 0$. This contradiction shows that $\mathfrak{Cm}\alpha(\mathfrak{G})$ is not representable. It follows that $\mathfrak{Cm}\mathcal{M}_n$ is not representable either since we have a relation algebra embedding of $\mathfrak{Cm}\alpha(\mathfrak{G})$ onto $\mathrm{Ra}\mathfrak{Cm}\mathcal{M}_n$. Thus representability of the latter implies representability of the former. □

For undefined terminolgy in the coming corollaries, the reader is referred to [11].

Corollary 6.3

(1) *There exist two atomic relation algebras with the same atom structure, only one of which is representable.*
(2) RRA *is not closed under completions and is not atom-canonical.*
(3) *There exists a non-representable relation algebra with a dense representable subalgebra.*
(4) [17] RRA *is not Sahlqvist axiomatizable.*
(5) *There exists an atomic relation algebra with no complete representation.*

Proof. Write α for $\alpha(\mathfrak{G})$.
(1) $\mathfrak{Tm}\alpha$ and $\mathfrak{Cm}\alpha$ have the same atom structure. $\mathfrak{Tm}\alpha$ is representable and $\mathfrak{Cm}\alpha$ is not.
(2) $\mathfrak{Cm}\alpha$ is the completion of $\mathfrak{Tm}\alpha$. $\mathfrak{Cm}(\mathrm{AtRRA})$ is not contained in RRA. Thus RRA is not atom-canonical.
(3) $\mathfrak{Tm}\alpha$ is dense in $\mathfrak{Cm}\alpha$.
(4) RRA is a conjugated variety that is not closed under completions, hence it is not Sahlqvist axiomatizable.
(5) $\mathfrak{Tm}\alpha$ has no complete represention; else $\mathfrak{Cm}\alpha$ would be representable. □

The analogous result holds for RCA_n. More precisely:

Corollary 6.4

(1) *Let* $2 < n < \omega$. *There exist two atomic cylindric algebras of dimension* n *with the same atom structure, only one of which is representable.*

(2) RCA_n *is not closed under completions and is not atom-canonical.*

(3) *There exists a non-representable* RCA_n *with a dense representable subalgebra.*

(4) [17] RCA_n *is not Sahlqvist axiomatizable.*

(5) *There exists an atomic representable binary generated* CA_n *with no complete representation.*

Proof. $\mathfrak{Tm}\mathcal{M}_n$ and $\mathfrak{Cm}\mathcal{M}_n$ are the cylindric algebras that do the job. □

In [9] and [12] the so-called rainbow construction is used to prove the above corollary. Here we show, by our simple choice of the graph $\mathfrak{G}$, that this degree of complexity is not really needed. The fact that our cylindric algebras are binary generated is also new. Next, we give an application to finite variable fragments of first order logic.

7 Omitting types for finite variable fragments

We work in usual first order logic. In the process, we use standard notation. As before L denotes a signature with no function symbols nor constants. Let T be a countable consistent L^n first order theory, or simply a theory, i. e., T is a set of first order formulas each of which is built up of at most n variables. In addition to that T has a model. Let Γ be a countable set of L^n formulas that is consistent over T, i. e., no contradiction is derivable from $T \cup \Gamma$, equivalently $T \cup \Gamma$ also has a model. For a formula φ and a first order structure $\mathcal{M}$ in the language of φ, write $\varphi^{\mathcal{M}}$ to denote the set of all assignments that satisfy φ in M, i. e.,

$$\varphi^{\mathcal{M}} = \{s \in {}^nM \,:\, \mathcal{M} \models \varphi(s)\}.$$

For example if $\mathcal{M} = (N, <)$ and φ is the formula $x_1 < x_2$, then any $s \in {}^nN$ is in φ^M if $s_1 < s_2$.

Unless otherwise specified T and Γ are as specified above.

Definition 7.1

(i) We say that Γ *is implicitly principal over* T if for all $\mathcal{M} \models T$,

$$\textstyle\bigcap_{\varphi\in\Gamma} \varphi^{\mathcal{M}} \neq \emptyset.$$

(ii) Let $k \in \omega$. We say that Γ *is explicitly* k*-principal over* T, or simply *explicitly* k*-principal*, if there exists a formula φ consistent with T such that φ is built up of at most k variables with at most the first n variables free, and φ isolates Γ, i. e.,

$$T \models \varphi \Rightarrow \psi \quad \text{for all } \psi \in \Gamma.$$

We sometimes say that Γ is explicitly k-principal, when T is clear from context.

The classical Henkin-Orey Omitting Types Theorem [6, Theorem 2.2.9], or rather the contrapositive thereof, implies that for countable languages, if Γ is implicitly principal (over T), then Γ is explicity k principal (over T) for some $k \in \omega$. The question we address here is:

Question 7.2 *Do we always guarantee that* $k \leq n$*, i. e., the formula isolating* Γ*,* φ *say, stays inside* L_n*, or do we have to occasionally "step outside"* L_n*?*

In other words, could φ be *always* chosen to be built up of at most n variables, or do we perhaps, in certain cases, need more than n variables? And if so, is there perhaps an upper bound on the number of variables needed? The following result was announced in [4]. It contrasts positive results on omitting types proved in [14].

Theorem 7.3 *For* $2 < n < k < \omega$ *there exists a countable* L^n*-theory* T*, a type* Γ *consistent over* T *such that* Γ *is implicitly principal but not* k*-explicitly principal over* T*.*

S k e t c h o f p r o o f. This theorem was proved in [15] modulo the existence of certain atom structures, which we now have. α_l denotes the atom structure built on $\mathfrak{G}$, where $n = l$ (see above). Now fix $n < k$. Let $\mathcal{M}_k$ be the cylindric basis for α_k. Let $\mathcal{M}_n$ be the cylindric bases for α_n. Then $\mathfrak{Tm}\mathcal{M}_n \cong \mathfrak{Nr}_n\mathfrak{Tm}\mathcal{M}_k$. Here $\mathfrak{Nr}_n\mathfrak{Tm}\mathcal{M}_k$ denotes the neat n-reduct of $\mathfrak{Tm}\mathcal{M}_k$ [7, Definition 2.6.28]. The rest follows from the properties of the atom structures constructed herein together with [15]. However, for the reader's convenience we repeat the argument in [15]. For brevity, let $\mathcal{A} = \mathfrak{Tm}\mathcal{M}_n$, and let $\mathcal{B} = \mathfrak{Tm}\mathcal{M}_k$. By a straighforward identification, we can assume without loss of generality that $\mathcal{A} = \mathfrak{Nr}_n\mathcal{B}$. By [8, Theorem 4.3.28(ii)] there is a countable first order language Λ such that

$$\mathcal{B} \cong (\mathrm{Fm}^{\Lambda_k}/T)$$

for some (countable) L_k-theory $T \subseteq \mathrm{Fm}^{\Lambda_k}$. Here Λ_k is the restriction of the language Λ to the first k variables and $\mathrm{Fm}^{\Lambda_k}/T$ is the Lindenbaum-Tarski representable k-dimensional cylindric algebra corresponding to T. It can be assumed that T consists of sentences only, i. e., that no free variables occur in formulas in T. It follows that

$$\mathcal{A} \cong \mathrm{Fm}^{\Lambda_n}/T.$$

Fix θ an isomorphism from $\mathrm{Fm}^{\Lambda_n}/T$ to $\mathcal{A}$ and let $\mathrm{At}\mathcal{A}$ denote the set of atoms of $\mathcal{A}$. Put

$$\Gamma = \bigcup\{\neg\varphi/T \,:\, \theta(\varphi/T) \in \mathrm{At}\mathcal{A}\}.$$

Recall that φ/T denotes the equivalence class of φ, consisting of all formulas equivalent to φ modulo T. Note that Γ is a set of L_n formulas. Since $\mathcal{A}$ is atomic and $A = Nr_nB$ then it follows that the supremum of $\mathrm{At}\mathcal{A}$ evaluated in $\mathcal{B}$ is equal to the unit, in symbols $\sum^B \mathrm{At}\mathcal{A} = 1$. Indeed, we have $\sum^{\mathcal{A}}\mathrm{At}\mathcal{A} = 1$ since $\mathcal{A}$ is atomic and the Boolean reduct of $\mathcal{A}$ is a complete subalgebra of that of $\mathcal{B}$. Thus we have that the following infimum is equal to the least element, in symbols

$$(+) \qquad \bigwedge\{\theta(\psi/T) \,:\, \psi \in \Gamma\} = 0.$$

Now we check that Γ is as desired, i. e., Γ is implicitly principal but not k-explicitly principal. To see that Γ is not explicitly k-principal, assume to the contrary that there exists $\varphi \in \mathrm{Fm}^{\Lambda_k}$ such that φ is consistent with T and φ isolates Γ. We can assume without loss of generality that the free variables occurring in φ are among the first n. But then we get that for all $\psi \in \Gamma$,

$$0 < \theta(\varphi/T) \leq \theta(\psi/T).$$

This contradicts $(+)$. To see that Γ is implicitly principal, assume to the contrary that Γ can be omitted, i. e., there exists $\mathcal{M} \models T$ such that $\bigcap_{\varphi\in\Gamma} \varphi^{\mathcal{M}} = \emptyset$. Then

$$\bigcup_{\varphi\in\Gamma}(\neg\varphi^{\mathcal{M}}) = {}^nM.$$

But then $\{\varphi^{\mathcal{M}} \,:\, \varphi \in \mathrm{Fm}^{\Lambda_n}\}$ would be the domain of a cylindric set algebra that is an atomic representation of $\mathcal{A}$ in the sense of [9, Definition 4], thus by [9, Theorem 5], it is a *complete* representation of $\mathcal{A}$. But this contradicts the fact that $\mathcal{A}$ is not completely representable. □

A different version of the above theorem is proved in [2]. We note that our construction quite easily leads to the (new) fact that the Omitting Types Theorem fails for finite first order definable extensions of finite variable fragments of first order logic studied in [5] and [16] as long as the number of variables is > 2.

References

[1] H. Andréka, J. D. Monk, and I. Németi (eds.), Algebraic Logic (North-Holland, 1991).
[2] H. Andréka, I. Németi, and T. Sayed Ahmed, Omitting types for finite variable fragments and complete representations for algebras. To appear in the Journal of Symbolic Logic.
[3] H. Andréka, J. Van Benthem, and I. Németi, Modal languages and bounded fragments of predicate logic. J. Philosophical Logic **27**, 217 – 274 (1998).

[4] H. Andréka and T. Sayed Ahmed, Omitting types in logics with finitely many variables. Bull. Symbolic Logic **5**, p. 88 (1999).
[5] B. Biró, Non-finite axiomatizability results in algebraic logic. J. Symbolic Logic **57**, 832 – 843 (1992).
[6] C. Chang and J. Keisler, Model Theory (North-Holland, 1994).
[7] L. Henkin, J. D. Monk, and A. Tarski, Cylindric Algebras, Part I (North-Holland, 1971).
[8] L. Henkin, J. D. Monk, and A. Tarski, Cylindric Algebras, Part II (North-Holland, 1985).
[9] R. Hirsch and I. Hodkinson, Complete representations in algebraic logic. J. Symbolic Logic **62**, 816 – 847 (1997).
[10] R. Hirsch and I. Hodkinson, Strongly representable atom structures. Proc. Amer. Math. Soc. **130**, 1819 – 1831 (2002).
[11] R. Hirsch and I. Hodkinson, Relation Algebras by Games. Studies in Logic and the Foundations of Mathematics 147 (Elsevier, 2002).
[12] I. Hodkinson, Atom structures of cylindric algebras and relation algebras. Annals Pure Appl. Logic **89**, 117 – 148 (1997).
[13] R. Maddux, Introductory course on relation algebras. In [1], pp. 361 – 392.
[14] T. Sayed Ahmed, Martin's axiom, omitting types and complete representations in algebraic logic. Studia Logica **72**, 1 – 25 (2002).
[15] T. Sayed Ahmed, Omitting types for finite variable fragments of first order logic. Bull. Section of Logic **32**, 117 – 122 (2003).
[16] T. Sayed Ahmed and B. Samir, A neat embedding theorem for expansions of cylindric algebras. Logic J. IGPL **15**, 41 – 51 (2007).
[17] Y. Venema, Atom structures and Sahlqvist equations. Algebra Universalis **38**, 185 – 199 (1997).

