Math. Log. Quart. **48** (2002) 3, 343 – 355

Mathematical Logic
Quarterly

A Modeltheoretic Solution to a Problem of Tarski

Tarek Sayed Ahmed[1)]

Department of Mathematics, Cairo University, Cairo, Egypt.[2)]

Abstract. Let $1 < n < \omega$ and $\beta > n$. We show that the class $\mathrm{Nr}_n\mathrm{CA}_\beta$ of n-dimensional neat reducts of β-dimensional cylindric algebras is not closed under forming elementary subalgebras. This solves a long-standing open problem of Tarski and his co-authors Andréka, Henkin, Monk and Németi. The proof uses genuine model-theoretic arguments.

Mathematics Subject Classification: 0G15, 03C10.

Keywords: algebraic logic, cylindric algebras, neat reducts, elementary.

0 Introduction and main result

The theory of cylindric algebras of dimension β, β an ordinal, is an abstract algebraic analogue of a first order logic with β-many variables x_i with $0 \le i < \beta$, just as the theory of Boolean algebras is an abstract algebraic analogue of sentential logic. In addition to the Boolean operations, cylindric algebras have unary *cylindric operations* c_i one for each index $i < \beta$. The operation c_i is an abstract version of the unary operation on first order formulas of existential quantification with respect to the ith variable x_i. Cylindric algebras also have distinguished constants d_{ij} for $i, j < \beta$. The constant d_{ij} is called a *diagonal element* and is an abstract version of the atomic identity formula $x_i = x_j$ in first order logic. The class of cylindric algebras of dimension β is denoted by CA_β and is axiomatized by finitely many equational schemata that aim at capturing the essential algebraic properties of existential quantification and atomic indentity formulas. We consider $\mathcal{A} \in \mathrm{CA}_\beta$ to be of the form $\langle A, \cdot, -, \mathsf{c}_i, \mathsf{d}_{ij}\rangle_{i,j\in\beta}$, where $\langle A, \cdot, -, \rangle$ – the *Boolean reduct of* $\mathcal{A}$ – is a Boolean algebra, with "$\cdot$" denoting Boolean *meet* and "$-$" denoting Boolean *complementation*. Suppose that $\mathcal{A}$ is a CA_β and suppose that $n < \beta$. Let A denote the universe of $\mathcal{A}$. Consider the set B of elements of A that are closed under the cylindric operations c_i for $n \le i < \beta$, that is, the elements $a \in A$ for which $\mathsf{c}_i a = a$ whenever $n \le i < \beta$. The algebra whose universe is B, and whose operations are the Boolean operations of $\mathcal{A}$ and the cylindric operations and constants with indices less than n is called *the neat reduct* of $\mathcal{A}$ to the dimension n, or simply the *neat n-reduct of* $\mathcal{A}$, and is denoted by $\mathcal{N}r_n\mathcal{A}$. If $\mathcal{A}$ represents a first order theory with β-many variables, then $\mathcal{N}r_n\mathcal{A}$ is the algebra of formulas using β many variables in which only the first n variables are allowed to be free. The class

[1)] The results in this article were presented by the author in Imperial College in October 1999. The author wishes to thank Ian Hodkinson and Robin Hirsch for making his visit to London possible. Thanks are due to Ian Hodkinson for writing down Lemma 2 as presented herein.

[2)] e-mail: rutahmed@rusys.eg.net

of all neat n-reducts of cylindric algebras of dimension β with $n < \beta$ is denoted by $\mathrm{Nr}_n\mathrm{CA}_\beta$. Thus $\mathrm{Nr}_n\mathrm{CA}_\beta = \{\mathcal{N}r_n\mathcal{A} : \mathcal{A} \in \mathrm{CA}_\beta\}$. The notion of neat reducts and the related notion of neat embeddings play a key role in the representation theory of cylindric algebras and variants thereof, like HALMOS' polyadic algebras. In fact, a classical result of HENKIN, the so-called *Neat Embedding Theorem*, or NET for short, in [8] states that the class of representable cylindric algebras of dimension α coincides with the class of algebras (of the same dimension) having the *neat embedding property.* $\mathcal{A} \in \mathrm{CA}_\alpha$ has the neat embedding property, if it can be *neatly embedded* in a $\mathrm{CA}_{\alpha+\omega}$, i.e., if $\mathcal{A}$ can be embedded into $\mathcal{N}r_\alpha\mathcal{B}$ for some $\mathcal{B} \in \mathrm{CA}_{\alpha+\omega}$ (see [8]). Representation results, like the NET of HENKIN, are proved in algebraic logic to be equivalent to completeness in the corresponding logics (cf. [9, Section 4.3.], [3], and [25]). Other metalogical properties, like interpolation and omitting types, are investigated (from an algebraic point of view) in connection to special neat embeddings in [32, 30, 31, 5]. In this note we shall prove:

Theorem 1 (Main Theorem). *Let $2 < n < \omega$, and let β be an ordinal $> n$. Then the class $\mathrm{Nr}_n\mathrm{CA}_\beta$ is not closed under elementary subalgebras. In particular, $\mathrm{Nr}_n\mathrm{CA}_\beta$ is not first order axiomatizable.*

In other words, if we start with the class of algebras that arise from a first order logic with β-many variables, and then restrict our attention to those algebras that concern formulas with n free variables with $2 < n < \omega \cap \beta$, then this class cannot be axiomatized by means of first order sentences. This solves the finite part of the long-standing open problem 4.4 in the monograph [9] on cylindric algebras by HENKIN, MONK and TARSKI. Theorem 1 also confirms a conjecture of NÉMETI formulated as Conjecture 1 in [21]. In this paper, NÉMETI proves that for any pair of ordinals $n < \beta$, the class $\mathrm{Nr}_n\mathrm{CA}_\beta$ is closed under ultraproducts. By the celebrated Keisler-Shelah Ultrapower Theorem (cf. [6, Theorem 6.15]) our Theorem 1 implies that for $1 < n < \beta \cap \omega$, the class $\mathrm{Nr}_n\mathrm{CA}_\beta$ is not closed under ultraroots. Various refinements of our result summarized in Corollary 3 below also solve the finite part of Problem 18 on [10, p. 312]. We note that ANDRÉKA and NÉMETI prove in 8.6 of [10, Part II] that $\mathrm{Nr}_2\mathrm{CA}_\beta$ is not elementary. Theorem 1 was announced in [27], and a different proof of Theorem 1 is given in [29]. A novelty that occurs here is the proof method, which – as indicated in the title – uses genuinely model-theoretic arguments inspired by a theorem of Fraïssé. The problem as to whether the class of neat reducts is closed under (elementary) subalgebras has a long history and dates back to Problem 2.11 in [8] and to related questions raised by PIGOZZI in [24] (cf. [24, Remark 2.2.21]). An implicit form of this problem appears as item (v) in the introduction of the monograph [9] as a conjecture of TARSKI whose proof could not be reconstructed. The reader is referred to [21, 32] for further elaboration of this problem.

1 Proofs

In this section we prove Theorem 1. For simplicity of notation we give the proof for $n = 3$. We start by formulating a Lemma (Lemma 2) stipulating the existence of a first order structure M. This first order structure M, among other conditions, enjoys quantifier elimination, meaning that every formula in the first order language of M is equivalent to a quantifier free one. Because M has quantifier elimination, the cylindric set algebra $\mathcal{A}$, consisting of all those relations definable in M using only 3 variables, will be a neat reduct. Using the other conditions, we then extract an elementary subalgebra $\mathcal{B}$ of $\mathcal{A}$ that is not a neat reduct. We proceed as follows. First we prove (modulo Lemma 2) that the class of neat reducts is not elementary, then we prove our Lemma at the end of this section. To formulate Lemma 2 we need the following notations:

By S_3 we denote the set of all permutations of 3. ${}^XY$ denotes the set of functions from X to Y. For u, $v \in {}^33$ and $i < 3$ we write u_i for $u(i) < 3$, and we write $u \equiv_i v$ if u and v agree except for i, i. e., if $u_j = v_j$ for all $j \in 3 \setminus \{i\}$. We identify (notationally) a structure M with its domain. For a symbol R of the signature of M we write R^M for the interpretation of R in M.

Now we are ready to formulate our Lemma.

Lemma 2. *Let L be a signature consisting of the unary relation symbols P_0, P_1, P_2 and uncountably many 3-ary predicate symbols. For $u \in {}^33$, let χ_u be the formula $\bigwedge_{i<3} P_{u_i}(x_i)$. Then there exists an L-structure M with the following properties:*

(1) *M has quantifier elimination, i. e., every L-formula is equivalent in M to a Boolean combination of atomic formulas.*

(2) *The sets P_i^M for $i < 3$ form a partition of M.*

(3) *$M \models \forall x_0x_1x_2\,(R(x_0, x_1x_2) \rightarrow \bigvee_{u \in S_3} \chi_u)$, for all $R \in L$.*

(4) *$M \models \exists x_0x_1x_2\,(\chi_u \wedge R(x_0, x_1, x_2) \wedge \neg S(x_0, x_1, x_2))$ for all distinct ternary R, $S \in L$ and $u \in S_3$.*

(5) *For $u \in S_3$ and $i < 3$, $M \models \forall x_0x_1x_2\,(\exists x_i\,\chi_u \leftrightarrow \bigvee_{v \in {}^33, v \equiv_i u} \chi_v)$.*

(6) *For $u \in S_3$ and any L-formula $\varphi(x_0, x_1, x_2)$, if $M \models \exists x_0x_1x_2\,(\chi_u \wedge \varphi)$, then $M \models \forall x_0x_1x_2\,(\exists x_i\,\chi_u \leftrightarrow \exists x_i\,(\chi_u \wedge \varphi))$ for all $i < 3$.*

Outline of proof. First we explain the idea behind the construction of such an M. In the process we give a sketchy outline of the proof that the class of neat reducts is not elementary in order to pave the way for a smooth (formal) proof of Theorem 1.

Property (1) of quantifier elimination postulates that the set of atomic formulas

$$\begin{aligned} J = &\{R(y_0, y_1, y_2) : \{y_0, y_1.y_2\} = \{x_0, x_1, x_2\} \text{ and } R \in L \text{ is a ternary relation}\} \\ &\cup \{P_i(x_j) : i, j < 3\} \cup \{x_i = x_j : i, j < 3\} \end{aligned}$$

is an elimination set for M. This means that every L-formula is equivalent in M to a Boolean combination of formulas in J. This implies that the cylindric set algebra consisting of those relations definable using only the first three variables is a neat reduct. In more detail, for an L-formula φ, let φ^M be the set of all assignments satisfying φ in M, i. e., $\varphi^M = \{s \in {}^\omega M : M \models \varphi[s]\}$. For an ordinal α, let Cs_α denote the class of cylindric set algebras of dimension α. Let $\mathcal{A}_\omega$ be the Cs_ω with domain $\{\varphi^M : \varphi \in L\}$ and operations (well-) defined by (cf. [9])[3)]

$$\varphi^M \cdot \psi^M = \varphi^M \cap \psi^M = (\varphi \wedge \psi)^M, \qquad -\varphi^M = (\neg\varphi)^M,$$

[3)] In [9, Section 4.3, Definition 4.3.4] $\mathcal{A}_\omega$ would be denoted by Cf_3^M, which is the set algebra based on M. In this connection we note that $\mathcal{A}$ is a regular locally finite ω-dimensional cylindric set algebra. The notion of a regular locally finite set algebra will be recalled below.

and for i, $j < \omega$

$$\mathsf{d}_{ij} = (x_i = x_j)^M, \qquad \mathsf{c}_i(\varphi^M) = (\exists x_i \varphi)^M.$$

Now we write L_3 for the set of all L-formulas using only the first three variables. A moment's reflection reveals that property (1) equates the Cs_3 $\mathcal{A}$ having domain $\{\varphi^M : \varphi \in L_3\}$ with the (possibly bigger) Cs_3 having domain

$$\{\varphi^M : \varphi \in L \text{ and } \varphi \text{ contains } x_0, x_1, x_2 \text{ as free variables}\},$$

with the cylindric operations defined for both as for the operations defined for $\mathcal{A}_\omega$. But the latter, as easily checked, is isomorphic to $\mathcal{N}r_3\mathcal{A}_\omega$, so property (1) guarantees that $\mathcal{A} \in \mathrm{Nr}_3\mathrm{CA}_\omega$. The rest of the conditions are designed to extract an elementary subalgebra of $\mathcal{A}$ that is not in $\mathrm{Nr}_3\mathrm{CA}_4$. But let us first understand the (abstract) structure of $\mathcal{A}$ based on M. For the sake of brevity, let 1_u denote χ_u^M. Then it follows from property (2) that the set $\{1_u : u \in {}^33\}$ forms a partition of ${}^3M$, the unit of $\mathcal{A}$. If $u \in S_3$, i.e. u is a permutation, then, by property (4), below every 1_u there are uncountably many pairwise distinct non-empty elements, namely the $R(x_0, x_1, x_2)^M$'s intersected with 1_u. Such elements are *big* as far as the cylindrifications are concerned, that is for $i < 3$ we have by properties (3), (6)

$$\mathsf{c}_i[R(x_0, x_1, x_2)^M \cap 1_u] = \mathsf{c}_i(1_u) = \bigcup\{1_u : u \equiv_i v\}.$$

Having explained the idea behind the conditions of Lemma 2, we explain how we can obtain an elementary subalgebra of $\mathcal{A}$ that is not a neat reduct. For $u \in {}^33$, let A_u stand for the *relativisation* of A to 1_u, i.e., $A_u = \{x \in A : x \leq 1_u\}$. A_u is the domain of a Boolean set algebra which we denote by $\mathcal{A}_u$. Then for $u \in S_3$, A_u is uncountable. Because $\{1_u : u \in {}^33\}$ is a partition of the unit of $\mathcal{A}$, it follows that the Boolean reduct of $\mathcal{A}$ is isomorphic to the Boolean product $\prod_{u \in {}^33} \mathcal{A}_u$. Moreover, we can (and will) expand the language of Boolean algebras by diagonal elements and the constants 1_u in such a way that the cylindric algebra $\mathcal{A}$ becomes *interpretable* in this product. Then we extract an elementary subalgebra $\mathcal{B}$ of $\mathcal{A}$ by an infinite cardinality twist. $\mathcal{B}$ is obtained from $\mathcal{A}$ by keeping only countably many elements below 1_{Id}, where Id is the identity function on 3, and discarding the rest of the elements below 1_{Id}. In the product, this corresponds to replacing the component $\mathcal{A}_{\mathrm{Id}}$ by an elementary countable Boolean subalgebra $\mathcal{B}_{\mathrm{Id}}$ of $\mathcal{A}_{\mathrm{Id}}$ and giving the resulting algebra the interpretation given to the Boolean product $\prod_{u \in {}^33} \mathcal{A}_u$. This will not be witnessed by first order logic and will ensure that the resulting structure $\mathcal{B}$ is *not* a neat reduct. In fact, $\mathcal{B}$ will not be even in $\mathrm{Nr}_3\mathrm{CA}_4$. The idea is that if $\mathcal{B}$ were a neat reduct, then using a substitution term definable in one extra dimension, we generate uncountably many elements in the component B_{Id}, which contradicts that the latter (by its very construction) is countable. Now we implement the details of the above sketch.

From now on, we follow the conventions of algebraists. We do not distinguish between an algebra $\mathcal{A}$ and its domain A, and denote both by A. If C is a cylindric algebra, we let $boole(C)$ denote its Boolean reduct of C. For a non-empty set I, and a family of algebras $\langle A_i : i \in I\rangle$ (of the same signature) indexed by I, we let $\prod_{i \in I} A_i$ denote the *product* of the A_i's. For a Boolean algebra $\langle B, \cdot, -\rangle$ and a finite subset X of B, we write $\bigwedge X$ and $\sum X$ for the *product* and the *sum* of elements of X, respectively. In particular $+$ denotes the Boolean join.

D e t a i l s o f p r o o f. Fix L and M as in Lemma 2. Let $A_\omega \in \mathrm{Cs}_\omega$, and $A \in \mathrm{Cs}_3$ be as specified above. That is $A_\omega = \{\varphi^M : \varphi \in L\}$ and $A = \{\varphi^M : \varphi \in L_3\}$. Then $A \cong \mathrm{Nr}_3 A_\omega$ via the map $i : A \longrightarrow A_\omega$ defined by $i(\varphi^M) = \varphi^M$. This map is of course well defined. Furthermore, quantifier elimination in M guarantees that this map is onto $\mathrm{Nr}_n A_\omega$. For $u \in {}^33$, let A_u denote the relativisation of A to χ_u^M, i.e., $A_u = \{x \in A : x \leq \chi_u^M\}$. Then A_u is uncountable for every $u \in S_3$, because by property (4) of Lemma 2, the sets $(\chi_u \wedge R(x_0, x_1, x_2))^M$, for $R \in L$ are distinct elements of A_u. Define a map $f : A \longrightarrow \prod_{u \in {}^33} A_u$ by $f(a) = \langle a \cdot \chi_u \rangle_{u \in {}^33}$. We will expand the language of the Boolean algebra $\prod_{u \in {}^33} \mathit{boole}(A_u)$ in such a way that the cylindric algebra A becomes interpretable in the expanded structure. For this we need

D e f i n i t i o n 3. Let P denote the following structure for the signature of Boolean algebras expanded by constant symbols 1_u for $u \in {}^33$ and d_{ij} for $i, j \in 3$:

(1) The Boolean part of P is the Boolean algebra $\prod_{u \in {}^33} \mathit{boole}(A_u)$,

(2) $1_u^P = f(\chi_u^M) = \langle 0, \ldots, 0, 1, 0, \ldots \rangle$ (with the 1 in the u^{th} place) for each $u \in {}^33$,

(3) $\mathsf{d}_{ij}^P = f(\mathsf{d}_{ij}^A)$ for $i, j < 3$.

We now show (cf. [18, 5.3, p. 212]), that A is interpretable in P. For this it is enough to show that f is one to one and that $\mathrm{ran}(f)$ (range of f) and the f-images of the graphs of the cylindric algebra functions in A are definable in P. Since the χ_u^M's form a partition of the unit of A, each $a \in A$ has a unique expression in the form $\sum_{u \in {}^33}(a \cdot \chi_u^M)$, and it follows that $f : \mathit{boole}(A) \longrightarrow \prod_{u \in {}^33} \mathit{boole}(A_u)$ is a Boolean isomorphism.

So the f-images of the graphs of the Boolean functions on A are trivially definable. f is bijective so $\mathrm{ran}(f)$ is definable by $x = x$. For the diagonals, $f(\mathsf{d}_{ij}^A)$ is definable by $x = \mathsf{d}_{ij}$. Finally, we consider cylindrifications. For $S \subseteq {}^33$, $i < 3$, let t_S be the closed term $\sum\{1_v : v \in {}^33,\ v \equiv_i u \text{ for some } u \in S\}$. Let

$$\eta_i(x, y) = \bigwedge_{S \subseteq {}^33}\Big(\bigwedge_{u \in S} x \cdot 1_u \neq 0 \wedge \bigwedge_{u \in {}^33 \setminus S} x \cdot 1_u = 0 \rightarrow y = t_S\Big).$$

We claim that for all $a \in A$, $b \in P$, we have $P \models \eta_i(f(a), b)$ iff $b = f(\mathsf{c}_i^A a)$. To see this, let $f(a) = \langle a_u \rangle_{u \in {}^33}$. So in A we have $a = \sum_{u \in {}^33} a_u$. Let u be given. Then a_u has the form $(\chi_i \wedge \varphi)^M$ for some $\varphi \in L^3$, so that $\mathsf{c}_i^A(a_u) = (\exists x_i\, (\chi_u \wedge \varphi))^M$. By property (6) of Lemma 2, if $a_u \neq 0$, this is $(\exists x_i\, \chi_u)^M$; by property (5), this is $(\bigvee_{v \in {}^33, v \equiv_i u} \chi_v)^M$. Let $S = \{u \in {}^33 : a_u \neq 0\}$. By normality and additivity of cylindrifications we have

$$\begin{aligned}\mathsf{c}_i^A(a) &= \sum_{u \in {}^33} \mathsf{c}_i^A a_u = \sum_{u \in S} \mathsf{c}_i^A a_u = \sum_{u \in S}\Big(\sum_{v \in {}^33,\, v \equiv_i u} \chi_v^M\Big)\\ &= \sum\{\chi_v^M : v \in {}^33,\ v \equiv_i u \text{ for some } u \in S\}.\end{aligned}$$

So $P \models f(\mathsf{c}_i^A a) = t_S$. Hence it follows that $P \models \eta_i(f(a), f(\mathsf{c}_i^A a))$. Conversely, if $P \models \eta_i(f(a), b)$, we require that $b = f(\mathsf{c}_i^A a)$. Towards this end, let S be the unique subset of ^{33} such that $P \models \bigwedge_{u \in S} f(a) \cdot 1_u \neq 0 \wedge \bigwedge_{u \in {}^33 \setminus S} f(a) \cdot 1_u = 0$. So we obtain $b = t_S = f(\mathsf{c}_i^A a)$.

We have proved that P is interpretable in A.[4)] Next, we extract an elementary subalgebra B of A that is not a neat reduct, i.e., B is not in $\mathrm{Nr}_3\mathrm{CA}_4$. This will

[4)] Furthermore it is easy to see that the interpretation is one dimensional and quantifier free in the sense of [18].

imply that $B \notin \mathrm{Nr}_3\mathrm{CA}_\beta$ for all $\beta \geq 4$, since for any such β it is easy to see that $\mathrm{Nr}_3\mathrm{CA}_\beta \subseteq \mathrm{Nr}_3\mathrm{CA}_4$. Let $\mathrm{Id} \in {}^33$ be the identity map on 3. Choose any countable elementary Boolean subalgebra of $boole(A_{\mathrm{Id}})$, B_{Id} say. Thus $B_{\mathrm{Id}} \preceq boole(A_{\mathrm{Id}})$. By the Feferman-Vaught Theorem [18, Corollary 9.6.5]

$$\begin{aligned} Q &= ((B_{\mathrm{Id}} \times \textstyle\prod_{u\in {}^33\setminus \mathrm{Id}} boole(A_u)), 1_u, \mathsf{d}_{ij})_{u\in {}^33,\, i,\, j<3} \\ &\preceq (\textstyle\prod_{u\in {}^33} boole(A_u)), 1_u, \mathsf{d}_{ij})_{u\in {}^33, i,j<3} = P. \end{aligned}$$

Let B be the result of applying the interpretation given above to Q. By [18, Theorem 5.3.2 and Remark 3], $B \equiv A$ as cylindric algebras. Indeed $B \preceq A$. Therefore $B \in \mathrm{RCA}_3$, where RCA_β denotes the class of representable CA_β (see [9]). In fact it is easy to see that B is simple (since simplicity is expressible by a first order formula) so that B is isomorphic to a Cs_3. Now we show that B cannot be a neat reduct, in fact we show that $B \notin \mathrm{Nr}_3\mathrm{CA}_\beta$ for any $\beta > 3$. Assume, seeking a contradiction, that $B = \mathrm{Nr}_3 D$ for some $D \in \mathrm{CA}_\beta$ with $\beta > 3$. Note that D may not be representable. In this paper, this is the only place where we deal with possibly non-representable algebras. Now $\chi_u^M \in B$ for each $u \in {}^33$. As is common practise, identifying functions with sequences, we let $v = \langle 1, 0, 2\rangle \in {}^33$. Let $t(x)$ be the unary CA_2 term $\mathsf{s}_1^0\mathsf{c}_1 x \cdot \mathsf{s}_0^1\mathsf{c}_0 x$, where $\mathsf{s}_i^j x = \mathsf{c}_i(\mathsf{d}_{ij}\cdot x)$ for $i \neq j$. Then we claim that $t^B(\chi_v^M) = \chi_{\mathrm{Id}}^M$. For better readability, in our calculations we omit the superscript B. For the sake of brevity, we denote χ_v^M by 1_{10} and χ_{Id}^M by 1_{01}. Then, by definition, we have $t(1_{01}) = \mathsf{c}_0(\mathsf{d}_{01}\cdot \mathsf{c}_1 1_{10}) \cdot \mathsf{c}_1(\mathsf{d}_{01}\cdot \mathsf{c}_0 1_{10})$. Computing we get

$$\mathsf{c}_0(\mathsf{d}_{01}\cdot \mathsf{c}_1 1_{10}) = \mathsf{c}_0(\mathsf{d}_{01}\cdot (\textstyle\sum\{1_u : u \equiv_1 1_{10}\}) = \mathsf{c}_0(d_{01}\cdot 1_{112}) = 1_{01} + 1_{112}.$$

Here 1_{112} denotes $\chi_{\langle 1,1,2\rangle}$. Note that we are using that the evaluation of the term $\mathsf{c}_1 1_{10}$ in B is equal to its value in A. This is so because B inherits the interpretation given to $\prod A_u$. A similar computation gives $\mathsf{c}_1(\mathsf{d}_{01}\cdot \mathsf{c}_0 1_{01}) = 1_{002} + 1_{01}$, where 1_{002} denotes $\chi_{\langle 0,0,2\rangle}$. Therefore, as claimed, $t^B(1_{10}) = 1_{01}$. Now let ${}_3\mathsf{s}(0,1)$ be the unary substitution term as defined in [8, 1.5.12], that is ${}_3\mathsf{s}(0,1)x = \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{s}_3^1(x)$. Then for any $\beta > 3$ we have $\mathrm{CA}_\beta \models {}_3s(0,1)\mathsf{c}_3 x \leq t(\mathsf{c}_3 x)$. Indeed, by [8, 1.5.12, 1.5.8 and 1.5.10(ii)], we get

$$\begin{aligned} {}_3\mathsf{s}(0,1)\mathsf{c}_3 x &\leq {}_3\mathsf{s}(0,1)\mathsf{c}_1\mathsf{c}_3 x = \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{s}_3^1\mathsf{c}_1\mathsf{c}_3 x = \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{c}_1\mathsf{c}_3 x = \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{c}_3\mathsf{c}_1 x \\ &= \mathsf{s}_0^3\mathsf{c}_3\mathsf{s}_1^0\mathsf{c}_1 x = \mathsf{c}_3\mathsf{s}_1^0\mathsf{c}_1 x = \mathsf{s}_0^1\mathsf{c}_1\mathsf{c}_3 x. \end{aligned}$$

Similarly ${}_3\mathsf{s}(0,1)\mathsf{c}_3 x \leq \mathsf{s}_1^0\mathsf{c}_0\mathsf{c}_3 x$. Therefore, ${}_3\mathsf{s}(0,1)\mathsf{c}_3 x \leq t(\mathsf{c}_3 x)$. It thus follows that

$$D \models {}_3\mathsf{s}(0,1)(1_{10}) \leq \mathsf{s}_1^0\mathsf{c}_1(1_{10}) \cdot \mathsf{s}_0^1\mathsf{c}_0(1_{10}) = 1_{01}.$$

Now ${}_3\mathsf{s}(0,1)$ preserves $\leq$ and permutes $\mathrm{Nr}_3 D$. In fact ${}_3\mathsf{s}(0,1)$ is a Boolean automorphism of $\mathrm{Nr}_3 D$ with inverse ${}_3\mathsf{s}(1,0)$. Now let

$$B_v = \{x \in B : x \leq 1_{10}\} \quad \text{and} \quad A_v = \{x \in A : x \leq 1_{10}\}.$$

Then $B_v = A_v$, and it follows (by property (4) of Lemma 2) that B_v is uncountable. Since ${}_3\mathsf{s}(0,1)$ is one to one, it follows that ${}_3\mathsf{s}(0,1)B_v = \{{}_3\mathsf{s}(0,1)x : x \in B_v\}$ has the same cardinality as B_v. Now let $x \in B_v$. Then $x \leq 1_{10}$. Since ${}_3\mathsf{s}(0,1)$ preserves order we have ${}_3\mathsf{s}(0,1)x \leq_3 \mathsf{s}(0,1)1_{10} = 1_{01}$. Thus ${}_3\mathsf{s}(0,1)B_v \subseteq B_{\mathrm{Id}} = \{b \in B : b \leq 1_{01}\}$, and so it follows that B_{Id} is uncountable too. But by construction, we have that $B_{\mathrm{Id}} = \{b \in B : b \leq 1_{01}\}$ is countable. This contradiction shows that $B \notin \mathrm{Nr}_3\mathrm{CA}_\beta$ for any $\beta > 3$. □

We now conclude:

Theorem 1. *Let* $\beta > 3$. *Then* $\mathrm{Nr}_3\mathrm{CA}_\beta$ *is not closed under forming elementary subalgebras, hence is not elementary.*

Now before proving Lemma 2, we consider certain refinements of Theorem 1, obtained by narrowing the class CA_β.

Definition 4. Let Ws_β stand for the class of weak set algebras of dimension β. We recall from [9] that these are CA_β's whose units are of the form

$${}^\beta U^{(p)} = \{s \in {}^\beta U : |\{i \in \beta : s_i \neq p_i\}| < \omega\}$$

for some $p \in {}^\beta U$. Let β be an infinite ordinal. Then $\mathrm{Cs}^{\mathrm{reg}}_\beta$ and Lf_β stand for the classes of regular set algebras and for the locally finite algebras of dimension β, respectively. We recall from [9, Definition 3.1.1(viii)] that $\mathcal{A} \in \mathrm{Cs}_\beta$ with unit ${}^\beta U$ is regular if for all $x \in A$ and for all $f, g \in {}^\beta U$, whenever $f \upharpoonright \Delta x \subseteq g$ and $f \in x$, then $g \in x$. $\mathcal{A} \in \mathrm{Cs}_\beta$ is locally finite if Δx is finite for every element in A. Here Δx, more commonly referred to as the *dimension set* of x is the set $\{i \in \beta : \mathsf{c}_i(x) = jx\}$ (cf. [8]).

We note that the class $\mathrm{Lf}_\omega \cap \mathrm{Cs}^{\mathrm{reg}}_\omega$ is the algebraic counterpart of first order models. For more on such connections the reader is referred to [4], [9, Section 4.3], and [21].[5)] By [9, Theorem 3.1.134], we have $\mathrm{Ws}_\beta \subseteq \mathsf{I}\mathrm{Cs}_\beta \cap \mathrm{Lf}_\beta$ for any $\beta > 3$. Here I stands for the operation of taking all isomorphic copies. For $\beta > 2$, RCA_β is a variety that coincides with the class of all isomorphic copies of subdirect products of weak set algebras. From the above we immediately get

Corollary 5. *Let* $\beta > 3$. *Let* $\mathsf{I}\mathrm{Ws}_\beta \subseteq L \subseteq \mathrm{CA}_\beta$. *Then* $\mathrm{Nr}_3 L = \{\mathrm{Nr}_3 A : A \in L\}$ *is not elementary. In particular,* $\mathrm{Nr}_3\mathrm{RCA}_\beta$ *is not elementary.*

Corollary 5 answers Problem 18 on [10, p. 312] but only for the finite dimensional case. The infinite version of Corollary 5 is proved in [29].

The rest of this section is devoted to proving Lemma 2. The construction is model-theoretic, so we need to review some model-theoretic notions from [18].

Definition 6.

(1) Let L be a signature and D an L structure. The *age* of D is the class $\boldsymbol{K}$ of all finitely generated structures that can be embedded in D. A class $\boldsymbol{K}$ is the *age* of D if the structures in $\boldsymbol{K}$ are up to isomorphism, exactly the finitely generated substructures of D.

(2) Let $\boldsymbol{K}$ be a class of structures.

(i) $\boldsymbol{K}$ has the *Hereditary Property*, HP for short, if whenever $A \in \boldsymbol{K}$ and B is a finitely generated substructure of A, then B is isomorphic to some structure in $\boldsymbol{K}$.

(ii) $\boldsymbol{K}$ has the *Joint Embedding Property*, JEP for short, if whenever $A, B \in \boldsymbol{K}$, then there is a $C \in \boldsymbol{K}$ such that both A and B are embeddable in C.

(iii) $\boldsymbol{K}$ has *Amalgamation Property*, AP for short, if $A, B, C \in \boldsymbol{K}$ and $e : A \longrightarrow B$, $f : A \longrightarrow C$ are embeddings, then there are D in $\boldsymbol{K}$ and embeddings $g : B \longrightarrow D$ and $h : C \longrightarrow D$ such that $g \circ e = h \circ f$.

[5)] In fact, one can define an isomorphism $h : \mathit{Models} \longrightarrow \mathrm{Lf}_\omega \cap \mathrm{Cs}^{\mathrm{reg}}_\omega$. This isomorphism is definable in ZF (Zermelo Fraenkel set theory) by an absolute formula without parameters (cf. [21, p. 406]). In [9, Section 4.3] $h(N)$ for a given model N is denoted by Cs^N_ω. Thus $A_\omega = \mathrm{Cs}^M_\omega$ constructed above is in $\mathrm{Cs}^{\mathrm{reg}}_\omega \cap \mathrm{Lf}_\omega$.

(3) We call a structure D *homogeneous* if every isomorphism between finitely generated substructures extends to an automorphism of D.

We recall from [18, Theorem 7.1.2] a theorem of FRAÏSSÉ that puts the above definitions in the context we need.

Fact 7 (FRAÏSSÉ). *Let L be a countable signature and let $\boldsymbol{K}$ be a non-empty finite or countable set of finitely generated L-structures which has* HP*,* JEP *and* AP*. Then there is an L structure D, unique up to isomorphism, such that*

(1) *D has cardinality $\leq \omega$,*

(2) *$\boldsymbol{K}$ is the age of D,*

(3) *D is homogeneous.*

Following HODGES [18], we refer to D as *Fraïssé limit* of the class $\boldsymbol{K}$. The next theorem, also due to FRAÏSSÉ, gives a sufficient condition for when the Fraïssé limit D of a class $\boldsymbol{K}$ of finitely generated structures has quantifier elimination. This condition is that $\boldsymbol{K}$ is *uniformly locally finite*, a condition that is satisfied when the signature of $\boldsymbol{K}$ is finite and has no function symbols, which is the form we need. Recall that L-structure M has *quantifier elimination* if every L formula $\varphi(\bar{x})$ is equivalent in M to a Boolean combination of quantifier free formulas. We now recall from [18] the form of Theorem 7.4.1 that we shall apply to construct our desired model.

Fact 8 (FRAÏSSÉ). *Suppose that the signature L is finite and has no function symbols. Suppose that $\boldsymbol{K}$ is a countable set of finite L structures with* HP*,* JEP *and* AP*. Let M be the Fraïssé limit of $\boldsymbol{K}$. Then M has quantifier elimination.*

Now we have all the necessary machinery to prove Lemma 2.

Proof of Lemma 2. Throughout the proof, we use the notation $\bar{x}, \bar{a}$ for finite sequences or tuples $\langle x_0, \dots x_{m-1}\rangle$, $\langle a_0, \dots a_{m-1}\rangle$. Given a structure M and a tuple $\bar{a}$, we often write, with a slight abuse of notation, $\bar{a} \in M$ instead of $\bar{a} \in {}^mM$, where m is the length of the tuple $\bar{a}$. The length of tuples will be clear from context. Let $\mathcal{L}$ be the relational signature containing unary relation symbols $P_0, \dots, P_3$ and a 4-ary relation symbol X. Let $\boldsymbol{K}$ be the class of all finite $\mathcal{L}$-structures D satsfying

(1) $\quad \forall x \bigvee_{i<j<4}(P_i(x) \wedge \bigwedge_{j\neq i} \neg P_j(x)),$

(2) $\quad \forall x_0 \dots x_3\, (X(x_0, \dots, x_3) \rightarrow P_3(x_3) \wedge \bigvee_{u\in S_3} \chi_u).$

Note that (1) simply says that the P_i's are disjoint. Then $\boldsymbol{K}$ contains countably many isomorphism types, because for each $n \in \omega$, there are countably many isomorphism types of finite L structures (satifying (1) and (2)) having cardinality $\leq n$. Also, it is easy to check that $\boldsymbol{K}$ is closed under substructures and that $\boldsymbol{K}$ has the AP. From the latter it follows that it has the JEP, since $\boldsymbol{K}$ contains the one element structure that is embeddable in any structure in $\boldsymbol{K}$ [6)]. By Fact 7 there is a countably infinite homogeneous $\mathcal{L}$-structure $\mathcal{M}$ with age $\boldsymbol{K}$. By Fact 8, $\mathcal{M}$ has quantifier elimination, and obviously, so does any elementary extension of $\mathcal{M}$. $\boldsymbol{K}$ contains structures with arbitrarily large P_3-part, so $P_3^{\mathcal{M}}$ is infinite. Let $\mathcal{M}^*$ be an elementary extension of $\mathcal{M}$ such that $|P_3^{\mathcal{M}^*}| = |L|$, and fix a bijection * from the set of ternary relation symbols of L to $P_3^{\mathcal{M}^*}$. Define an L-structure M with domain $P_0^{\mathcal{M}^*} \cup P_1^{\mathcal{M}^*} \cup P_2^{\mathcal{M}^*}$,

6) It is not always true that AP implies JEP; think of fields.

by $P_i^M = P_i^{\mathcal{M}^*}$ for $i < 3$ and such that for ternary $R \in L$, $M \vDash R(a_0, a_1, a_2)$ iff $\mathcal{M}^* \vDash X(a_0, a_1, a_2, R^*)$. If $\varphi(\bar{x})$ is any L-formula, let $\varphi^*(\bar{x}, \bar{R})$ be the $\mathcal{L}$-formula with parameters $\bar{R}$ from $\mathcal{M}^*$ obtained from φ by replacing each atomic subformula $R(x, y, z)$ by $X(x, y, z, R^*)$ and relativizing quantifiers to $\neg P_3$. In more detail, replace $\exists x\, \varphi(x)$ and $\forall x\, \varphi(x)$ by $\exists x\, (\neg P_3(x) \wedge \varphi(x))$ and $\forall x\, (\neg P_3(x) \rightarrow \varphi(x))$, respectively. A straightforward induction on complexity of formulas gives that for $\bar{a} \in M$, $M \vDash \varphi(\bar{a})$ iff $\mathcal{M}^* \vDash \varphi^*(\bar{a}, \bar{R})$. We show that M is as required, i. e., that M so constructed satisfies conditions (1) to (6) of Lemma 2.

For quantifier elimination, if $\varphi(\bar{x})$ is an L-formula, then $\varphi^*(\bar{x}, \bar{R}^*)$ is equivalent in $\mathcal{M}^*$ to a quantifier free $\mathcal{L}$-formula $\psi(\bar{x}, \bar{R}^*)$. Then, replacing ψ's atomic subformulas $X(x, y, z, R^*)$ by $R(x, y, z)$, replacing all $X(t_0, \dots, t_3)$ not of this form by $\bot$, replacing subformulas $P_3(x)$ by $\bot$, and $P_i(R^*)$ by $\bot$ if $i < 3$ and $\top$ if $i = 3$, gives a quantifier free L-formula ψ equivalent in M to φ.

Now we check that property (2) holds. For this end, let

$$\sigma = \forall x\, (\neg P_3(x) \rightarrow \textstyle\bigvee_{i<3}(P_i(x) \wedge \bigwedge_{j \neq i} \neg P_j(x))).$$

Then $\boldsymbol{K} \vDash \sigma$, so $\mathcal{M} \vDash \sigma$ and $\mathcal{M}^* \vDash \sigma$. It follows from the definition that M satisfies (2).

Checking property (3) is similar.

For property (4), let $u \in S_3$ and let r, $s \in P_3^M$ be distinct. Take a finite $\mathcal{L}$-structure D with points $a_i \in P_{u_i}^D$ $(i < 3)$ and distinct r', $s' \in P_3^D$ with

$$D \vDash X(a_0, a_1, a_2, r') \wedge \neg X(a_0, a_1, a_2, s').$$

Then $D \in \boldsymbol{K}$, so D embeds into $\mathcal{M}$. By homogeneity, we can assume that the embedding takes r' to r and s' to s. Therefore $\mathcal{M} \vDash \exists \bar{x}\, (\chi_u \wedge X(\bar{x}, r) \wedge \neg X(\bar{x}, s))$, where $\bar{x} = \langle x_0, x_1, x_2 \rangle$. Since r, s were arbitrary and $\mathcal{M}^*$ is an elementary extension of $\mathcal{M}$, we get that

$$\mathcal{M}^* \vDash \forall yz\, (P_3(y) \wedge P_3(z) \wedge y \neq z \rightarrow \exists \bar{x}\, (\chi_u \wedge X(\bar{x}, y) \wedge \neg(X(\bar{x}, z))).$$

The result for M now follows. Note that it follows from properties (3) and (4) of Lemma 2, that $P_i^M \neq \emptyset$ for each $i < 3$. So it is clear that

$$M \vDash \forall x_0 x_1 x_2\, (\exists x_i\, \chi_u \leftrightarrow \textstyle\bigvee_{v \in {}^3 3,\, v \equiv_i u} \chi_v),$$

giving (5).

Finally consider (6). Clearly, it is enough to show that for any $\mathcal{L}$-formula $\varphi(\bar{x})$ with parameters $\bar{r} \in P_3^M$, $u \in S_3$ and $i < 3$, we have

$$\mathcal{M} \vDash \exists \bar{x}\, (\chi_u \wedge \varphi) \rightarrow \forall \bar{x}\, (\exists x_i\, (\chi_u \rightarrow \exists x_i\, (\chi_u \wedge \varphi)).$$

In order to simplify notation assume that $i = 2$. Let $\bar{a}$, $\bar{b} \in \mathcal{M}$ with

$$\mathcal{M} \vDash (\chi_u \wedge \varphi)(\bar{a}) \text{ and } \mathcal{M} \vDash \exists x_2\, (\chi_u(\bar{b})).$$

We require $\mathcal{M} \vDash \exists x_2\, (\chi_u \wedge \varphi)(\bar{b})$. It follows from the assumptions that

$$\mathcal{M} \vDash P_{u_0}(a_0) \wedge P_{u_1}(a_1) \wedge a_0 \neq a_1 \text{ and } \mathcal{M} \vDash P_{u_0}(b_0) \wedge P_{u_1}(b_1) \wedge b_0 \neq b_1.$$

These are the only relations on $a_0 a_1 \bar{r}$ and on $b_0 b_1 \bar{r}$ (cf. property (3) of Lemma 2), so $\theta^- = \{(a_0, b_0)(a_1, b_1)(r_l, r_l) : l < |\bar{r}|\}$ is a partial isomorphism of $\mathcal{M}$. By homogeneity, this partial isomorphism is induced by an automorphism θ of $\mathcal{M}$. Let $\bar{c} = \theta(\bar{a}) = (b_0, b_1, \theta(a_2))$. Then $\mathcal{M} \vDash (\chi_u \wedge \varphi)(\bar{c})$. Since $\bar{c} \equiv_2 \bar{b}$, we have $\mathcal{M} \vDash \exists x_2\, (\chi_u \wedge \varphi)(\bar{b})$ as required. □

By proving Lemma 2, our proof of Theorem 1 and Corollary 3 is complete.

In what follows we comment on various refinements of Theorem 1, concerning other algebras of logic and higher dimensions. Also we briefly discuss the significance of the notion of neat reducts in (algebraic) logic.

3 Concluding remarks

(i) Let SC stand for the class of Pinter's substitution algebras. These are defined in the appendix of [22] or [32]. Let $\beta > 3$. Then it is proved in [32] that $\mathrm{Nr}_3\mathrm{SC}_\beta$ is not closed under forming subalgebras. It follows from the above construction that $\mathrm{Nr}_3\mathrm{SC}_\beta$ is not closed under forming *elementary subalgebras*. This can be seen by passing to reducts. That is, let $\mathrm{Rd}_{\mathrm{SC}}C$ be the SC reduct of a given $C \in \mathrm{CA}_3$. Then $\mathrm{Rd}_{\mathrm{SC}}A \in \mathrm{Nr}_3\mathrm{SC}_\omega$ and $\mathrm{Rd}_{\mathrm{SC}}B \notin \mathrm{Nr}_3\mathrm{SC}_4$. In fact, the complete analogue of Corollary 3 holds for SC's, since reducts of weak set algebras are weak set algebras. However, the construction adopted herein (in its present form) does not settle the QA (quasipolyadic) and QEA (quasipolyadic equality) cases. This follows from the fact that the term ${}_3\mathsf{s}(0,1)$ which played the key role in showing that B is not a neat reduct, is a basic operation in these algebras, whereas the proof of Theorem 1 depended on the fact that this term is not even term definable in cylindric algebras. We refer the reader to [27], where it is proved that when K is QA or QEA, and $\beta > 2$, then Nr_2K_β is not first order axiomatizable.

(ii) Let RA be the class of Tarski's relation algebras defined in e. g. [9, Definition 5.3.1]. For $n > 2$, RaCA_n is the class of RA reducts of CA_n defined in [9, 5.3.7] or [15]. It is known that $\mathrm{RaCA}_n \subseteq \mathrm{RA}$ iff $n > 3$ and that RaCA_n is not closed under forming subalgebras for $n > 2$. This is proved by Maddux [19] and independently by Németi and Simon (cf. [33, Chapter 5]). Ian Hodkinson pointed out to the author that the ideas adopted herein may be used to prove that the class RaCA_n, $n > 2$, is not closed under *elementary* subalgebras. It is known that $\mathrm{SRaCA}_4 = \mathrm{RA}$, and that both of these classes coincide with $\mathrm{SRaNr}_3\mathrm{CA}_4 = \mathrm{RaSNr}_3\mathrm{CA}_4$, a result of Maddux [19]. However SRaCA_n is properly contained in RA for $n > 4$. It is worth noting that an important question in CA and RA theory is: which subclasses of CA_3 give rise to relation algebras when applying the Ra operator. In this connection the reader is referred to [13, 15, 23] and [33, Chapter 5].

(iii) It seems likely that property (3) in Lemma 2 can be strengthened to show that the relations in question are *disjoint* rather than distinct. In this case, the resulting algebras A and B would be atomic and completely representable in the sense of [12]. Also, for each $u \in S_3$, the set of atoms below 1_u would be an *uncountable splitting of* 1_u in the sense of [2]. That is, letting $\mathrm{At}\,A$ denote the set of atoms of A, we get that for all $u \in S_3$ and for all $x \in \mathrm{At}\,A$ such that $x \le 1_u$, x is *cylindrically equivalent to* 1_u, formally $(\forall i < 3)\,(\mathsf{c}_i x = \mathsf{c}_i 1_u = \sum\{1_v : v \equiv_i u\})$. Andréka [2] refers to such atoms as "big atoms". The method of splitting elements (which is an instance of Henkin's method of *dilation* in the sense of [9, 3.2.69], see also [33, Chapter 4][7)]) was used by

[7)] Henkin's own methods of splitting, introduced in [8, Lemma 2.6.12] to construct non-representable CA's, is different than Andréka's splitting.

Andréka in [2] to show that for $\omega > n > 2$ and $\beta > n+1$ the class $\mathrm{SNr}_n\mathrm{CA}_\beta$ cannot be axiomatized by a set of universal formulas containing only finitely many variables.

(iv) We have proved that first order logic cannot distinguish between the algebras $A \in \mathrm{Nr}_3\mathrm{CA}_\omega$ and $B \notin \mathrm{Nr}_3\mathrm{CA}_4$. It seems likely that one can prove that stronger logics like $L_{\kappa\omega}$, where κ is a regular cardinal, also cannot distinguish between the algebras A and B. We did not check that. In [29] we prove that for any pair of infinite ordinals $\alpha < \beta$, the class $\mathrm{Nr}_\alpha\mathrm{CA}_\beta$ is also not elementary.

(v) The tie between special neat embeddings and amalgamation properties in cylindric-like algebras of relations is studied in [24, 32, 30]. Closure of the class of neat reducts under forming subalgebras plays a crucial role in determining which subclasses $L \subseteq \mathrm{RCA}_n$, n a countable ordinal, have the so called *strong amalgamation property.* This connection was first observed by Pigozzi. (cf. [24, p. 325, Lemma 2.2.12(iii)]). The (strong) amalgamation property is proved in algebraic logic to be the algebraic counterpart of (strong) interpolation theorems (like Beth-Definability) in the corresponding logics. This tie between the algebraic property of closure of the class of neat reducts under forming subalgebras and the metalogical one of interpolation results is further emphasized and investigated in [32, 30]. While Henkin shows that the Neat Embedding Theorem (NET) corresponds to completeness of several variants of first order logic (cf. [8, 3]), Pigozzi [24] shows that variation on NET corresponds to interpolation properties in such variants. In [31] sharpenings of NET is investigated in connection to the algebraic property of complete representations ([12]) and the metalogical one of omitting types [5]. It is worth noting that in [12] Hirsch and Hodkinson prove that the class of completely representable RCA_3, or CR_3 for short, is not elementary either. In [31] it is proved that algebras in CR_3 coincides with the atomic algebras in the class $\mathrm{S_cNr_3CA}_\omega$, where $\mathrm{S_c}$ stands for the operation of forming *complete subalgebras.*[8)]

(vi) Let B_n be the class of relation algebras having n-relational basis in the sense of Maddux [20]. Then B_n coincides with the class CRA_n of relation algebras having n-square complete representations in the sense of [14]. In [14] it is pointed out that CRA_n is an n analogue of the class CRA of completely representable relation algebras. Also it is proved that the class CRA_n, $n \geq 5$, is not elementary. In [32] it is proved that the classes $\mathrm{S_cRaCA}_\omega$ and CRA coincide on countable algebras. In view of this characterization, a candidate – other than the class CRA_n – for an n-approximation or n-analogue of CRA is the class of atomic algebras in $\mathrm{S_cRaCA}_n$, obtained by truncating the dimension ω to n. It turns out in [17] that, for $n \geq 4$, this class coincides with the class of relation algebras having n-dimensional hyperbasis in the sense of [14]. Now let $\mathrm{RA}_n = \mathrm{SB}_n$. While RA_n and SRaCA_n are n-analogues of RRA, CRA_n and atomic algebras in $\mathrm{S_cRaCA}_n$ are n-analogues of the class of completely representable relation algebras. Recently Robin Hirsch [11] proved that, like CRA_n, the class of relation algebras having an n-dimensional hyperbasis is not elementary, either. In fact, Robin Hirsch constructs an algebra $B \notin \mathrm{RaCA}_n$ such that an ultrapower of B is in RaCA_n. His (unpublished) construction, it seems, proves our Theorem 1 as well.

8) Let A be a Boolean algebra. A subalgebra B of A is a complete subalgebra of A if for all $X \subseteq A$, whenever ΣX exists in A, then ΣX exists in B, and they are equal.

(vii) Let $1 < \alpha < \beta$. It is not hard to show that the class $\mathrm{Nr}_\alpha\mathrm{CA}_\beta$ is a *pseudo-elementary class*, i. e, a PC_Δ class in the sense of [18, p. 207]. A psuedo-elementary class is roughly a reduct of an elementary class. Let $\mathrm{ELNr}_\alpha\mathrm{CA}_\beta$ denote the *elementary closure of* $\mathrm{Nr}_\alpha\mathrm{CA}_\beta$, i. e, $\mathrm{ELNr}_\alpha\mathrm{CA}_\beta = \mathsf{UpUr}\mathrm{Nr}_\alpha\mathrm{CA}_\beta$ where Up and Ur stand for the operations of forming ultraproducts and ultraroots, respectively. By [32], [29] and Theorem 1 herein, the inclusions $\mathrm{Nr}_\alpha\mathrm{CA}_\beta \subset \mathrm{ELNr}_\alpha\mathrm{CA}_\beta \subset \mathrm{SNr}_\alpha\mathrm{CA}_\beta$ are proper. In particular, $\mathrm{ELNr}_\alpha\mathrm{CA}_\beta$ is not closed under forming subalgebras hence is not (pseudo-) universal. By [16] one can synthesize (and consequently obtain) an axiomatization of the elementary class $\mathrm{ELNr}_\alpha\mathrm{CA}_\beta$ by games, in the spirit of Robinson's finite forcing in Model theory. It seems plausible that the class $\mathrm{ELNr}_\alpha\mathrm{CA}_\beta$ is finitely axiomatizable over the variety $\mathrm{SNr}_\alpha\mathrm{CA}_\beta$. The latter is finitely axiomatizable if and only if $\beta \leq \alpha+1$, a result of Andréka [2]. In particular, we conjecture that $\mathrm{ELNr}_\alpha\mathrm{CA}_\beta$ is finitely axiomatizable if and only if $\beta \leq \alpha + 1$.

References

[1] Amer, M., Cylindric algebras of sentences (Abstract). J. Symbolic Logic **58** (1993), 743.

[2] Andréka, H., Complexity of equations valid in algebras of relations. Annals Pure Appl. Logic **89** (1997), 149 – 209.

[3] Andréka, H., T. Gergely, and I. Németi, On universal algebraic constructions of logics. Studia Logica **36** (1977), 9 – 47.

[4] Andréka, H., I. Németi, I. Sain, Algebraic Logic. In: Handbook of Philosophical Logic, second edition, Vol. 1. (M. Gabbay and F. Guenthner, eds.) Kluwer, Dordrecht 2000.

[5] Andréka, H., and Sayed Ahmed, T., Omitting types in logics with finitely many variables. Presented in the Logic Colloquium 1998. Abstract appeared in: Bulletin of Symbolic Logic **5** (1999), p. 88.

[6] Chang, C. C., and H. J. Keisler, Model Theory. North-Holland Publ. Comp., Amsterdam 1994.

[7] Hirsch, R., and I. Hodkinson, Complete representations in algebraic logic. J. Symbolic Logic **62** (1997), 816 – 847.

[8] Henkin, L., J. D. Monk, and A. Tarski, Cylindric Algebras, Part I. North-Holland Publ. Comp., Amsterdam 1971.

[9] Henkin, L., J. D. Monk, and A. Tarski, Cylindric Algebras, Part II. North-Holland Publ. Comp., Amsterdam 1985.

[10] Henkin, L., J. D. Monk, A. Tarski, H. Andréka, and I. Németi, Cylindric Set Algebras. Lecture Notes in Math. 883, Springer-Verlag, Berlin-Heidelberg-New York 1981.

[11] Hirsch, R., Personal communication.

[12] Hirsch, R., and I. Hodkinson, Step-by-step building representations in algebraic logic. J. Symbolic Logic **62** (1997), 225 – 278.

[13] Hirsch, R., and I. Hodkinson, Relation algebras from cylindric algebras. J. Symbolic Logic (to appear).

[14] Hirsch, R., and I. Hodkinson, Relation algebras with n dimensional relational bases. Annals Pure Appl. Logic **101** (2000), 227 – 274.

[15] Hirsch, R., I. Hodkinson, and R. Maddux, Relation algebra reducts of cylindric algebras and an application to proof theory. J. Symbolic Logic (to appear).

[16] Hirsch, R., and I. Hodkinson, Synthesising axioms by games. A CD-ROM of essays dedicated to Johan Van Benthem on occasion of his 50th birthday.

[17] HIRSCH, R., and T. SAYED AHMED, The class of algebras having n-dimensional hyperbasis is elementary only for $n \leq 3$. Manuscript.

[18] HODGES, W., Model Theory. Cambridge University Press, Cambridge 1993.

[19] MADDUX, R., Topics in relation algebras. Ph.D. dissertation. University of California, Berkley 1978.

[20] MADDUX, R., Non-finite axiomatizability results for cylindric and relation algebras. J. Symbolic Logic **54**, (1989), 951 – 974.

[21] NÉMETI, I., The class of neat reducts of cylindric algebras is not a variety but is closed w. r. t. HP. Notre Dame J. Formal Logic **24** (1983), 399 – 409.

[22] NÉMETI, I., Algebraization of quantifier logics: An introductory overview. Math. Inst. Budapest, Preprint, No. 13-1996. A shortened version appeared in Studia Logica **50** (1991), 465 – 569.

[23] NÉMETI, I., and A. SIMON, Relation algebras from cylindric and polyadic algebras. Logic Journal of IGPL, **5** (1997), 575 – 588.

[24] PIGOZZI, D., Amalgamation, congruence extension, and interpolation properties in algebras. Algebra Universalis **1** (1971), 269 – 349.

[25] SAIN, I., Searching for a finitizable algebraization of first order logic. Logic Journal of IGPL **8** (2000), 495 – 589.

[26] SAYED AHMED, T., Algebras of sentences of logic. Masters Thesis, Cairo University 1998.

[27] SAYED AHMED, T., The class of 2-dimensional neat reducts of polyadic algebras is not elementary. Submitted 1999.

[28] SAYED AHMED, T., The class of neat reducts is not elementary (Abstract). Bulletin of Symbolic Logic **5** (1999), 407 – 408.

[29] SAYED AHMED, T., For any ordinal $\beta > \omega$, the class $\mathrm{Nr}_\omega\mathrm{CA}_\beta$ is not elementary. Submitted.

[30] SAYED AHMED, T., On amalgamation of polyadic alge- bras. Algebra Universalis (to appear).

[31] SAYED AHMED, T., Martin's axiom, omitting types and complete representations in algebraic logic. Studia Logica (to appear).

[32] SAYED AHMED, T., and I. NÉMETI, On neat reducts of algebras of logic. Studia Logica (to appear).

[33] SIMON, A., Non representable algebras of relations. Ph.D. Dissertation. Budapest 1997.

(Received: February 6, 2001; Revised: July 11, 2001)