DIAGONALIZATION EXHIBITED IN THE LIAR PARADOX, RUSSELL’S PARADOX AND GÖDEL’S INCOMPLETENESS THEOREM

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Abstract

By formulating a property on a class of relations on the set of natural numbers, we make an attempt to provide alternative proof to the insolvability of Hilbert’s tenth problem, Gödel’s incompleteness theorem, Tarski’s definability theorem and Turing’s halting problem.

1. Introduction

In this paper we give a new unified proof of four important theorems, solved during the period 1930 - 1970. These problems in historical order are Gödel’s incompleteness theorem, Tarski’s definability theorem, Turing’s halting problem [5] and Hilbert’s tenth problem [8], [2]. The latter was solved 70 years after it was posed by Hilbert in 1900. Martin Davis, under the supervision of Emil Post, was the first to consider the possibility of a negative solution to this problem which “begs for insolvability” as stated by Post. Later, Julia Robinson, Martin Davis, Hilary Putnam worked on the
problem for almost twenty years, and the problem was reduced to showing that the exponential function is Diophantine. This latter result was proved by Matiyasevich in 1970.

Gödel Theorems are one of the most important intellectual achievements in the 20th century and beyond. The opening lines of Gödel’s 1931 paper speaking of Russell and Whitehead’s ‘Principia Mathematica’ and Zermelo Fraenkel set theory are ‘these two systems are so overwhelmingly extensive that all methods of proof used in mathematics today have been formalized in them, that is to say, reduced to a few axioms and rules of inference that can be effectively decided’. What Gödel actually articulates here was a universal credo at the time, and so his revelation of PM’s incompleteness, in the 25 pages that followed came like a sudden thunderbolt from the bluest of skies. To add insult to injury Gödel’s conclusion sprang not from a weakness in PM but from a strength; that strength is that numbers are so flexible that their patterns can mimic patterns of formal reasoning. Gödel exploited the simple but marvelous fact that the familiar whole numbers can dance just the same way as the unfamiliar symbol patterns in PM can dance. This ‘isomorphism of dances’ is implemented via Gödel numbering or coding. The Prim numbers that he invented, which are the Gödel numbers of symbols, formulas, finite strings of formulas i.e., proof etc., using two very basic essential facts about numbers. First there are infinitely many prime numbers, so that one can code all finite strings from a countable signature in an effective way, given a number we can tell whether it is in Prim or not by simply factorizing it and so he also needed the unique factorization theorem. Other than that he used some fairly basic number theory like the Chinese remainder theorem to complete his fantastic coding. He was able to mirror meta-mathematics in mathematics, Formulas are not these dead just meaningless strings of symbols; ‘formulas speak’ they speak about other formulas, or strings of formulas like their provability. So a formula whose code number is in Prim has two interpretations. A blunt basic one as a list of symbols, and a more subtle one on a higher liver speaking about the provability of another formula possible itself. The formula ‘I am not provable’ is true but not provable because precisely it is not provable by
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definition. In other words, the Prim numbers he invented acted indistinguishably from provable strings, and the main strength of PM is that it can talk about Prim numbers. In this respect, it is quite articulate but introspective, it can talk about itself albeit in code. In a word PM, and any formal system aspiring to do the same thing, it is their expressive that gives rise to their non-completeness. In what follows we give a unifying perspective of this phenomenon. We formulate a general theorem on \( P \), from which we get as corollaries the insolvability of the four famous problems stated above. The property \( P \) together with a “Cantor’s diagonalization” process emphasizes that all the above theorems are a variation on a theme, that of self reference and diagonalization combined.

2. Properties of Relations on \( \mathbb{N} \)

**Definition 1.** A property \( P \) is defined to be a subset of \( \bigcup_{r=1}^{\infty} \mathcal{P}(\mathbb{N}^r) \). We say that a set \( X \) satisfies \( P \) if \( X \in P \) and a function \( f : \mathbb{N}^r \to \mathbb{N}^s \) satisfies \( P \) if as a relation it satisfies \( P \).

**Definition 2.** Let \( n \in \mathbb{N} \), \( S \subseteq \{1, ..., n\} \) and let \( l \) be the cardinality of \( S \). We define the function \( \pi^n_S : \mathbb{N}^n \to \mathbb{N}^l \) as follows \( \pi^n_S(x_1, ..., x_n) = (x_{s_1}, ..., x_{s_l}) \), where \( S = \{s_1, ..., s_l\} \), \( 1 \leq s_1 < s_2 < \cdots < s_l \leq n \).

**Definition 3.** Let \( P \) be a property, then we say that \( P \) is broad if it satisfies the following conditions.

(i) The sets \( \mathbb{N}^r, \{x\} \) satisfy \( P \) for every \( r \in \mathbb{N}, x \in \mathbb{N}^r \).

(ii) The functions, namely, addition and multiplication \( +, \cdot : \mathbb{N}^2 \to \mathbb{N} \), and the identity \( i : \mathbb{N} \to \mathbb{N} \) satisfy \( P \).

(iii) The property \( P \) is closed under cross product and intersection.

(iv) The function \( \pi^n_S \) satisfies \( P \) for every \( n \in \mathbb{N}, S \subseteq \{1, ..., n\} \).

(v) For every \( n \in \mathbb{N}, S \subseteq \{1, ..., n\}, V \subseteq \mathbb{N}^n \) satisfies \( P \), \( \pi^n_S(V) \) satisfies \( P \).
Examples 1. (1) Take $P = \bigcup_{r=1}^{\infty} \mathcal{P}(\mathbb{N}^r)$.

(2) Take $P = \{D : D$ is a Diophantine set\}.

Lemma 1. If $P$ is a broad property and $f : \mathbb{N}^r \to \mathbb{N}^s$ satisfies $P$, $V$ satisfies $P$, then $f^{-1}(V)$ satisfies $P$.

Proof. If $P$ is a strong property and $f : \mathbb{N}^r \to \mathbb{N}^s, V$ satisfies $P$, then we claim that

$$f^{-1}(V) = \pi^{r+s}_{[1,\ldots,r]}(f \cap (\mathbb{N}^r \times V)).$$

To prove the claim, we have $\bar{x} \in f^{-1}(V)$ iff $f(\bar{x}) \in V$ iff $\exists \bar{y}$ such that $\bar{y} \in V, \bar{y} = f(\bar{x})$ iff $\exists \bar{y}$ such that $\bar{y} \in V, (\bar{x}, \bar{y}) \in f$. Here we treat $f$ as a relation, so $\bar{x} \in f^{-1}(V)$ iff $\exists \bar{y}$ such that $(\bar{x}, \bar{y}) \in f, (\bar{x}, \bar{y}) \in \mathbb{N}^r \times V$ and this is equivalent to $(\bar{x}, \bar{y}) \in f \cap (\mathbb{N}^r \times V)$ which, in turn, is equivalent to $\bar{x} \in \pi^{r+s}_{[1,\ldots,r]}(f \cap (\mathbb{N}^r \times V))$ and hence the claim is proved. Now by (iii), (iv), (v) it follows that $f^{-1}(V)$ satisfies $P$.

Theorem 2 (composition). Let $P$ be a broad property, and the functions $h_1, \ldots, h_n : \mathbb{N}^k \to \mathbb{N}$ satisfy $P$ and the function $g : \mathbb{N}^n \to \mathbb{N}^s$ satisfies $P$, then the function $f = g(h_1, \ldots, h_n) : \mathbb{N}^k \to \mathbb{N}^s$ satisfies $P$.

Proof. For simplification take $n = 2$. By definition; to prove that $f$ satisfies $P$ we treat $f$ as a relation and prove that it satisfies $P$, as was done in the previous lemma. We have $f = g(h_1, h_2)$, so we have

$$f = \pi^{k+s+2}_{[1,\ldots,k,k+3,\ldots,k+s+2]}((h_1 \times \mathbb{N}^{s+1}) \cap ((\pi^k_{[1,\ldots,k,k+2]} \times g)) \cap (\mathbb{N}^k \times g)).$$
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Since $h_1, h_2, g$ satisfy $P$, we have by Lemma 1 and by (iii), (iv), (v), that $f$ satisfies $P$. 

**Corollary 3.** If $P$ is a broad property, then every polynomial in any number of variables satisfies $P$.

**Proof.** The proof is divided into 3 parts. First, the constant function satisfies $P$. Let $r, s \in \mathbb{N}, c \in \mathbb{N}^r$, then the function $f : \mathbb{N}^s \to \mathbb{N}^r$ defined as $f(x) = c$ for all $x \in \mathbb{N}^k$ satisfies $P$. Because it is as a relation equal $\mathbb{N}^s \times \{c\}$ which satisfies $P$ by (i), (iii) it follows by definition that $f$ satisfies $P$. Second, let $r \in \mathbb{N}$ then the function $f : \mathbb{N}^k \to \mathbb{N}^k$ defined by $f(x) = x$ for all $x \in \mathbb{N}^k$ satisfies $P$ by (ii).

Third, by induction we can use part 1, 2, Theorem 2 to show that every polynomial in any number of variables satisfies $P$.

**Theorem 4.** Let $P$ be any broad property, then every Diophantine set satisfies $P$.

**Proof.** Let $r \in \mathbb{N}, D \subseteq \mathbb{N}^r$ be a Diophantine set, then by definition of Diophantine sets there exist a positive integer $m$ and a polynomial $Q(x_1, ..., x_r, y_1, ..., y_m)$ such that

$$\overline{x} \in D \iff \exists y_1, ..., y_m, \text{ such that } Q(\overline{x}, y_1, ..., y_m) = 0.$$ 

So $\overline{x} \in D \iff \exists \overline{y}$ such that $(\overline{x}, \overline{y}, 0) \in Q$ which is equivalent to $\overline{x} \in \pi_{\{1, ..., r\}}^{r+m+1}(Q \cap (\mathbb{N}^{r+m} \times \{0\}))$. So

$$D = \pi_{\{1, ..., r\}}^{r+m+1}(Q \cap (\mathbb{N}^{r+m} \times \{0\}))$$

and hence by Corollary 3, (iii), (v) it follows that $D$ satisfies $P$.

**Theorem 5.** The set of all Diophantine sets forms the smallest strong property.
Proof. To prove that the Diophantine sets form a strong property, we just need to check the conditions.

(i) The zero polynomial proves that the set $\mathbb{N}^r$ is diophantine and the polynomial $p(x_1, ..., x_n) = \prod_{k=1}^{n} (x_k - c_k)$ defines the singleton $c = (c_1, ..., c_n) \in \mathbb{N}^n$.

(ii) The addition, multiplication and identity functions are defined by the polynomials $+(x, y, z) = x + y - z$, $*(x, y, z) = x * y - z$, $i(x, y) = x - y$ receptively.

(iii) Assume $A \subseteq \mathbb{N}^r$, $B \subseteq \mathbb{N}^s$ are Diophantine, then there exist two polynomials $P, Q$ such that $\bar{x} \in A \Leftrightarrow \exists y_1, ..., y_m$, such that $P(\bar{x}, y_1, ..., y_m) = 0$, $\bar{y} \in B \Leftrightarrow \exists y_1, ..., y_s$, such that $Q(\bar{x}, y_1, ..., y_l) = 0$, then the polynomial $PQ$ has the property that $(\bar{x}, \bar{y}) \in A \times B \Leftrightarrow \exists k_1, ..., k_m$, $z_1, z_l$ such that $P(\bar{x}, \bar{z}) \times Q(\bar{y}, z) = 0$, so $A \times B$ is Diophantine, if $r = s$ then the polynomial $P(x_1, ..., x_r, y_1, ..., y_m) \times Q(x_1, ..., x_r, z_1, ..., z_l)$ defines the intersection, so again $A \cap B$ is Diophantine.

(iv) The polynomial that defines the graph of the function $\pi^n_S$ is $\prod_{k=1}^{l} (x_{s_k} - x_{m+k})$, where $S = \{s_1, ..., s_l\}$, $1 \leq s_1 < s_2 < \cdots < s_l \leq n$.

(v) Let $A$ be a Diophantine set so there exist a polynomial $P$ such that $\bar{x} \in A \Leftrightarrow \exists y_1, ..., y_m$, such that $P(\bar{x}, y_1, ..., y_m) = 0$ so the set $\pi^n_S(A)$ is Diophantine and defined by the polynomial $Q = P$ but with a permutation of arguments.

\[\blacksquare\]

3. A Unifying Theorem (Diagonalization)

Theorem 6. Let $P$ be a broad property such that $P \cap \mathcal{P}(\mathbb{N})$ is countable. Let $D_1, D_2, ...$ be an enumeration of subsets of $\mathbb{N}$ that satisfy $P$, then the
function $g : \mathbb{N}^2 \to \mathbb{N}$ defined as the follows
\[
g(n, x) = \begin{cases} 
0 & \text{if } x \in D_n, \\
1 & \text{if } x \notin D_n
\end{cases}
\]
is not recursive, and $g$ does not satisfy $P$.

**Proof.** Assume $g$ satisfies $P$, then by Theorem 2 and by (ii) it follows that the function $g(n, n) : \mathbb{N} \to \mathbb{N}$ satisfies $P$ and by Lemma 1, the set $V = \{ n | n \notin D_n \} = g^{-1}([1])$ satisfies $P$. Since $V \subseteq \mathbb{N}$ satisfies $P$ and the sequence of the sets $(D_n)_{n \in \mathbb{N}}$ form an enumeration of the subsets of $\mathbb{N}$ satisfying $P$. So $\exists n$ such that $V = D_n$ but then $n \in D_n \iff n \notin V$ but by definition of $V$, we have $n \in V \iff n \notin D_n$ and this is a contradiction.

So $g$ does not satisfy $P$. Now assume that $g$ is recursive, since a function is Diophantine if and only if it is recursive [2], hence $g$ is Diophantine. By Theorem 4 it follows that $g$ satisfies $P$ which leads to a contradiction. Hence $g$ is not recursive.

**Theorem 7** (Pairing Function Theorem 1). There are Diophantine functions $P(x, y), L(z), R(z)$ such that

1. for all $x, y$, $L(P(x, y)) = x$, $R(P(x, y)) = y$, and
2. for all $z$, $P(L(z), R(z)) = z$, $L(z) \leq z$, $R(z) \leq z$.

**Proof.** See [2] p. 203 Theorem 1.1. \qed

**Corollary 8.** Hilbert’s tenth problem $(H10)$ is unsolvable.

**Proof.** Take the following enumeration of polynomial with positive coefficients
\[
P_1 = 1 \\
P_{3i-1} = x_{i-1} \\
P_{3i} = P_{L(i)} + P_{R(i)}
\]
\[ P_{n+1} = P_{L(i)} \cdot P_{R(i)}, \]

where \( L, R \) are the left, right functions respectively. Those are defined in [2] p. 202. Finally, let

\[ D_n = \{ x_0 \mid (\exists x_1, \ldots, x_n) [P_{L(n)}(x_0, x_1, \ldots, x_n) = P_{R(n)}(x_0, x_1, \ldots, x_n)] \} \]

then \( D_n \) is an enumeration of Diophantine sets [2] p. 206. Assume that \( H^{10} \) is solvable then the function

\[ g(n, x) = \begin{cases} 0 & \text{if } x \in D_n, \\ 1 & \text{if } x \notin D_n \end{cases} \]

is recursive which contradicts Theorem 6.

\[ \square \]

**Corollary 9** (Halting problem). There is no algorithm such that, given a program and an input to that program, determines if the program halts at the given input or not.

**Proof.** Let \( D_n = \{ x \mid \text{the program number } n \text{ halt at input } x \} \). Then it is easy to see that \( D_n \) form an enumeration of listable sets, but listable sets are just Diophantine sets so by Theorems 4, 6 we get that the function

\[ g(n, x) = \begin{cases} 1 & \text{if the program number } n \text{ halt at input } x \\ 0 & \text{if otherwise} \end{cases} \]

is not recursive which is the required. \( \square \)

**Corollary 10.** There exists a listable subset of \( \mathbb{N} \) which is not recursive.

**Proof.** In the previous proof and the unifying theorem proof we proved that the function \( g(n, n) \) is not recursive, so the set \( g(n, n)^{-1} \) is not recursive, but it is easy to see that it is listable. \( \square \)

**Definition 4.** A relation \( R \subseteq \mathbb{N}^r \) on \( \mathbb{N} \) is said to be definable if there exists a first order formula \( \alpha(v_1, \ldots, v_r) \) such that

\[ \bar{x} \in R \iff \alpha(\bar{x}) \text{ is true in } \mathbb{N}. \]
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Examples 2. Every Diophantine set is definable.

Corollary 11 (One version of Gödel’s Incompleteness Theorem). Let \( A \) be a recursive set of sentences true in \( \mathbb{N} \), then there exist a sentence \( \sigma \) such that \( \sigma \) is true in \( \mathbb{N} \) but \( A \not
\sigma \).

Proof. By Theorem 10 there exists a listable set subset of \( \mathbb{N} \) which is not recursive. Let \( K \) be that set. Since every listable set is definable so the set \( K \) is definable, that is there is a formula \( \chi(v) \) which defines \( K \), then \( \neg \chi(v) \) defines \( K^c \). Thus we have

\[ a \in K^c \iff \neg \chi(a) \text{ is true.} \]

Let \( A \) be a recursive set sentences true in \( \mathbb{N} \). Let \( J = \{ a \in \mathbb{N} \mid A \vdash \neg \chi(a) \} \).

Then we have

1. \( J \) is recursively enumerable
2. \( J \subseteq K^c \).

We write \( r.e. \) instead of recursively enumerable. Now \( J \) is a proper subset of \( K^c \) because if \( J = K^c \), then \( K^c \) is \( r.e. \) and since \( K \) is also \( r.e. \) so it follows that \( K \) is recursive which is a contradiction. Let \( q \in K^c \) such that \( q \notin J \) then take \( \sigma = \neg \chi(1) \), then \( q \in K^c \) says that \( \sigma \) is true in \( \mathbb{N} \) and \( q \notin J \) says that \( A \not\vdash \sigma \) and the theorem is proved.

4. A Theorem of Tarski on Definability

Definition 5. A property \( P \) is said to be very broad if it is broad and it is closed under complement.

Lemma 12. Let \( P \) be a very broad property, \( \alpha(v_1, ..., v_r) \) be any first order formula, then the set \( \{ \overline{x} \mid \alpha(\overline{x}) \text{ is true} \} \) satisfies \( P \).

Proof. The proof is by straightforward induction on the complexity of formulas. Assume that \( \alpha \) is an atomic formula then the corresponding set
satisfies $P$ by condition (ii). If \( \alpha(v_{i \in I}), \beta(v_{k \in K}) \) are two formulas such that the sets \( A = \{ \overline{x} | \alpha(\overline{x}) \text{ is true} \}, B = \{ \overline{x} | \beta(\overline{x}) \text{ is true} \} \) satisfy $P$ where $I, K$ are two finite subsets of $\mathbb{N}$ then let \( I = |I|, s = |K|, j = |I \cup K| \). The two formulas $\alpha \land \beta, -\alpha$ satisfy the lemma because the corresponding sets are \((\pi_{I})^{-1}(A) \cap (\pi_{K'})^{-1}(B), A' \) respectively, and those satisfy $P$ by Lemma 1 and (iv), where \( I' = \{ x - \min(I) + 1 | x \in I \} \) the same for $K'$. The formula \( \exists v_{k} \alpha \) satisfies the lemma because its corresponding set is $A$ if $k \not\in I$ and it is $\pi_{m}(A)$, where \( I = \{ x_{1}, ..., x_{m} = k, ..., x_{j} \} \) so it satisfies $P$ by Lemma 1. Hence all formulas satisfy the lemma.

**Theorem 13.** If $P$ is a very broad property, then every definable set satisfies $P$.

**Proof.** Let $V$ be any definable set, then there exists a formula $\alpha$ such that \( \overline{x} \in V \iff \alpha(\overline{x}) \) is true.

So by Lemma 12 it follows that $V$ satisfies $P$. \( \square \)

**Theorem 14.** The set of all definable sets form a very broad property.

**Proof.** To prove that the definable sets form a very broad property, we just need to check that the conditions of very broad properties are satisfied.

(i) The formula $v_{1} = v_{1} \land \cdots \land v_{r} = v_{r}$ defines the set $\mathbb{N}^{r}$ so it is definable and the formula $v_{1} = S^{c_{1}} \land \cdots \land v_{n} = S^{c_{n}}$ defines the singleton $c = (c_{1}, ..., c_{n}) \in \mathbb{N}^{n}$ hence the first condition is satisfied.

(ii) The addition, multiplication and identity functions are defined by the formulas \( v_{1} + v_{2} = v_{3}, v_{1} \ast v_{2} = v_{3}, v_{1} = v_{2} \) receptively, hence the second condition is also satisfied.

(iii) Assume $A \subseteq \mathbb{N}^{r}, B \subseteq \mathbb{N}^{s}$ are definable, then there exist two formulas $\alpha, \beta$ such that \( \overline{x} \in A \iff \alpha(\overline{x}) \) is true, \( \overline{x} \in B \iff \beta(\overline{x}) \) is true, then
Diagonalization Exhibited in the Liar paradox, Russell’s Paradox … the formula \( \alpha(v_1, ..., v_r) \land \beta(v_{r+1}, ..., v_{r+s}) \) has the property that \((\bar{x}, \bar{y}) \in A \times B \iff \alpha(\bar{x}) \land \beta(\bar{y}) \) is true, so \(A \times B\) is definable, if \( r = s \) then the formula \( \alpha(v_1, ..., v_r) \land \beta(v_1, ..., v_r) \) defines \(A \cap B\) so the third condition is satisfied.

(iv) The formula that defines the graph of the function \( \pi^n_S \) is 
\[ \bigwedge_{k=1}^l (v_{sk} = v_{m+k}), \] where \( S = \{s_1, ..., s_l\}, 1 \leq s_1 < s_2 < \cdots < s_l \leq n, \) so the fourth condition is satisfied.

(v) Let \( V \subseteq \mathbb{N}^n \) be a definable set so there exists a formula \( \alpha \) such that \( \bar{x} \in A \iff \alpha(\bar{x}) \). So the set \( \pi^n_S(A) \) is definable by the formula 
\[ \exists v_i \in I \alpha(v_1, ..., v_n), \] where \( S = \{s_1, ..., s_l\}, 1 \leq s_1 < s_2 < \cdots < s_l \leq n, I = \{1, ..., n\}\backslash S. \)

(vi) Let \( V \) be a definable set so there exist a formula \( \alpha \) such that \( \bar{x} \in A \iff \alpha(\bar{x}) \), then the set \( V^c \) is definable by the formula \( \neg \alpha(\bar{x}) \).

Hence the set of all definable sets forms a very broad property. \( \square \)

**Theorem 15.** Let \( P \) be a very strong property and \( P \cap \mathcal{P}(\mathbb{N}) \) is countable. So let \( D_1, ... \) be the enumeration of all subsets of \( \mathbb{N} \) and satisfies \( P \), then the set \( V = \{(a, b) \mid a \in D_b\} \) is not definable.

**Proof.** Assume it is definable, since \( P \) is closed under complement, then \( V^c \) is definable, hence the exist a formula \( \alpha(v_1, v_2) \) defines \( V^c \), so the formula \( \alpha(v_1, v_1) \) defines the set \( Z = \{a \mid a \notin D_a\} \) but since \( D_1, ... \) form an enumeration of all subsets of \( \mathbb{N} \) that satisfies \( P \) but since \( Z \) is a subset of \( \mathbb{N} \) that satisfies \( P \), so there exists \( n \) such that \( Z = D_n \) but by definition of \( Z \) \( n \notin D_n \iff n \in Z \), so \( Z \) cannot equal to \( D_n \) which is a contradiction. Hence \( V \) is not definable. \( \square \)
Theorem 16 (Tarski definability theorem). The set \{ (a, b) | the statement \( \alpha(v_1) \) with Gödel number \( b \) is true at \( a \) \} is not definable.

Proof. Take \( P \) be the class of all definable sets, then by Theorem 14 this is a very broad property. Take \( D_b = \{ a \in \mathbb{N} : \) the formula with Gödel number \( b \) is true at \( a \} \) then this is an enumeration of all subsets of \( \mathbb{N} \) that satisfy \( P \), so by Theorem 15, the set \{ (a, b) | a \in D_b \} is not definable, but this is precisely the conclusion of the theorem.

In the paper Lawvere [7] shows the Gödel’s incompleteness theorem and the truth-definition theory of Tarski are consequences of some very simple algebra in the cartesian-closed setting. Here is his main theorem.

Theorem 17. In any cartesian closed category, if there exists an object \( A \) and a weakly point surjective morphism

\[ g : A \to Y^A. \]

Then \( Y \) has the fixed point property.

In this part, we closely follow [9]. Let \( g : T \times T \to Y \). \( g \) is said to be universal if for all \( f : T \to Y \) there exists \( t \in T \) such that \( g(s, t) = s \) for all \( s \in T \). In this case we say that \( f \) is representable by \( g \) and \( t \) or simply by \( g \).

Theorem 18. Let \( \alpha : 2 \to 2 \) be the function defined by \( \alpha(0) = 1 \) and \( \alpha(1) = 0 \), then for all set \( T \) and all functions \( g : T \times T \to 2 \), there exists a function \( f : T \to Y \), such that for all \( t \in T \)

\[ f(-) \neq g(-, t). \]

Proof. Let \( \delta : T \to T \times T \) be the function

\[ t \mapsto (t, t). \]

Then let \( f = \alpha \circ g \circ \Delta \), that is

\[ g(t) = \alpha(f(t, t)). \]

Then clearly \( g \) is as required. \( \square \)
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(1) Our diagonalization lemma is an instance of the above theorem. Let $D_1, D_2, \ldots$ be an enumeration of subsets of $\mathbb{N}$ that satisfy $P$, let $g : \mathbb{N} \times \mathbb{N} \to 2$ be defined by

$$g(n, x) = \begin{cases} 0 & \text{if } x \in D_n, \\ 1 & \text{if } x \not\in D_n. \end{cases}$$

For each $n$, $g(-, n)$ is the characteristic function of $D_n$.

$$g(-, n) = \chi_{D_n}$$

$f$ as constructed of the theorem is the characteristic function of

$$V = \{n \in \mathbb{N} : n \not\in D_n\}$$

and $f$ cannot be represented by $g$.

(2) Cantor’s diagonalization lemma. Let $D_1, D_2, \ldots$ be an enumeration of subsets of $\mathbb{N}$ that satisfy $P$, let $g : \mathbb{N} \times \mathbb{N} \to 2$ be defined by

$$g(n, x) = \begin{cases} 0 & \text{if } x \in D_n, \\ 1 & \text{if } x \not\in D_n. \end{cases}$$

For each $n$, $g(-, n)$ is the characteristic function of $D_n$.

$$g(-, n) = \chi_{D_n}$$

$f$ as constructed of the theorem is the characteristic function of

$$V = \{n \in \mathbb{N} : n \not\in D_n\}$$

and $f$ cannot be represented by $g$.

(3) Russell’s paradox [9] p. 369. Let $g : Sets \times Sets \to 2$ be defined as follows:

$$g(s, t) = \begin{cases} 1 & \text{if } s \in t, \\ 1 & \text{if } s \not\in t. \end{cases}$$

Then $g$ is the characteristic function of those sets that are not members of themselves.
(4) Grelling paradox [9] p.370. There are some adjectives in English that describe themselves and some that do not. For example “English” is English and French is not a French word. Call words that do not apply to themselves heterological. Now is “heterological” heterological. Let $g : \text{Adj} \times \text{Adj} \rightarrow 2$ be defined as follows:

$$g(s, t) = \begin{cases} 
1 & \text{if } s \text{ describes } t \\
1 & \text{if } s \text{ does not describe } t.
\end{cases}$$

Then $f$ constructed as above is the characteristic function of a property of adjectives that cannot be described by any adjectives.

(5) Related to the above is the Liar paradox and the strong Liar paradox and The Barber’s paradox. The solution of the last, is that there is a village where everyone who does not shave themselves is shaved by the barber simply does not exist. We formulate famous limitative theorems of Gödel, Turing, Tarski and Church in a unified “Liar formalism” We should mention that the above paradoxes can be overcome as above. A related paradox is Quine’s paradox which is:

“yields falsehood when appended to its own quotation” yields falsehood when appended to its own quotation.

A common solution to the Liar paradox is saying that there are sentences that are neither true nor false but are meaningless “I am lying” would be such a statement. Consider now the sentence “yields falsehood or meaninglessness when appended to it own quotation” yields falsehood or meaninglessness when appended to its own quotation. This can be resolved by enlarging the set 2 to 3 and defining the function $g : \text{Sent} \times \text{Sent} \rightarrow 3$ as follows:

$$g(s, t) = \begin{cases} 
T & \text{if } s \text{ describes } t \\
M & \text{if } s \text{ does not describe } t.
\end{cases}$$

Then $f$ is the characteristic function of sentences that are neither false nor meaningless when describing themselves.
Diagonalization Exhibited in the Liar paradox, Russell’s Paradox …

Extending further the previous definition, it is easy to see that Theorem 19 remains valid if 2 is replaced by any set $Y$ and $\alpha : Y \to Y$ has no fixed points. Now taking the contrapositive of this, we get

**Theorem 19.** Let $Y$ be a set and there exists a set $T$ and a function $g : T \times T$ such that for all $f : T \to Y$ are representable by $f$, then every $\alpha : Y \to Y$ has a fixed point.

**Proof.** Like the above proof. See also [9] Theorem 3.

Using this, we give a proof of the so called Diagonalization lemma in textbooks on Gödel’s incompleteness theorems. For a formula $\alpha(x), [\alpha(x)]$ denotes its Gödel number. We assume that we are working in a theory where there is a recursive function $D : \mathbb{N} \to \mathbb{N}$, such that

$$D[\alpha(x)] = [\alpha([\alpha(x)])].$$

**Theorem 20.** For any formula $\alpha(x)$ with $x$ as its only free variable, then there exists a sentence $\sigma$ such that

$$\vdash \sigma \leftrightarrow \alpha([\sigma]).$$

**Proof.** [9] Theorem 4. Let $Fm_n$ denote the set of formulas with free variables among the first $n$ and let $Sn$ denote the set of sentences. Let $g : Fm_1 \times Fm_1 \to Sn$ be defined by

$$(\mu(x), \beta(x)) \mapsto \beta([\mu(x)]).$$

Let $\Phi_\alpha : Sn \to Sn$ be defined by

$$\tau \mapsto \alpha([\tau]).$$

Let

$$f = \Phi \circ g \circ \Delta.$$ 

Then $f$ is representable by

$$g(x) = \alpha(D(x)).$$

So there is a fixed point of $\Phi_\alpha$ at $\sigma = g([g(x)])$, which is the required.
The best part of the schema worked out in Theorem 19 and exemplified above is that it shows that there are really no paradoxes. There are, rather, limitations. Paradoxes are ways of showing that violating a imitation leads to an inconsistency. The Liar paradox shows that if you permit natural languages to talk about its own truthfulness then we will have inconsistencies. Russell’s paradox shows that if we permit one to talk about any set without limitations, we will get an inconsistency. Gödel’s theorem shows that if we permit a system to talk about its truth then we get an inconsistency, however replacing truth by theoremhood is a very rewarding thing to do, it gives the justly celebrated incompleteness theorems of Gödel. Though the provability relation is recursively enumerable, hence definable, Tarski showed that truth is not even definable. Gödel was smart enough to draw a line between theoremhood and truth, arriving at a positive limitative result, and not an inconsistency. The above scheme in Theorem 19 exhibits the inherent limitations of similar systems. The constructed or diagnolized out \( f \) is a limitation that your system \( g \) cannot deal with. If the system does deal with it, then there will be an inconsistency (a fixed point \( f \)).

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References


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