ATOM CANONICITY, FINITE AXIMATIZABILITY
AND COMPLETE REPRESENTATIONS FOR
ALGEBRA OF RELATIONS

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Abstract

Let \( 2 < n < \omega \). Let \( V \subseteq W \) be two classes of Boolean algebras with
operators. We say that \( V \) is atom-canonical with respect to \( W \) if
whenever \( \mathfrak{A} \in V \) is atomic, then its Dedekind-MacNeille completon,
in symbols \( \mathfrak{mAt}\mathfrak{A} \) is in \( W \). We show that \( \mathsf{S\mathcal{N}_n\mathcal{C}}\mathcal{A}_t(n) \) is not atom
canonical with respect to \( \mathcal{RCA}_n \) for \( t(n) = n(n + 1)/2 \). Using another
construction showing that \( \mathcal{RCA}_n \cap \mathsf{N\mathcal{R}_n\mathcal{C}}\mathcal{A}_m \) is not atom canonical
with respect to \( \mathcal{RCA}_n \), we show that the varieties of relation and
cyindric algebras of dimension \( n \) are not finitely axiomatizable.
Finally, we show that the class of completely representable relations
and cylindric algebras of dimension \( n \) is not first order definable,
reproving classical results of Monk, Maddux, Biro and Hirsch and
Hodkinson. The last result is proved by showing that
\( \mathsf{N\mathcal{R}_n\mathcal{C}}\mathcal{A}_\omega \subseteq \mathcal{LCA}_n \) and then constructing an atomless \( \mathfrak{C} \in \mathcal{C}\mathcal{A}_\omega \) such
that \( \mathsf{N\mathcal{R}_n\mathfrak{C}} \) and \( \mathfrak{R}\mathfrak{a}\mathfrak{C} \) are atomic, having uncountably many atoms, but
lacking a complete representation. Here $\text{LCA}_n$ is the elementry cloure of $\text{CRCA}_n$ known as the elementary class of algebras satisfying the first order sentences known as the Lyndon conditions.

1. Introduction

A class $K$ of Boolean algebras with operators (BAOs) is atom-canonical if whenever $\mathfrak{A} \in K$ is atomic and completely additive, then its Dedekind-MacNeille completion, namely, the complex algebra of its atom structure, namely, $\mathcal{CmAt}\mathfrak{A}$ is also in $K$. (We use in what follows instances of the so-called blow up and blur construction.) But first a Lemma proved as [18, Lemma 5.]

**Lemma 1.1.** Let $2 < n < m \leq \omega$. Let $\mathfrak{A} \in \text{CA}_n$. If $\forall$ has winning strategy in $G^m_\omega(\text{At}\mathfrak{A})$, then $\mathfrak{A}$ does not have an $m$-square representation.

**Definition 1.2.** A $\text{CA}_n$ atom structure $\text{At}$ is weakly representable if there is an atomic $\mathfrak{A} \in \text{RCA}_n$ such that $\text{At} = \text{At}\mathfrak{A}$; it is strongly representable if $\mathcal{CmAt}\mathfrak{A} \in \text{RCA}_n$.

These two notions are distinct for $2 < n < \omega$, cf. [9] for the CA case and the next Theorem. The most general exposition of CA rainbow constructions is given in [7, Section 6.2, Definition 3.6.9] in the context of constructing atom structures from classes of models. Our models are just coloured graphs [5]. (Such constructions as the name might sugget involve colours.) In fact rainbows involve playing simple Ehrenfeucht-Fraïssé pebble pair games on two relational structures determined by $G$ (the greens) and $R$ (the reds), lifted to games on coloured complete irreexive graphs played on the complex algebra of certain (rainbow) atom structures. Such rainbow games are deterministic games played between $\exists$ Iloise and $\forall$ belard. One of the payers $\exists$ or $\forall$ has to win. There are no draws, and the number of rounds can be transfinite. In this game $\exists$ tries to prove that $G$ and $R$ are alike having the same structure by pebbling one of the structures, while her enemy $\forall$ tries to show that they are not by pebbling the other structure using the second
pebble n the pair whose first pebble was used by ∀. The pebble pairs are outside the board of play. This is a private game, where winning strategy’s can be lifted to networks (coloured graphs) of atom structures of (rainbow) algebras. It is always the case that, when these simple Ehrenfeucht-Fraïssé games are lifted to coloured graphs, ∀ wins on a red clique, if he has a winning strategy. His is only way to win is by bombarding ∃ with special coloured graphs called cones, having common base and green tints, asking for a red label between the appexes of the last two played cones. If ∃ can respond in every round with a suitable red, then she wins. Else ∀’s winning strategy works and he wins if ∃ runs out of reds satisfying certain consistency conditions involving ‘red triangles’.

Definition 1.3. Let V ⊆ W be varieties of BAOs. We say that V is atom-canonical with respect to W if for any atomic A ∈ V, its Dedekind-MacNeille completion, namely, AtmA is in W.

We refer the reader to [18] for the notions of locally classical m-representations where 2 < n < m ≤ ω, namely, m-square and m at representations. In our main construction we use rainbow constructions as presented in [5, 9]. The game Gm(AtA) is the usual ω rounded atomic game played on atomic networks of CA_n as defined in [7]. The game Gm is like Gm except that ∀ has the extra advantage of reusing the m nodes in play, The following Theorem is proved in [18, 20].

Lemma 1.4. Let 2 < n < m ≤ ω. If A ∈ S_nS_nCA_m, then ∃ has a winning strategy in Gm(AtA). Furthermore, A has a complete m-square representation ⇔ ∃ has a winning strategy in Gm(AtA).

Such games translate to games on coloured graphs in rainbow constructions [5, Section 4.3].

2. Main Result

Theorem 2.1. Let 2 < n < ω and t(n) = n(n + 1)/2 + 1. The variety
\( \text{RCA}_n \) is not-atom canonical with respect to \( \text{SNr}_n \text{CA}_{t(n)} \). In fact, there is a countable atomic simple \( \mathfrak{A} \in \text{RCA}_n \) such that \( \mathfrak{CmAt} \mathfrak{A} \) does not have an \( t(n) \)-square, a fortiori \( t(n) \)-flat, representation.

**Proof.** The proof is long and uses many ideas in [9]. We will highlight only the differences in detail from the proof in [9] needed to make our result work. When parts of the proof coincide we will be more sketchy. The proof is divided into four parts:

1. **Blowing up and blurring** \( \mathcal{B}_f \) **forming a weakly representable atom structure** \( \mathfrak{A}_f \). Take the finite rainbow \( \mathcal{C}_n \), \( \mathcal{B}_f \), where the reds \( \mathcal{R} \) is the complete irreflexive graph \( n \), and the greens are \( \{g_i : 1 \leq i < n-1\} \cup \{g_0^i : 1 \leq i \leq n(n-1)/2\} \), endowed with the cylindric operations. We will show \( \mathcal{B} \) detects that \( \text{RCA}_n \) is not atom-canonical with respect to \( \text{SNr}_n \text{CA}_{t(n)} \) with \( t(n) \) as specified in the statement of the theorem. Denote the finite atom structure of \( \mathcal{B}_f \) by \( \mathfrak{A}_f \); so that \( \mathfrak{A}_f = \mathfrak{A}(\mathcal{B}_f) \). One then defines a larger the class of coloured graphs like in [9, Definition 2.5]. Let \( 2 < n < \omega \). Then the colours used are like above except that each red is ‘split’ into \( \omega \) many having ‘copies’ the form \( r^i_{ij} \) with \( i < j < n \) and \( l \in \omega \), with an additional shade of red \( \rho \) such that the consistency conditions for the new reds (in addition to the usual rainbow consistency conditions) are as follows:

- \( (r^i_{j*}, r^i_{jk}, r^i_{j*k}) \) unless \( i = i' = i' \) and \( |\{(j, k), (j', k'), (j^*, k^*)\}| = 3 \)

- \( (r, \rho, \rho) \) and \( (r, r^*, \rho) \), where \( r, r^* \) are any reds.

The consistency conditions can be coded in an \( L_{\omega_1, \omega} \) theory \( T \) having signature the reds with \( \rho \) together with all other colours like in [7, Definition 3.6.9]. The theory \( T \) is only a first order theory (not an \( L_{\omega_1, \omega} \) theory) because the number of greens is finite which is not the case with [7] where the number of available greens are countably infinite coded by an infinite
disjunction. One construct an \( n \)-homogeneous model \( M \) is as a countable limit of finite models of \( T \) using a game played between \( \exists \) and \( \forall \) like in [9, Theorem 2.16]. In the rainbow game \( \forall \) challenges \( \exists \) with cones having green tints\((g^0_i)\), and \( \exists \) wins if she can respond to such moves. This is the only way that \( \forall \) can force a win. \( \exists \) has to respond by labelling apexes of two successive cones, having the same base played by \( \forall \). By the rules of the game, she has to use a red label. She resorts to \( \rho \) whenever she is forced a red while using the rainbow reds will lead to an inconsistent triangle of reds; [9, Proposition 2.6, Lemma 2.7]. The number of greens make [9, Lemma 3.10] work with the same proof. Using only finitely many green and not infinitely many. The winning strategy is implemented by \( \exists \) using the red label \( \rho \) that comes to her rescue whenever she runs out of ‘rainbow reds’, so she can always and consistently respond with an extended coloured graph. This proof will is implemented in the framework of an entirely analogous blow up and blur constructions applied to strikingly similar finite rainbow atom structures in [6]. In both cases, the relational structures \( G \) and \( R \) used satisfy \( |G| = |R| + 1 \). For \( RA, R = 3 \) and for \( CA_n, R = n \) (the dimension), where the finite ordinals 3 and \( n \) are viewed as complete irreflexive graphs.\(^1\) From Hodkinson’s construction in [9], we know that \( \exists m\mathbb{B}b(\mathbb{B}_f, r, \omega) \notin SNr_mCA_m \) for some finite \( m > n \), where \( \mathbb{B}b(\mathbb{B}_f, r, \omega) \) denotes the result of blowing up \( \mathbb{B}_f \) by splitting each red atom into \( \omega \)-many ones, to be denoted henceforth by \( \mathfrak{A} \). The (semantical) argument used in [9] does not give any information on the value of such \( m \). By truncating the greens to be \( n(n + 1)/2 \) (instead of the ‘overkill’ of infinitely many in [9]), and using a syntactical blow up and blur construction, we could pin down such a value of \( m \), namely, \( m = n + t(n) \) (=number of greens + \( t(n) \)) by showing in a moment that that although \( \mathbb{B}b(\mathbb{B}_f, r, \omega) \)

\(^1\) Worthy of note, is that it is commonly accepted that relation algebras have dimension three being a natural habitat for three variable first order logic. Nevertheless, sometimes it is argued that the dimension should be three and a half in the somewhat loose sense that RAAs lie ‘halfway’ between \( CA_3 \) and \( CA_4 \) manifesting behaviour of each.
containing the term algebra is representable, but not completely representable. On the other hand, its completion, namely, \( \mathcal{C} \) will be outside \( \mathcal{S}_n \mathcal{C} \mathcal{A}_{n(n)} \). Proving representability \( \mathcal{A} \) can be done by completely representing its canonical extension, in a fairly simple step by step manner. The atom structure \( \mathcal{A} \) consists of principal ultrafilters generated by atoms, together with only one non-principle ultrafilter, that can be identified with the shade of red \( \rho \). This is needed for representing \( \mathcal{A} \), but not completely; the atom structure \( \mathcal{A} \) is not and cannot be completely representable; it is not even strongly representable. As a matter of fact, it is just weakly representable, with all these notions of representability for atom structures are taken from [7].

2. Representing a term algebra (and its completion) as (generalized) set algebras. Having \( \mathcal{M} \) at hand, one constructs two atomic \( n \)-dimensional set algebras based on \( \mathcal{M} \), sharing the same atom structure and having the same top element. The atoms of each will be the set of coloured graphs, seeing as how, quoting Hodkinson [9] such coloured graphs are ‘literally indivisible’. Now \( L_n \) and \( L_{n, \omega} \) are taken in the rainbow signature (without \( \rho \)). Continuing like in op.cit., deleting the one available red shade, set \( W = \{ \overline{a} \in \mathcal{M} : \mathcal{M} \models (\bigwedge_{i<j<n} -\rho(x_i, x_j))(\overline{a}) \} \), and for \( \phi \in L_{n, \omega} \), let \( \phi^W : \{ \overline{s} \in W : \mathcal{M} \models \phi[\overline{s}] \} \). Here \( W \) is the set of all \( n \)-ary assignments in \( \mathcal{M} \), that have no edge labeled by \( \rho \). Let \( \mathfrak{A} \) be the relativized set algebra with domain \( \{ \phi^W : \phi \ \text{a first-order } L_n \text{-formula} \} \) and unit \( W \), endowed with the usual concrete quasipolyadic operations read off the connectives. Classical semantics for \( L_n \) rainbow formulas and their semantics by relativizing to \( W \) coincide [9, Proposition 3.13] but not with respect to \( L_{n, \omega} \) rainbow formulas. Hence the set algebra \( \mathfrak{A} \) is isomorphic to a cylindric set algebra of dimension \( n \) having top element \( \mathcal{M} \), so \( \mathfrak{A} \) is simple, in fact its \( \mathcal{D} \) \( \mathcal{F} \) reduct is simple. Let \( \mathcal{F} = \{ \phi^W : \phi \in L_{n, \omega} \} \) [9, Definition 4.1] with the operations
defined like on $\mathcal{A}$ the usual way. $\mathcal{CmAt}$ is a complete $\mathcal{CA}_n$ and, so like in [9, Lemma 5.3] we have an isomorphism from $\mathcal{CmAt}$ to $\mathcal{C}$ defined via $X \mapsto \cup X$. Since $\mathcal{At}\mathcal{A} = \mathcal{At}\mathcal{Im}(\mathcal{At}\mathcal{A})$, which we refer to only by $\mathcal{At}$, and $\mathcal{ImAt}\mathcal{A} \subset \mathcal{A}$, hence $\mathcal{ImAt}\mathcal{A} \subset \mathcal{TmAt}$ is representable. The atoms of $\mathcal{A}$, $\mathcal{ImAt}\mathcal{A}$ and $\mathcal{ImAt}\mathcal{A} = \mathcal{CmAt}$ are the coloured graphs whose edges are not labelled by $\rho$. These atoms are uniquely determined by the interpretation in $\mathcal{M}$ of so-called MCA formulas in the rainbow signature of $\mathcal{At}$ as in [9, Definition 4.3].

3. **Embedding** $\mathcal{A}_{n+1,n}$ **into** $\mathcal{Cm}$$\mathcal{At}(\mathcal{At}(\mathcal{A}_{n+1,n}, r, \omega))$. Let $\mathcal{CRG}_f$ be the class of coloured graphs on $\mathcal{At}_f$ and $\mathcal{CRG}$ be the class of coloured graph on $\mathcal{At}$. We can (and will) assume that $\mathcal{CRG}_f \subseteq \mathcal{CRG}$. Write $M_a$ for the atom that is the (equivalence class of the) surjection $a : n \rightarrow M, M \in \mathcal{CRG}$. Here we identify $a$ with $[a]$; no harm will ensue. We define the (equivalence) relation $\sim$ on $\mathcal{At}$ by $M_a \sim N_a, (M, N \in \mathcal{CRG})$:

- $a(i) = a(j) \Leftrightarrow b(i) = b(j),$
- $M_a(a(i), a(j)) = r^l \Leftrightarrow N_b(b(i), b(j)) = r^k$, for some $l, k \in \omega$,
- $M_a(a(i), a(j)) = N_b(b(i), b(j))$, if they are not red,
- $M_a(a(k_0), ..., a(k_{n-2})) = N_b(b(k_0), ..., b(k_{n-2}))$, whenever defined.

We say that $M_a$ is a copy of $N_b$ if $M_a \sim N_b$ (by symmetry $N_b$ is a copy of $M_a$). Indeed, the relation "copy of" is an equivalence relation on $\mathcal{At}$. An atom $M_a$ is called a red atom, if $M_a$ has at least one red edge. Any red atom has $\omega$ many copies, that are cylindrically equivalent, in the sense that, if $N_a \sim M_b$ with one (equivalently both) red, with $a : n \rightarrow N$ and $b : n \rightarrow M$, then we can assume that $\text{nodes}(N) = \text{nodes}(M)$ and that for all $i < n, a \upharpoonright n \sim \{i\} = b \upharpoonright n \sim \{i\}$. In $\mathcal{CmAt}$, we write $M_a$ for $\{M_a\}$ and we denote suprema taken in $\mathcal{CmAt}$, possibly finite, by $\sum$. Define the map $\Theta$ from $\mathcal{A}_{n+1,n} = \mathcal{CmAt}_f$ to $\mathcal{CmAt}$, by specifying first its values on $\mathcal{At}_f$,
via $M_a \mapsto \sum_j M_a^{(j)}$, where $M_a^{(j)}$ is a copy of $M_a$. So each atom maps to the suprema of its copies. This map is well-defined because $\mathcal{CmAt}$ is complete. We check that $\Theta$ is an injective homomorphism. Injectivity is easy. We check preservation of all the $\mathcal{CA}_n$ extra Boolean operations.

- Diagonal elements. Let $1 < k < n$. Then:

\[
M_x < \Theta(d_{lk}^{\mathcal{CmAt}_f}) \iff M_x \leq \bigcup_j M_a^{(j)}
\]

\[
\iff M_x \leq \bigcup_{a_l = a_k} \sum_j M_a^{(j)}
\]

\[
\iff M_x = M_a^{(j)} \text{ for some } a : n \to M \text{ such that } a(l) = a(k)
\]

\[
\iff M_x \in d_{lk}^{\mathcal{CmAt}}.
\]

- Cylindrifiers. Let $i < n$. By additivity of cylindrifiers, we restrict our attention to atoms $M_a \in \mathcal{At}_f$ with $a : n \to M$ and $M \in \mathcal{CRG}_f \subseteq \mathcal{CRG}$. Then:

\[
\Theta(c_i^{\mathcal{CmAt}_f} M_a) = f \left( \bigcup_{[c] \neq [a]} M_c \right) = \bigcup_{[c] \neq [a]} \Theta(M_c)
\]

\[
= \bigcup_{[c] \neq [a]} \sum_j M_c^{(j)} = \sum_{[c] \neq [a]} \bigcup_j M_c^{(j)} = \sum_j c_i^{\mathcal{CmAt}} M_a^{(j)}
\]

\[
= c_i^{\mathcal{CmAt}} \left( \sum_j M_a^{(j)} \right) = c_i^{\mathcal{CmAt}} \Theta(M_a).
\]

- Modlities. Coinciding with the identity map on both sides are trivially preserved.
4. **∀ has a winning strategy in** \( G^{t(n)} \mathsf{At}(\mathfrak{B}_f) \); **and the required result.** It is straightforward to show that **∀** has winning strategy first in the Ehrenfeucht-Fraïssé forth private game played between **∃** and **∀** on the complete irreflexive graphs \( n + 1 \leq n(n - 1)/2 + 1 \) and \( n \) in \( n + 1 \) rounds \( \mathsf{EF}_{n+1}^{n+1}(n + 1, n) \) [7, Definition 16.2] since \( n + 1 \) is ‘longer’ than \( n \). Using (any) \( p > n \) many pairs of pebbles available on the board **∀** can win this game in \( n + 1 \) many rounds. **∀** lifts his winning strategy from the 1st private Ehrenfeucht-Fraïssé forth game to the graph game on \( \mathsf{At}_f = \mathsf{At}(\mathfrak{B}_f) \) [5, pp. 841] forcing a win using \( t(n) \) nodes. One uses the \( n(n - 1)/2 + 2 \) green relations in the usual way to force a red clique \( C \), say with \( n(n - 1)/2 + 2 \).

Pick any point \( x \in C \). Then there are > \( n(n - 1)/2 \) points \( y \in C \{x\} \). There are only \( n(n - 1)/2 \) red relations. So there must be distinct \( y, z \in C \{x\} \) such that \( (x, y) \) and \( (x, z) \) both have the same red label (it will be some \( r_{ij}^m \) for \( i < j < n \)). But \( (y, z) \) is also red, and this contradicts [9, Definition 2.5(2), 4th bullet point]. In more detail, **∀** bombards **∃** with cones having common base and distinct green tints until **∃** is forced to play an inconsistent red triangle (where indicies of reds do not match). He needs \( n - 1 \) nodes as the base of cones, plus \( |P| + 2 \) more nodes, where \( P = \{ (i, j) : i < j < n \} \) forming a red clique, triangle with two edges satisfying the same \( r_{ij}^m \) for \( p \in P \). Calculating, we get \( t(n) = n - 1 + n(n - 1)/2 + 2 = n(n + 1)/2 + 1 \).

By Lemma 1.4, \( \mathfrak{B}_f \not\subseteq \mathsf{SNr}_{\omega}CA_{t(n)} \) when \( 2 < n < \omega \). Since \( \mathfrak{B}_f \) is finite, then \( \mathfrak{B}_f \not\subseteq \mathsf{SNr}_nCA_{t(n)} \), because \( \mathfrak{B}_f \) coincides with its canonical extension and for any \( D \in CA_n \), \( D \in \mathsf{SNr}_nCA_{2n} \Rightarrow D^+ \in S_n \mathsf{SNr}_nCA_{2n} \). But \( \mathfrak{B}_f \) embeds into \( \mathcal{R}_{ca} \mathcal{CmAt} \mathcal{A}^{\text{top}} \), hence \( \mathcal{R}_{ca} \mathcal{CmAt} \mathcal{A}^{\text{top}} \) is outside the variety \( \mathsf{SNr}_nCA_{t(n)} \), as well. By the second part of Lemma 1.4, the required follows.

We refer the reader to [18] for the definition of m-square representation.
of an algebra \( \mathfrak{A} \) (models of a theory \( T \)). If \( T \) is a first order complete \( L_n \) theory, then an \( m \)-square model of \( T \), only locally classic on \( m \)-squares, gives rise to an isomorphism between the Tarski Lindenbaum quotient algebra \( \mathfrak{A}/m_T \) and a certain concrete algebra having base \( \mathbb{M} \) reflecting so called \( m \) clique guarded semantics, generalizing clique guarded semantics as defined and discussed in some detail for relation algebras in [6]. Using the reasoning in [2] summarized in the proof of [16, Theorem 3.1.1], we get:

**Corollary 2.2.** Let \( 2 < n < \omega \). Then there is an atomic \( L_n \) theory \( T \) such that the non principal type of coatoms \( \Gamma \) is realizable in every \( t(n) \) square model, but \( \Gamma \) is not isolated.

Let \((C)RRA\) denote the class of completely representable relation algebras. Let \( CRCA_n \) denote the class of completely representable \( CA_n \)s and \( LCA_n = EICRCA_n \), be its elementary closure, namely, the class of algebras whose atom structures satisfy the Lyndon conditions as defined in [7].

**Theorem 2.3** [13, 14, 12, 6, 3, 5, 19]. Let \( 2 < n < \omega \) and \( \alpha \) be an infinite ordinal. Then the following hold:

1. The varieties \( RRA \) and \( RCA_n \) are not finitely axiomatizable, and so is the elementary class \( LCA_n \).

2. The classes \( CRRA \) and \( CRCA_n \) are not elementary.

3. The class of completely representable Halmos’ polyadic algebras of dimension \( \alpha \) is elementary. Furthermore for any \( \beta > \alpha \), the variety \( SN_{\alpha}CA_{\beta} \) is finitely axiomatizable, closed and Dedekind-MacNeille completions, hence atom-canonical and Sahqvist axiomatizable.

**Proof.** 1. Let \( R_I \) be the finite Maddux algebra \( E_{f(l)}(2, 3) \), as defined on [2, p.83, Section 5, in the proof of Theorem 5.1] with \( l \)-blur \((J_I, E_I)\) as defined in [2, Definition 3.1] and \( f(l) \geq l \) as specified in [2, Lemma 5.1] (denoted by \( k \) therein). Let \( \mathcal{R}_I = \mathfrak{B}(R_I, J_I, E_I) \in RRA \), where \( \mathcal{R}_I \) is the
relation algebra having atom structure denoted $At$ in [2, p. 73] when the blown up and blurred algebra denoted $R_l$ happens to be the finite Maddux algebra $E_{f(l)}(2, 3)$ and let $\mathfrak{A}_l = \mathfrak{N}_n \mathfrak{B}_l (R_l, J_l, E_l) \in \mathcal{RCA}_n$ as defined in [2, Top of p.80] (with $R_l = E_{f(l)}(2, 3)$). Then $(\mathfrak{A}_l: l \in \omega \sim n)$, and $(\mathfrak{A}_l: l \in \omega \sim n)$ are sequences of weakly representable atom structures that are not strongly representable with a completely representable ultra product.

2. The idea is showing by Lemma 1.4, that $\mathfrak{N}_n \mathcal{CA}_\omega \subseteq \mathcal{LCA}_n = \mathcal{EICRCA}_n$ as follows. If $\mathfrak{A} \in \mathfrak{N}_n \mathcal{CA}_\omega$, then by Lemma 1.4, $\exists$ has a winning strategy in $G^{\omega} \mathfrak{A}_l$ hence it has a winning strategy in the $\omega$ rounded game $G^{\omega} \mathfrak{A}_l$ without the need to reuse any of the nodes in play, since infinity many are available, hence a fortiori $\exists$ has a winning strategy in the $k$ rounded atomic game $G_k (\mathfrak{A}_l)$ for all $k < \omega$, so by definition, cf. [7], $\mathfrak{A} \in \mathcal{LCA}_n$. However, $\mathfrak{B} = \mathfrak{N}_n \mathfrak{C}$ constructed in [20, Theorem 4.5] where $\mathfrak{C}$ is an atomless $\mathcal{CA}_\omega$ having an $\omega$ dimensional cylindric basis, satisfies that $\mathfrak{B} \in \mathfrak{N}_n \mathcal{CA}_\omega (\subseteq \mathcal{LCA}_n)$ while $\mathfrak{B} \not \in \mathcal{CRCA}_n$. The analogous result for relation algebras can be proved using the same idea by replacing $\mathfrak{B} = \mathfrak{N}_n \mathfrak{C}$ by $\mathfrak{A} = \mathfrak{R} \mathfrak{C}$ and easily working out the RA analogue of the atomic game $G^m$ with $m$ nodes and Lemma 1.4.

3. [17]

**Definition 2.4.** For an $n$-dimensional atomic network $N$ on an atomic $\mathcal{CA}_n$ and for $x, y \in \text{nodes}(N)$, set $x \sim y$ if there exists $\bar{x}$ such that $N(x, y, \bar{x}) \leq d_{01}$. Define the equivalence relation $\sim$ over the set of all finite sequences over $\text{nodes}(N)$ by $\bar{x} \sim \bar{y}$ iff $|\bar{x}| = |\bar{y}|$ and $x_i \sim y_i$ for all $i < |\bar{x}|$. (It can be easily checked that this indeed an equivalence relation). A hypernetwork $N = (N^a, N^h)$ over an atomic $\mathcal{CA}_n$ consists of an $n$-dimensional network $N^a$ together with a labelling function for hyperlabels.
\[N^h : \llcorner \text{nodes}(N) \to \Lambda \] (some arbitrary set of hyperlabels \( \Lambda \)) such that for \( \bar{x}, \bar{y} \in \llcorner \text{nodes}(N) \) if \( \bar{x} \sim \bar{y} \Rightarrow N^h(\bar{x}) = N^h(\bar{y}) \). If \( \| \bar{x} \| = k \in \mathbb{N} \) and \( N^h(\bar{x}) = \lambda \), then we say that \( \lambda \) is a \( k \)-ary hyperlabel. \( \bar{x} \) is referred to as a \( k \)-ary hyperedge, or simply a hyperedge. A hyperedge \( \bar{x} \in \llcorner \text{nodes}(N) \) is short, if there are \( y_0, ..., y_{n-1} \) that are nodes in \( N \), such that \( N(x_i, y_0, \bar{z}) < d_{01} \text{ or } N(x_i, y_{n-1}, \bar{z}) \leq d_{01} \) for all \( i < |x| \), for some (equivalently for all) \( \bar{z} \). Otherwise, it is called long. This game involves, besides the standard cylindrifier move, two new amalgamation moves. Concerning his moves, this game with \( m \) rounds \( (m \leq \omega) \), call it \( H_m \), \( \forall \) can play a cylindrifier move, like before but now played on \( \lambda \)-neat hypernetworks \( (\lambda \text{ a constant label}) \). Also \( \forall \) can play a transformation move by picking a previously played hypernetwork \( N \) and a partial, finite surjection \( \theta : \omega \to \text{nodes}(N) \), this move is denoted \( (N, \theta) \). \( \exists \)'s response is mandatory. She must respond with \( N\theta \). Finally, \( \forall \) can play an amalgamation move by picking previously played hypernetworks \( M, N \) such that \( M \upharpoonright \text{nodes}(M) \cap \text{nodes}(N) = N \upharpoonright \text{nodes}(M) \cap \text{nodes}(N) \) and \( \text{nodes}(M) \cap \text{nodes}(N) \neq \emptyset \). This move is denoted \( (M, N) \). To make a legal response, \( \exists \) must play a \( \lambda_0 \)-neat hypernetwork \( L \) extending \( M \) and \( N \), where \( \text{nodes}(L) = \text{nodes}(M) \cup \text{nodes}(N) \).

The next Lemma will be needed to prove Theorem 2.5 and Corollary 2.8, which are the main results in this section. With Theorem 2.1, they constitute the core of this article.

**Theorem 2.5.** Let \( \alpha \) be a countable atom structure. If \( \exists \) has a winning strategy in \( H_\omega(\alpha) \), then there exists a complete \( \mathcal{D} \in \text{RCA}_\omega \) such that \( \mathcal{C}_\alpha \equiv \mathcal{N}_\mathcal{D} \). In particular, \( \mathcal{C}_\alpha \in \text{Nr}_\omega \text{CA}_\omega \).

**Proof.** Fix some \( a \in \alpha \). The game \( H_\omega \) is designed so that using \( \exists \)'s winning strategy in the game \( H_\omega(\alpha) \) one can define a nested sequence \( M_0 \)
\( \subseteq M_1, \ldots \) of \( \lambda \)-neat hypernetworks where \( M_0 \) is \( \exists \)'s response to the initial \( \forall \)-move \( a \), such that: If \( M_r \) is in the sequence and \( M_r(\bar{x}) \leq c_ia \) for an atom \( a \) and some \( i < n \), then there is \( s \geq r \) and \( d \in \text{nodes}(M_s) \) such that \( M_s(\bar{y}) = a, \bar{y} = d \) and \( \bar{y} \equiv_i \bar{x} \). In addition, if \( M_r \) is in the sequence and \( \theta \) is any partial isomorphism of \( M_r \), then there is \( s \geq r \) and a partial isomorphism \( \theta^+ \) of \( M_s \) extending \( \theta \) such that \( \text{rng}(\theta^+) \supseteq \text{nodes}(M_r) \).

(This can be done using \( \exists \)'s responses to amalgamation moves). Now let \( M_a \) be the limit of this sequence, that is \( M_a = \bigcup M_i \), the labelling of \( n - 1 \) tuples of nodes by atoms, and hyperedges by hyperlabels done in the obvious way using the fact that the \( M_i \)'s are nested. Let \( L \) be the signature with one \( n \)-ary relation for each \( b \in \alpha \), and one \( k \)-ary predicate symbol for each \( k \)-ary hyperlabel \( \lambda \). Now we work in \( L_{\infty, \omega} \). For fixed \( f_a \in \text{o} \text{nodes}(M_a) \), let \( \mathfrak{A}_a = \{ f_a \in \text{o} \text{nodes}(M_a) : \{ i < \omega : g(i) \neq f_a(i) \} \text{ is finite} \} \). We make \( \mathfrak{A}_a \) into the base of an \( L \) relativized structure \( \mathcal{M}_a \) allowing a clause for infinitary disjunctions. In more detail, for \( b \in \alpha, l_0, \ldots, l_{n-1}, i_0, \ldots, i_{k-1} < \omega \), \( k \)-ary hyperlabels \( \lambda \), and all \( L \)-formulas \( \phi, \phi_i, \psi \) and \( f \in U_a : \)

\[
\mathcal{M}_a, f \models (x_{l_0}, \ldots, x_{l_{n-1}}) \Leftrightarrow \mathcal{M}_a(f(l_0), \ldots, f(l_{n-1})) = b,
\]

\[
\mathcal{M}_a, f \models (x_{i_0}, \ldots, x_{i_{k-1}}) \Leftrightarrow \mathcal{M}_a(f(i_0), \ldots, f(i_{k-1})) = \lambda,
\]

\[
\mathcal{M}_a, f \models \neg \phi \Leftrightarrow \mathcal{M}_a, f \models \phi,
\]

\[
\mathcal{M}_a, f \models \left( \bigvee_{i \in I} \phi_i \right) \Leftrightarrow (\exists i \in I)(\mathcal{M}_a, f \models \phi_i),
\]

\[
\mathcal{M}_a, f \models \exists x_{i/m} \phi \Leftrightarrow \mathcal{M}_a, f[i/m] \models \phi, \text{ some } m \in \text{nodes}(\mathcal{M}_a).
\]

For any such \( L \)-formula \( \phi \), write \( \phi^\mathcal{M}_a \) for \( \{ f \in \mathfrak{A}_a : \mathcal{M}_a, f \models \phi \} \). Let \( D_a = \{ \phi^\mathcal{M}_a : \phi \text{ is an } L \text{-formula} \} \) and \( \mathfrak{D}_a \) be the weak set algebra with universe \( D_a \). Let \( \mathfrak{D} = \prod_{a \in \alpha} \mathfrak{D}_a \). Then \( \mathfrak{D} \) is a generalized complete weak set.
algebra [4, Definition 3.1.2 (iv)]. Now we show \( Cm \alpha \equiv \mathcal{N}_n \mathcal{D} \). Let \( X \subseteq \mathcal{N}_n \mathcal{D} \). Then by completeness of \( \mathcal{D} \), we get that \( d = \sum_{x \in X} X \) exists. Assume that \( i \notin n \), then \( c_i d = c_i \sum X = \sum_{x \in X} c_i x = \sum X = d \), because the \( c_i \)'s are completely additive and \( c_i x = x \), for all \( i \notin n \), since \( x \in \mathcal{N}_n \mathcal{D} \). We conclude that \( d \in \mathcal{N}_n \mathcal{D} \), hence \( d \) is an upper bound of \( X \) in \( \mathcal{N}_n \mathcal{D} \). Since \( d = \sum_{x \in X} X \) there can be no \( b \in \mathcal{N}_n \mathcal{D}(\subseteq \mathcal{D}) \) with \( b \leq d \) such that \( b \) is an upper bound of \( X \) for else it will be an upper bound of \( X \) in \( \mathcal{D} \). Thus \( \sum_{x \in X} \mathcal{N}_n \mathcal{D} X = d \). We have shown that \( \mathcal{N}_n \mathcal{D} \) is complete. Making the legitimate identification \( \mathcal{N}_n \mathcal{D} \subseteq \mathcal{D} Cm \alpha \) by density, we get that \( \mathcal{N}_n \mathcal{D} = \mathcal{D} \) (since \( \mathcal{N}_n \mathcal{D} \) is complete), hence \( \mathcal{D} \in \mathcal{N}_n \mathcal{CA}_\alpha \).

We start with an easy lemma. If \( \mathcal{B} \) is a Boolean algebra and \( b \in \mathcal{B} \), then \( \mathcal{R}_b \mathcal{B} \) denotes the Boolean algebra with domain \( \{ x \in \mathcal{B} : x \leq b \} \), top element \( b \), and other Boolean operations those of \( \mathcal{B} \) relativized to \( b \).

**Lemma 2.6.** In the following \( \mathcal{A} \) and \( \mathcal{D} \) are Boolean algebras.

1. If \( \mathcal{A} \) is atomic and \( 0 \neq a \in \mathcal{A} \), then \( \mathcal{R}_a \mathcal{A} \) is also atomic. If \( \mathcal{A} \subseteq d \mathcal{D} \) and \( a \in A \), then \( \mathcal{R}_a \mathcal{A} \subseteq d \mathcal{R}_a \mathcal{D} \),

2. If \( \mathcal{A} \subseteq d \mathcal{D} \) then \( \mathcal{A} \subseteq c \mathcal{D} \). In particular, for any class \( \mathcal{K} \) of BAOs, \( \mathcal{K} \subseteq S_d \mathcal{K} \subseteq S_c \mathcal{K} \). If furthermore \( \mathcal{A} \) and \( \mathcal{D} \) are atomic, then \( \mathcal{A} \mathcal{D} \subseteq \mathcal{A} \mathcal{D} \).

**Proof.** (1) Let \( b \in \mathcal{R}_a \mathcal{D} \) be non-zero. Then \( b \leq a \) and \( b \) is non-zero in \( \mathcal{D} \). By atomicity of \( \mathcal{D} \) there is an atom \( c \) of \( \mathcal{D} \) such that \( c \leq b \). So \( c \leq b \leq a \), thus \( c \in \mathcal{R}_c \mathcal{D} \). Also \( c \) is an atom in \( \mathcal{R}_c \mathcal{D} \) because if not, then it will not be an atom in \( \mathcal{D} \). The second part is similar.

(2). Assume that \( \sum \mathcal{A} S = 1 \) and for contradiction that there exists \( b' \in \mathcal{D} \), \( b' < 1 \) such that \( s \leq b' \) for all \( s \in S \). Let \( b = 1 - b' \) then \( b \neq 0 \), hence by assumption (density) there exists a non-zero \( a \in \mathcal{A} \) such that
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If \( a \cdot s \neq 0 \) for some \( s \in S \), then \( a \) is not less than \( b' \). If \( a \cdot s = 0 \) for every \( s \in S \), implying that \( a = 0 \), contradiction. Now we prove the second part. Assume that \( \mathfrak{A} \subseteq_d \mathfrak{D} \) and \( \mathfrak{D} \) is atomic. Let \( b \in \mathfrak{D} \) be an atom. We show that \( b \in \text{At}\mathfrak{A} \). By density there is a non-zero \( a' \in \mathfrak{A} \), such that \( a' \leq b \) in \( \mathfrak{D} \). Since \( \mathfrak{A} \) is atomic, there is an atom \( a \in \mathfrak{A} \) such that \( a \leq a' \leq b \). But \( b \) is an atom of \( \mathfrak{D} \), and \( a \) is non-zero in \( \mathfrak{D} \), too, so it must be the case that \( a = b \in \text{At}\mathfrak{A} \). Thus \( \text{At}\mathfrak{B} \subseteq \text{At}\mathfrak{A} \) and we are done.

Lemma 2.7. Let \( 1 < n < \omega \). There are two atomic cylindric algebras \( \mathfrak{A} \) and \( \mathfrak{B} \) having uncountably many atoms, such that \( \mathfrak{A} \in \text{Nr}_n \mathcal{CA}_{\omega} \), \( \mathfrak{A} \equiv_{\omega, \infty} \mathfrak{B} \) and \( \mathfrak{B} \not\in \text{Sr}_d \text{Nr}_n \mathcal{CA}_{n+1} \).

Proof. We first need to slightly modify the construction in [16, Lemma 5.1.3, Theorem 5.1.4] reformulating it as a ‘splitting argument’. The algebras \( \mathfrak{A} \) and \( \mathfrak{B} \) constructed in \textit{op.cit} satisfy that \( \mathfrak{A} \in \text{Nr}_n \mathcal{CA}_{\omega} \), \( \mathfrak{B} \not\in \text{Nr}_n \mathcal{CA}_{n+1} \) and \( \mathfrak{A} \equiv \mathfrak{B} \). As they stand, \( \mathfrak{A} \) and \( \mathfrak{B} \) are not atomic, but they it can be fixed that they are to be so giving the same result, by interpreting the uncountably many tenary relations in the signature of \( M \) defined in [16, Lemma 5.1.3], which is the base of \( \mathfrak{A} \) and \( \mathfrak{B} \) to be disjoint in \( M \), not just distinct. The construction is presented this way in [?], where (the equivalent of) \( M \) is built in a more basic step-by-step fashion. We work with \( 2 < n < \omega \) instead of only \( n = 3 \). The proof presented in \textit{op.cit} lift verbatim to any such \( n \). Let \( u \in {}^n n \). Write \( \mathbf{1}_u \) for \( \chi_u^M \) (denoted by \( 1_u \) (for \( n = 3 \)) in [16, Theorem 5.1.4]).) We denote by \( \mathfrak{A}_u \) the Boolean algebra \( \mathfrak{R}_u \mathfrak{A} = \{ x \in \mathfrak{A} : x \leq \mathbf{1}_u \} \) and similarly for \( \mathfrak{B} \), writing \( \mathfrak{B}_u \) short hand for the Boolean algebra \( \mathfrak{R}_u \mathfrak{B} = \{ x \in \mathfrak{B} : x \leq \mathbf{1}_u \} \). We show that \( \exists \) has a winning strategy in an Ehrenfeucht-Fraïssé-game over \( (\mathfrak{A}, \mathfrak{B}) \) concluding that \( \mathfrak{A} \equiv_{\infty} \mathfrak{B} \). At any stage of the game, if \( \forall \) places a pebble on one of \( \mathfrak{A} \) or \( \mathfrak{B} \), \( \exists \) must place a matching pebble, on the other algebra. Let \( \bar{a} = \langle a_0, a_1, ..., a_{n-1} \rangle \) be the position of the pebbles played so far (by either player) on \( \mathfrak{A} \) and let \( \bar{b} = \)
$\langle b_0, \ldots, b_{n-1} \rangle$ be the position of the pebbles played on $\mathcal{B}$. $\exists$ maintains the following properties throughout the game: For any atom $x$ (of either algebra) with $x \cdot 1_{Id} = 0$ then $x \in a_i \iff x \in b_i$ and $\overline{a}$ induces a finite partition of $1_{Id}$ in $\mathfrak{A}$ of $2^n$ (possibly empty) parts $p_i : i < 2^n$ and $\overline{b}$ induces a partition of $1_{Id}$ in $\mathfrak{B}$ of parts $q_i : i < 2^n$. Furthermore, $p_i$ is finite $\iff q_i$ is finite and, in this case, $|p_i| = |q_i|$. That such properties can be maintained is fairly easy to show. Using that $M$ has quantifier elimination we get, using the same argument in op.cit that $\mathfrak{A} \in \text{NR}_n \mathfrak{CA}_{\omega_3}$. The property that $\mathfrak{B} \notin \text{NR}_n \mathfrak{CA}_{n+1}$ is also still maintained. To see why consider the substituting operator $n_s(0, 1)$ (using one spare dimension) as defined in the proof of [16, Theorem 5.1.4]. Assume for contradiction that $\mathfrak{B} = \text{NR}_n \mathfrak{C}$, with $\mathfrak{C} \in \mathfrak{CA}_{n+1}$. Let $u = (1, 0, 2, \ldots, n - 1)$. Then $\mathfrak{A}_u = \mathfrak{B}_u$ and so $|\mathfrak{B}_u| > \omega$. The term $n_s(0, 1)$ acts like a substitution operator corresponding to the transposition $[0, 1]$; it ‘swaps’ the first two co-ordinates. Now one can show that $n_s(0, 1)^\mathfrak{g} \mathfrak{B}_u \subseteq \mathfrak{B}_u[0, 1] = \mathfrak{B}_u$ so $|n_s(0, 1)^\mathfrak{g} \mathfrak{B}_u|$ is countable because $\mathfrak{B}_u$ was forced by construction to be countable. But $n_s(0, 1)$ is a Boolean automorphism with inverse $n_s(0, 1)$, so that $|\mathfrak{B}_u| = |n_s(0, 1)^\mathfrak{g} \mathfrak{B}_u| > \omega$, contradiction.

Now we show that the algebra $\mathfrak{B}$ outside $S_d \text{NR}_n \mathfrak{CA}_{\omega_3} \cap \text{At} \supseteq S_d \text{NR}_n \mathfrak{CA}_{\omega_3} \cap \text{CRCA}_{\omega_3}$. Take $\kappa$ the signature of $M$ to be $2^{2^{\omega_3}}$ and assume for contradiction that $\mathfrak{B} \in S_d \text{NR}_n \mathfrak{CA}_{\omega_3} \cap \text{At}$. Then $\mathfrak{B} \subseteq_d \text{NR}_n \mathfrak{D}$, for some $\mathfrak{D} \in \mathfrak{CA}_{\omega_3}$ and $\text{NR}_n \mathfrak{D}$ is atomic. For brevity, let $\mathfrak{C} = \text{NR}_n \mathfrak{D}$. Then by the first item of Lemma 2.6 $\mathfrak{R}_{Id} \mathfrak{B} \subseteq_d \mathfrak{R}_{Id} \mathfrak{C}$. Since $\mathfrak{C}$ is atomic, then by the following item of the same Lemma $\mathfrak{R}_{Id} \mathfrak{C}$ is also atomic. Using the same reasoning as above, we get that $|\mathfrak{R}_{Id} \mathfrak{C}| > 2^{\omega_3}$ (since $\mathfrak{C} \in \text{NR}_n \mathfrak{CA}_{\omega_3}$). By the choice of $\kappa$, we get that $|\text{At} \mathfrak{R}_{Id} \mathfrak{C}| > \omega$. By density using Lemma 2.6, $\text{At} \mathfrak{R}_{Id} \mathfrak{C} \subseteq \text{At} \mathfrak{R}_{Id} \mathfrak{B}$. But by the construction of $\mathfrak{B}$, we have $|\mathfrak{R}_{Id} \mathfrak{B}| = \omega$. The property that $\mathfrak{B} \notin \text{NR}_n \mathfrak{CA}_{n+1}$ is also still maintained. To see why consider the substituting operator $n_s(0, 1)$ (using one spare dimension) as defined in the proof of [16, Theorem 5.1.4]. Assume for contradiction that $\mathfrak{B} = \text{NR}_n \mathfrak{C}$, with $\mathfrak{C} \in \mathfrak{CA}_{n+1}$. Let $u = (1, 0, 2, \ldots, n - 1)$. Then $\mathfrak{A}_u = \mathfrak{B}_u$ and so $|\mathfrak{B}_u| > \omega$. The term $n_s(0, 1)$ acts like a substitution operator corresponding to the transposition $[0, 1]$; it ‘swaps’ the first two co-ordinates. Now one can show that $n_s(0, 1)^\mathfrak{g} \mathfrak{B}_u \subseteq \mathfrak{B}_u[0, 1] = \mathfrak{B}_u$ so $|n_s(0, 1)^\mathfrak{g} \mathfrak{B}_u|$ is countable because $\mathfrak{B}_u$ was forced by construction to be countable. But $n_s(0, 1)$ is a Boolean automorphism with inverse $n_s(0, 1)$, so that $|\mathfrak{B}_u| = |n_s(0, 1)^\mathfrak{g} \mathfrak{B}_u| > \omega$, contradiction.
\[ |At\mathcal{R}_{\text{ld}}^{\mathcal{B}}| = \omega, \] which is a contradiction and we are done.

We conclude:

**Corollary 2.8.** Let \( 2 < n < \omega \). Then any class between \( \text{CRCA}_n \cap \text{Nr}_n \text{CA}_\omega \) and \( K \), where \( K \) is either \( S_c \text{Nr}_n \text{CA}_{n+3} \) or \( S_d \text{Nr}_n \text{CA}_{n+1} \) is not elementary.

**Theorem 2.9.** Any class \( K \) such that \( \text{Nr}_n \text{CA}_\omega \subseteq K \subseteq S_c \text{Nr}_n \text{CA}_{n+3} \) is not first order definable.

**Proof.** (1) **Defining a rainbow-like atom structure \( \alpha \).** We use the algebra in [18, Theorem 5.12]. The algebra \( \mathcal{C}_{\mathbb{Z}, \mathbb{N}}(\in \text{RCA}_n) \) based on \( \mathbb{Z} \) (greens) and \( \mathbb{N} \) (reds) denotes the rainbow-like algebra used in *op.cit* which is defined as follows: The reds \( R \) is the set \( \{r_{ij} : i < j < \omega (= \mathbb{N}) \} \) and the green colours used constitute the set \( \{g_i : 1 \leq i < n - 1\} \cup \{g^i_0 : i \in \mathbb{N}\} \). In complete coloured graphs the forbidden triples are like the usual rainbow constructions based on \( \mathbb{Z} \) and \( \mathbb{N} \), with a significant addition: First the colours used are:

- **greens:** \( g_i (1 \leq i \leq n - 2), g^i_0, i \in \mathbb{Z}, \)
- **whites:** \( w_i : i \leq n - 2, \)
- **reds:** \( r_{ij} (i, j \in \mathbb{N}), \)
- **shades of yellow:** \( y_S : S \) a finite subset of \( \omega \) or \( S = \omega. \)

The rainbow algebra depending on \( \mathbb{N} \) and \( \mathbb{Z} \) from the class \( K \) consisting of all coloured graphs \( M \) such that:

1. \( M \) is a complete graph and \( M \) contains no triangles (called forbidden triples) of the following types:

\[
(g, g', g^*), (g_i, g_i, w_i) \text{ any } 1 \leq i \leq n - 2, \tag{1}
\]

\[
(g^j_0, g^k_0, w_0) \text{ any } j, k \in A, \tag{2}
\]
(r_{ij}, r_{j'k'}, r_{i'k'}) unless i = i^*, j = j^* and k^* = k^*.

Observe that this 1.7 is not as item 1.3 in the proof of Theorem 2.1. Here inconsistent triples of reds are defined differently.

2. The triple \((g_0^i, g_0^j, r_{kl})\) is also forbidden if \(\{(i, k), (j, l)\}\) is not an order preserving partial function from \(\mathbb{Z} \rightarrow \mathbb{N}\).

It is proved in op.cit that \(\exists\) has a winning strategy in \(G_k(\text{AtC}_{\mathbb{Z}, \mathbb{N}})\) for all \(k \in \omega\), so that \(\mathcal{C}_{\mathbb{Z}, \mathbb{N}} \in \text{EICRCA}_{\omega}\). With some more effort it can be proved that \(\exists\) has a winning strategy \(\sigma_k\) say in \(H_k(\text{AtC}_{\mathbb{Z}, \mathbb{N}})\) for all \(k \in \omega\). Let \(\alpha = \text{AtC}_{\mathbb{Z}, \mathbb{N}}\).

(2) \(\exists\) has a winning strategy in \(H_\omega(\alpha)\). We describe \(\exists\)'s strategy in dealing with labelling hyperedges in \(\lambda\)-neat hypernetworks, where \(\lambda\) is a constant label kept on short hyperedges and, not to interrupt the main stream, we defer the rest of the highly technical proof to the appendix. In a play, \(\exists\) is required to play \(\lambda\)-neat hypernetworks, so she has no choice about the short edges, these are labelled by \(\lambda\). In response to a cylindrifier move by \(\forall\) extending the current hypernetwork providing a new node \(k\), and a previously played coloured hypernetwork \(M\) all long hyperedges not incident with \(k\) necessarily keep the hyperlabel they had in \(M\). All long hyperedges incident with \(k\) in \(M\) are given unique hyperlabels not occurring as the hyperlabel of any other hyperedge in \(M\). In response to an amalgamation move, which involves two hypernetworks required to be amalgamated, say \((M, N)\) all long hyperedges whose range is contained in \(\text{nodes}(M)\) have hyperlabel determined by \(M\), and those whose range is contained in \(\text{nodes}(N)\) have hyperlabels determined by \(N\). If \(\bar{x}\) is a long hyperedge of \(\exists\)'s response \(L\), where \(\text{rng}(\bar{x}) \subseteq \text{nodes}(M), \text{nodes}(N)\) then \(\bar{x}\) is given a new hyperlabel, not used in any previously played hypernetwork and not used within \(L\) as the label of any hyperedge other than \(\bar{x}\). This completes her strategy for labelling hyperedges. In [18] it is shown that \(\exists\) has a winning strategy in
\( G_k(\text{Atc}_{\mathbb{Z}, \mathbb{N}}) \), where \( 0 < k < \omega \) is the number of rounds. With some more effort it can be prove that \( \exists \) has a winning strategy in \( H_k(\text{Atc}) \) for each \( k < \omega \), call it \( \sigma_k \). We can assume that \( \sigma_k \) is deterministic. Let \( \mathcal{D} \) be a non-principal ultrapower of \( \mathcal{C}_{\mathbb{Z}, \mathbb{N}} \). Then \( \exists \) has a winning strategy \( \sigma \) in \( H_\omega(\text{Atc}) \)-essentially she uses \( \sigma_k \) in the \( k \)th component of the ultraproduct so that at each round of \( H_\omega(\text{Atc}) \), \( \exists \) is still winning in co-finitely many components, this suffices to show she has still not lost. We can also assume that \( \mathcal{C}_{\mathbb{Z}, \mathbb{N}} \) is countable by replacing it by the term algebra. Now one can use an elementary chain argument to construct countable elementary subalgebras \( \mathcal{C}_{\mathbb{Z}, \mathbb{N}} = \mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \cdots \preceq \cdots \mathcal{D} \) in this manner. One defines \( \mathcal{A}_{i+1} \) be a countable elementary subalgebra of \( \mathcal{D} \) containing \( \mathcal{A}_i \) and all elements of \( \mathcal{D} \) that \( \sigma \) selects in a play of \( G(\text{Atc}) \) in which \( \forall \) only chooses elements from \( \mathcal{A}_i \). Now let \( \mathcal{B} = \bigcup_{i<\omega} \mathcal{A}_i \). This is a countable elementary subalgebra of \( \mathcal{D} \), hence necessarily atomic, and \( \exists \) has a winning strategy in \( H_\omega(\text{Atc}) \) and \( \mathcal{B} \equiv \mathcal{C}_{\mathbb{Z}, \mathbb{N}} \). Thus \( \text{Atc} \in \text{AtNr}_n \text{CA}_\omega \) and \( \mathcal{C}_\text{mAtc} \in \text{Nr}_n \text{CA}_\omega \). (This does not imply that \( \mathcal{B} \in \text{Nr}_n \text{CA}_\omega \), cf. example ??). Since \( \mathcal{B} \subseteq \mathcal{C}_\text{mAtc} \), \( \mathcal{B} \in \text{S}_{d1} \text{Nr}_n \text{CA}_\omega \), so \( \mathcal{B} \in \text{S}_{c1} \text{Nr}_n \text{CA}_\omega \). Being countable, it follows by [16, Theorem 5.3.6] that \( \mathcal{B} \in \text{CRCA}_n \).

(3) \( \forall \) has a winning strategy in \( G^{n+3}(\alpha) \). We now show that hat \( \forall \) has a winning strategy in \( G^{n+3}(\text{Atc}_{\mathbb{Z}, \mathbb{N}}) \) (denoted in \textit{op.cit} by \( F^{n+3}(\text{Atc}_{\mathbb{Z}, \mathbb{N}}) \)), hence by Lemma 1.4, \( \mathcal{C}_{\mathbb{Z}, \mathbb{N}} \not\in \text{S}_c \text{Nr}_n \text{CA}_{n+3} \). It can be shown that \( \forall \) has a winning strategy in the graph version of the game \( G^{n+3}(\text{Atc}) \) played on coloured graphs [5]. The rough idea here, is that, as is the case with winning strategy’s of \( \forall \) in rainbow constructions, \( \exists \) bombards \( \forall \) with cones having distinct green tints demanding a red label from \( \exists \) to apexes of successive cones. The number of nodes are limited but \( \forall \) has the option to re-use them, so this process will not end after finitely many rounds.
The added order preserving condition relating two greens and a red, forces \( \exists \) to choose red labels, one of whose indices form a decreasing sequence in \( \mathbb{N} \).

In \( \omega \) many rounds \( \forall \) forces a win, so by Lemma 1.4, \( \mathcal{C} \not\in S_cN_rCA_{n+3} \).

More rigorously, \( \forall \) plays as follows: In the initial round \( \forall \) plays a graph \( M \) with nodes \( 0, 1, \ldots, n - 1 \) such that \( M(i, j) = w_0 \) for \( i < j < n - 1 \) and \( M(i, n - 1) = g_j(i = 1, \ldots, n - 2), M(0, n - 1) = g_0^0 \) and \( M(0, 1, \ldots, n - 2) = y_Z \). This is a 0 cone. In the following move \( \forall \) chooses the base of the cone \( (0, \ldots, n - 2) \) and demands a node \( n \) with \( M_2(i, n) = g_j(i = 1, \ldots, n - 2) \) and \( M_2(0, n) = g_0^{-1} \). \( \exists \) must choose a label for the edge \( (n + 1, n) \) of \( M_2 \).

It must be a red atom \( r_{mk}, m, k \in \mathbb{N} \). Since \( -1 < 0 \), then by the ‘order preserving’ condition we have \( m < k \). In the next move \( \forall \) plays the face \( (0, \ldots, n - 2) \) and demands a node \( n + 1 \), with \( M_3(i, n) = g_j(i = 1, \ldots, n - 2) \), such that \( M_3(0, n + 2) = g_0^0 \). Then \( M_3(n + 1, n) \) and \( M_3(n + 1, n - 1) \) both being red, the indices must match. \( M_3(n + 1, n) = r_{kl}^1 \) and \( M_3(n + 1, r - 1) = r_{km}^l \) with \( l < m \in \mathbb{N} \). In the next round \( \forall \) plays \( (0, 1, \ldots, n - 2) \) and re-uses the node 2 such that \( M_4(0, 2) = g_0^{-3} \). This time we have \( M_4(n, n - 1) = r_{jl} \) for some \( j < l < m \in \mathbb{N} \). Continuing in this manner leads to a decreasing sequence in \( \mathbb{N} \). We have proved the required.

(4) **Proving the required.** Let \( K \) be a class between \( S_dN_rCA_{\omega} \cap CRCA_n \) and \( S_cN_rCA_{n+3} \). Then \( K \) is not elementary, because \( \mathcal{C}_{\mathbb{Z}, \mathbb{N}} \not\in S_cN_rCA_{n+3}(\equiv K) \), \( \mathcal{B} \in S_dN_rCA_{\omega} \cap CRCA_n(\subseteq K) \) and \( \mathcal{C}_{\mathbb{Z}, \mathbb{N}} = \mathcal{B} \). It clearly suffices to show that \( K_k = \{ \mathcal{A} \in CA_n \cap At : \mathcal{C}mAt\mathcal{A} \in ONr_rCA_k \} \) is not elementary. \( \exists \) has a winning strategy in \( H_\omega(\alpha) \) for some countable atom structure \( \alpha \), \( \exists \mathfrak{m}\alpha \subseteq_d \mathcal{C}m\alpha \in Nr_rCA_{\omega} \) and \( \mathfrak{m}\alpha \in CRCA_n \). Since \( \mathcal{C}_{\mathbb{Z}, \mathbb{N}} \not\in S_cN_rCA_{n+3} \), then \( \mathcal{C}_{\mathbb{Z}, \mathbb{N}} = \mathcal{C}mAt\mathcal{C}_{\mathbb{Z}, \mathbb{N}} \not\in K_k \), \( \mathcal{C}_{\mathbb{Z}, \mathbb{N}} = \mathfrak{m}\alpha \) and \( \mathfrak{m}\alpha \in K_k \) because \( \mathcal{C}m\alpha \in Nr_rCA_{\omega} \subseteq S_dN_rCA_{\omega} \subseteq S_cN_rCA_{n+3} \). We have shown that \( \mathcal{C}_{\mathbb{Z}, \mathbb{N}} \in ElK_k \sim K_k \), proving the required. To prove the second
item. From the first item, we have excluded any first order definable class between $\mathcal{S}_d^1\mathcal{N}_{r_n}\mathcal{C}A_{\omega} \cap \mathcal{C}R\mathcal{C}A_n$ and $\mathcal{S}_c^1\mathcal{N}_{r_n}\mathcal{C}A_{n+3}$. So hoping for a contradiction, we can only assume that there is a class $\mathcal{M}$ between $\mathcal{N}_{r_n}\mathcal{C}A_{\omega} \cap \mathcal{C}R\mathcal{C}A_n$ and $\mathcal{S}_d^1\mathcal{N}_{r_n}\mathcal{C}A_{\omega} \cap \mathcal{C}R\mathcal{C}A_n$ that is first order definable. Then $\mathcal{E}(\mathcal{N}_{r_n}\mathcal{C}A_{\omega} \cap \mathcal{C}R\mathcal{C}A_n) \subseteq \mathcal{M} \subseteq \mathcal{S}_d^1\mathcal{N}_{r_n}\mathcal{C}A_{\omega} \cap \mathcal{C}R\mathcal{C}A_n$. We have $\mathcal{B} \equiv \mathfrak{A}$, and $\mathfrak{A} \in \mathcal{N}_{r_n}\mathcal{C}A_{\omega} \cap \mathcal{C}R\mathcal{C}A_n$, hence $\mathcal{B} \in \mathcal{E}(\mathcal{N}_{r_n}\mathcal{C}A_{\omega} \cap \mathcal{C}R\mathcal{C}A_n) \subseteq \mathcal{S}_d^1\mathcal{N}_{r_n}\mathcal{C}A_{\omega} \cap \mathcal{C}R\mathcal{C}A_n$ which contradicts that $\mathcal{B} \notin \mathcal{S}_d^1\mathcal{N}_{r_n}\mathcal{C}A_{\omega}$.

References


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