Notions of representability for cylindric algebras–some algebras are more representable than others

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Abstract . The theory of cylindric algebras was introduced by Tarski in the fifties of the 20th century, and its intensive study was further pursued by pioneers such as Henkin and Monk and, by the Hungarian mathematicians Andréka, Németi and Sain, and many of their students; to name only a few: Madarász, Marx, Kurucz, Simon, Mikulás, and Sági and many others (outside Hungary) including the author of this paper, where we introduce and investigate new notions of representability for cylindric algebras and investigate various connections between such notions. Let $2 < n \leq l < m \leq \omega$. Let CA_n denote the variety of cylindric algebras of dimension n and let RCA_n denote the variety of representable CA_ns. We say that an atomic algebra $\mathfrak{A} \in \mathsf{CA}_n$ has the complex neat embedding property up to l and m if $\mathfrak{A} \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$ and $\mathfrak{CmAt}\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$. Fixing the prarameters l at the value n, this is a measure of how much the algebra is representable. The yardstick is how far can its Dedekind-MacNeille completion be dilated, that is to say, counting the number of more extra dimensions its Dedekind-MacNeille completion neatly embeds into. If $\mathfrak{A}, \mathfrak{B} \in \mathsf{RCA}_n$ are atomic, $\mathfrak{CmAt}\mathfrak{B} \in S\mathsf{Nr}_n\mathsf{CA}_l$ and $\mathfrak{CmAt}\mathfrak{A} \in S\mathsf{Nr}_n\mathsf{CA}_m$, then we say that $\mathfrak{A}$ is more representable than $\mathfrak{B}$. When $m = \omega$, we say that $\mathfrak{A}$ is strongly representable; this is the maximum degree of representability; the algebra in question cannot be 'more representable' than that. In this case the atom structure of $\mathfrak{A}$, namely $\mathsf{At}\mathfrak{A}$, is strongly representable in the sense of Hirsch and Hodkinson. This notion gives an infinite potential spectrum of 'degrees' of representability. In this connection, we exhibit various atomic algebras in $\mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$ that do no not have the complex neat embedding property for infinitely many values of l and m. It is known that the class of Kripke frames $\mathsf{Str}(\mathsf{RCA}_n) = \{\mathfrak{F} : \mathfrak{Cm}\mathfrak{F} \in \mathsf{RCA}_n\}$ is not elementary. From this it follows that there is some $n < m < \omega$ such that $\mathsf{Str}(\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m) = \{\mathfrak{F} : \mathfrak{Cm}\mathfrak{F} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m\}$ is not elementary. Replacing $\mathbf{S}$ by $\mathbf{S}_c$ (forming complete subalgebras), $\mathbf{S}_d$ (forming dense subalgebras) and $\mathbf{I}$ (forming isomorphic copies), respectively, we show that for any $\mathbf{O} \in \{\mathbf{S}_c, \mathbf{S}_d, \mathbf{I}\}$, the class of frames $\mathsf{Str}(\mathbf{O}\mathsf{Nr}_n\mathsf{CA}_{n+3}) = \{\mathfrak{F} : \mathfrak{Cm}\mathfrak{F} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_{n+3}\}$ is not elementary. Metalogical applications are given to n-variable fragments of first order logic endowed wth so-called clique guarded semantics. The last semantics capture the new notions of representations introduced and studied in this paper. [1]

1 Introduction

It is well known that every Boolean algebra (satisfying a finite set of equations) is isomorphic to a field of sets, that is to say, every Boolean algebra is representable in some concrete sense, where the Boolean meets and joins are intepreted respectively as set theoretic intersections and unions. This result, betten known in the literature as Stone's Theorem, is equivalent (in ZFC) to the completeness of propositional logic. But in the case of cylindric and polyadic algebras of various dimensions the 'representation problem' is somewhat more involved. For example not every cylindric algebra of dimension > 1 is representable as a genuine field of sets with cylindrifications interpreted as forming cylinders, and in fact, the class of representable cylindric algebras of dimension > 2, though a variety, cannot be axiomatized by finite schema

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of equations. Nevertheless, Tarski proved that in certain significant cases, the finitely many cylindric algebra axioms may be adequate and strong enough to enforce representability.

In this connection, Tarski proved that every locally finite dimensional cylindric algebra of infinite dimension is representable, and this, in turn, is equivalent to the completeness theorem of first order logic proved earlier by Malcev and Henkin (who generalized Gödel's original completeness proof for only countable first order languages using the technique of Skolem functions). Here the condition of 'local finite dimensional' is an algebraic condition reflecting the fact that formulas considered have finite length. The condition of being locally finite dimensional is not a first order one, nor can it indeed be replaced equivalently by a condition that is first order definable, because it can be quite easily shown that the ultraproduct of locally finite dimensional cylindric algebras (of infinite dimension) may not be locally finite dimensional.

In the realm of representable algebras, there are several types of representations. Ordinary representations are just isomorphisms from Boolean algebras with operators to a more concrete structure (having the same signature) whose elements are sets endowed with set-theoretic operations like intersection and complementation and forming cylinders. Complete representations, on the other hand, are representations that preserve arbitrary conjunctions whenever defined. More generally consider the following question: Given an algebra and a set of meets, is there a representation that carries this set of meets to set theoretic intersections? A complete representation would thus be one that preseves *all existing* meets (finite of course and infinite). Here we are assuming that our semantics is specified by set algebras, with the concrete Boolean operation of intersection among its basic operations.

When the algebra in question is countable, and we have only countably many meets; this is an algebraic version of an omitting types theorem; the representation omits the given set meets or non-principal types. When the algebra in question is atomic, then a representation omitting the non-principal type consisting of co-atoms, turns out to be a complete representation. This follows from the following result due to Hirsch and Hodkinson: A Boolean algebra $\mathfrak{A}$ has a complete representation $f : \mathfrak{A} \to \langle \wp(X), \cup, \cap, \sim, \emptyset, X \rangle$ (f is a 1-1 homomorphism and X a set) $\iff$ $\mathfrak{A}$ atomic and $\bigcup_{x \in \mathsf{At}\mathfrak{A}} f(x) = X$, where $\mathsf{At}\mathfrak{A}$ is the set of atoms of $\mathfrak{A}$. The notion of complete representations has been linked to the algebraic notion of atom-canonicity (a well known persistence property in modal logic) and to the metalogical notions of Martin's axiom, omitting types theorems and the existence of atomic models for atomic theories in various fragments and extensions of first order logic [14, 16].

On the face of it, the notion of complete representations seems to be strikingly a second order one. This intuition is confirmed in [6] where it is proved that the classes of completely representable cylindric algebras of dimension at least three and that of relation algebras are not elementary. These results were proved by Hirsch and Hodkinson using so-called rainbow algebras [6]; in this paper we present entirely different proofs for all such results and some more closely related ones using so called Monk-like algebras. Our proof depends essentially on some form of an infinite combinatorial version of Ramsey's Theorem. But running to such conclusions–concerning (non-)first order definablity– can be reckless and far too hasty; for in other non-trivial cases the notion of complete representations turns *not to be* a genuinely second order one; it is definable in first order logic.

The class of completely representable Boolean algebras is elementary; it simply coincides with the atomic ones. A far less trivial example is the class of completely representable infinite dimensional polyadic algebras; it coincides with the class of atomic, completely additive algebras. It is not hard to show that, like atomicity, complete additivity for atomic algebras can indeed be defined in first order logic as is explained in detail in [15]. Complete additivity of extra Boolean operations defined on atomic algebras is a notion that can be captured in first order logic; and surprisingly quite simply [15]. It is commonly accepted that the cylindric paradigm and polyadic paradigm belong to different worlds, often exhibiting condflicting behaviour, with the last highlighted by the presence of the operations of substitutions [4] that occur in polyadic jargon under the name of transformation systems.

The elementary closure of the class of completely representable relation and cylindric algebras of dimension > 2 has been studied in some depth by Hirsch and Hodkinson. This class is characterized by the so-called Lyndon conditions. For each k, there is a kth Lyndon condition, ρ_k which is a first order senternce coding a winning strategy in a zero-sum k-rounded Ehrenfeucht–Fraïssé game between two players $\exists$ and $\forall$; the ρ_ks taken together axiomatize this class. All of the ρ_ks are needed for the axiomatization of this class, for it is not finitely axiomatizable.

Fix finite $n > 2$. Let CRCA_n denote the class of completely representable CA_ns and $\mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n$ be the class of algebras satisfying the Lyndon conditions. For a class K of Boolean algebras with operators, let $\mathsf{K} \cap \mathbf{At}$ denote the class of atomic algebras in K. By modifying the games coding the Lyndon conditions allowing $\forall$ to reuse the pebble pairs on the board, we will show that $\mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n = \mathbf{El}\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$. Define $\mathfrak{A} \in \mathsf{CA}_n$ to be *strongly representable* $\iff$ $\mathfrak{A}$ is atomic and the complex algebra of its atom structure, equivalently its Dedekind-MacNeille completion, in symbols $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ is in RCA_n. This is a strong form of representability; of course $\mathfrak{A}$ itself will be in RCA_n, because $\mathfrak{A}$ embeds into $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ and RCA_n is a variety, *a fortiori* closed under forming subalgebras. We denote the class of strongly representable atomic algbras of dimension n by SRCA_n. Nevertheless, there are atomic simple countable algebras that are representable, but not strongly representable. In fact, we shall see that there is a countable simple atomic algebra in RCA_n such that $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+3}(\supset \mathsf{RCA}_n)$.

So in a way some algebras are more representable than others. In fact, the following inclusions are known to hold:

$$\mathsf{CRCA}_n \subsetneq \mathsf{LCA}_n \subsetneq \mathsf{SRCA}_n \subsetneq \mathsf{RCA}_n \cap \mathbf{At}.$$

In this paper we delve into a new notion, that of *degrees of representability.* Not all algebras are representable in the same way or strength. If $\mathfrak{C} \subseteq \mathsf{Nr}_n\mathfrak{D}$, with $\mathfrak{D} \in \mathsf{CA}_m$ for some ordinal (possibly infinite) m, we say that $\mathfrak{D}$ is an m-dilation of $\mathfrak{C}$ or simply a dilation if m is clear from context. Using this jargon of 'dilating algebras' we say that $\mathfrak{A} \in \mathsf{RCA}_n$ is *strongly representable up to* $m > n$ $\iff$ $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$. This means that, though $\mathfrak{A}$ itself is in RCA_n, the Dedekind-MacNeille completion of $\mathfrak{A}$ is not representable, but nevertheless it has some neat embedding property; it is 'close' to bieng representable. Using this jargon, $\mathfrak{A}$ admits a dilation of a bigger dimension. The bigger the dimension of the dilation of the representable algebra the more representable the algebra is, the closer it is to being strongly representable. Through the unfolding of this paper, we will investigate and make precise the notion of an algebra being more representable than another. It is known that LCA_n is an elementary class, but SRCA_n is not. We shall prove below that $\mathsf{Str}(\mathbf{O}\mathsf{Nr_n}\mathsf{CA_{n+3}} = \{\mathfrak{F} : \mathfrak{Cm}\mathfrak{F} \in \mathbf{O}\mathsf{Nr_n}\mathsf{CA_{n+3}}\}$ is not elementary with $\mathbf{O} \in \mathbf{S}_c, S_d, I$ as defind in the abstract.

Layout: After the preliminaries, we show that there exists an atomic, countable and simple $\mathfrak{A} \in \mathsf{RCA}_n$, such that its Dedekind-MacNeille completion, namely, the complex algebra of its atom structure, briefly $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$, is outside the variety $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+3}$, cf. Theorem 3.2. For any $2 < n < l < \omega$, we show there exists an atomic algebra $\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$, such that its Dedekind-MacNeille completion $\mathfrak{Cm}\mathsf{At}\mathfrak{B}$ is not representable, cf. Theorem 3.5. We show that there is an atomic algebra $\mathfrak{E} \in \mathsf{RCA}_n$ such that its Dedekind-MacNeille completion

$\mathfrak{CmAtE}$ is in $\mathsf{Nr}_n\mathsf{CA}_\omega$, but the algebra $\mathfrak{E}$ itself, is not even in $\mathsf{Nr}_n\mathsf{CA}_{n+1}$, cf. Theorem 3.6. We show that for $2 < n < \omega$ a version of the omitting types fails for L_n 'almost everywhere'– a notion to be made precise. We show that for any $\mathbf{O} \in \{\mathbf{S}_c, \mathbf{S}_d, \mathbf{I}\}$, the class of frames $\mathsf{Str}(\mathbf{O}\mathsf{Nr}_n\mathsf{CA}_{n+3}) = \{\mathfrak{F} : \mathfrak{CmF} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_{n+3}\}$ is not elementary, cf. Theorem 5.5. Our proof constructs a completely representable algebra $\mathfrak{B}$ and an atomic representable algebra $\mathfrak{C}$ such that $\mathsf{At}\mathfrak{B} \in \mathsf{AtNr}_n\mathsf{CA}_\omega$, $\mathfrak{CmAtB} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, $\mathfrak{B} \equiv \mathfrak{C}$ and $\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$, cf. Theorem 5.4. We relate notions of representablity formulated for atomic algebras such as, complete, strong, weak, and satisfying the Lyndon condition, to atomic algebras having special neat embedding properties, cf. Theorems 6.1, 6.3, 6.3.

2 Preliminaries

We follow the notation of [1] which is in conformity with the notation in the monograph [5]. In particular, for any pair of ordinal $\alpha < \beta$, $\mathsf{Nr}_\alpha\mathsf{CA}_\beta (\subseteq \mathsf{CA}_\alpha)$ denotes the class of α–neat reducts of CA_βs. The last class is studied extensively in the chapter [13] of [1] as a key notion in the representation theory of cylindric algebras.

Definition 2.1. Assume that $\alpha < \beta$ are ordinals and that $\mathfrak{B} \in \mathsf{CA}_\beta$. Then the *$\alpha$–neat reduct* of $\mathfrak{B}$, in symbols $\mathfrak{Nr}_\alpha\mathfrak{B}$, is the algebra obtained from $\mathfrak{B}$, by discarding cylindrifiers and diagonal elements whose indices are in $\beta \setminus \alpha$, and restricting the universe to the set $Nr_\alpha B = \{x \in \mathfrak{B} : \{i \in \beta : \mathsf{c}_i x \neq x\} \subseteq \alpha\}$.

It is straightforward to check that $\mathfrak{Nr}_\alpha\mathfrak{B} \in \mathsf{CA}_\alpha$. Let $\alpha < \beta$ be ordinals. If $\mathfrak{A} \in \mathsf{CA}_\alpha$ and $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$, with $\mathfrak{B} \in \mathsf{CA}_\beta$, then we say that $\mathfrak{A}$ *neatly embeds* in $\mathfrak{B}$, and that $\mathfrak{B}$ is a *β–dilation of* $\mathfrak{A}$, or simply a *dilation* of $\mathfrak{A}$ if β is clear from context. For $\mathbf{K} \subseteq \mathsf{CA}_\beta$, we write $\mathsf{Nr}_\alpha\mathbf{K}$ for the class $\{\mathfrak{Nr}_\alpha\mathfrak{B} : \mathfrak{B} \in \mathbf{K}\}$.

Let $2 < n < \omega$. Following [5], Cs_n denotes the class of *cylindric set algebras of dimension* n, and Gs_n denotes the class of *generalized cylindric set algebra of dimension* n; $\mathfrak{C} \in \mathsf{Gs}_n$, if $\mathfrak{C}$ has top element V a disjoint union of cartesian squares, that is $V = \bigcup_{i \in I} {}^nU_i$, I is a non-empty indexing set, $U_i \neq \emptyset$ and $U_i \cap U_j = \emptyset$ for all $i \neq j$. The operations of $\mathfrak{C}$ are defined like in cylindric set algebras of dimension n relativized to V. It is known that $\mathbf{I}\mathsf{Gs}_n = \mathsf{RCA}_n = \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_\omega = \bigcap_{k \in \omega} \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k}$. We often identify set algebras with their domain referring to an injection $f : \mathfrak{A} \to \wp(V)$ ($\mathfrak{A} \in \mathsf{CA}_n$) as a complete representation of $\mathfrak{A}$ (via f) where V is a Gs_n unit.

Definition 2.2. An algebra $\mathfrak{A} \in \mathsf{CA}_n$ is *completely representable* $\iff$ there exists $\mathfrak{C} \in \mathsf{Gs}_n$, and an isomorphism $f : \mathfrak{A} \to \mathfrak{C}$ such that for all $X \subseteq \mathfrak{A}$, $f(\sum X) = \bigcup_{x \in X} f(x)$, whenever $\sum X$ exists in $\mathfrak{A}$. In this case, we say that $\mathfrak{A}$ is *completely representable via f.*

It is known that $\mathfrak{A}$ is completely representable via $f : \mathfrak{A} \to \mathfrak{C}$, where $\mathfrak{C} \in \mathsf{Gs}_n$ has top element V say $\iff$ $\mathfrak{A}$ is atomic and f is *atomic* in the sense that $f(\sum \mathsf{At}\mathfrak{A}) = \bigcup_{x \in \mathsf{At}\mathfrak{A}} f(x) = V$ [6]. We denote the class of completely representable CA_ns by CRCA_n. To define certain deterministic games to be used in the sequel, we recall the notions of *atomic networks* and *atomic games* [7, 8]. Let $i < n$. For n–ary sequences $\bar{x}$ and $\bar{y}$ $\iff$ $\bar{y}(j) = \bar{x}(j)$ for all $j \neq i$.

Definition 2.3. Fix finite $n > 2$ and assume that $\mathfrak{A} \in \mathsf{CA}_n$ is atomic.

(1) An *n–dimensional atomic network* on $\mathfrak{A}$ is a map $N : {}^n\Delta \to At\mathfrak{A}$, where Δ is a non–empty set of *nodes*, denoted by $\mathsf{nodes}(N)$, satisfying the following consistency conditions for all $i < j < n$:

- If $\bar{x} \in {}^n\mathsf{nodes}(N)$ then $N(\bar{x}) \leq \mathsf{d}_{ij} \iff x_i = x_j$,
- If $\bar{x}, \bar{y} \in {}^n\mathsf{nodes}(N)$, $i < n$ and $\bar{x} \equiv_i \bar{y}$, then $N(\bar{x}) \leq \mathsf{c}_i N(\bar{y})$.

For n–dimensional atomic networks M and N, we write $M \equiv_i N \iff M(\bar{y}) = N(\bar{y})$ for all $\bar{y} \in {}^n(n \sim \{i\})$.

(2) Assume that $m, k \leq \omega$. The *atomic game* $G^m_k(\mathsf{At}\mathfrak{A})$*, or simply* G^m_k, is the game played on atomic networks of $\mathfrak{A}$ using m nodes and having k rounds [8, Definition 3.3.2], where $\forall$ is offered only one move, namely, *a cylindrifier move*: Suppose that we are at round $t > 0$. Then $\forall$ picks a previously played network N_t ($\mathsf{nodes}(N_t) \subseteq m$), $i < n$, $a \in \mathsf{At}\mathfrak{A}$, $x \in {}^n\mathsf{nodes}(N_t)$, such that $N_t(\bar{x}) \leq \mathsf{c}_i a$. For her response, $\exists$ has to deliver a network M such that $\mathsf{nodes}(M) \subseteq m$, $M \equiv_i N$, and there is $\bar{y} \in {}^n\mathsf{nodes}(M)$ that satisfies $\bar{y} \equiv_i \bar{x}$ and $M(\bar{y}) = a$. We write $G_k(\mathsf{At}\mathfrak{A})$, or simply G_k, for $G^m_k(\mathsf{At}\mathfrak{A})$ if $m \geq \omega$.

(3) The ω–rounded game $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$ or simply $\mathbf{G}^m$ is like the game $G^m_\omega(\mathsf{At}\mathfrak{A})$ except that $\forall$ has the option to reuse the m nodes in play.

Let $2 < n < m \leq \omega$. The notion of an algebra $\mathfrak{A}$ having signature CA_n possesing an m-square representation is define in detail [16]. An m-square representation only locally classic. Given $2 < l < m \leq \omega$, an m - square representation is l-square but the converse may fail dramatically. An ω-square rpresentation–the limiting case-is an ordinary representation, such a representation is m-square for each finite m. Roughly, if we zoom in by a movable window to an m-square represention, there will come a point determined by the parameter m, were we mistake this locally classic represenation for a genuine ordinary Tarskian one. However, when we zoom out 'contradictions' reappear. We will return to such issues in some detail in a moment. The following lemma is proved in [17, Lemma 4.6] and [16, Lemma 5.8].

Lemma 2.4. *Let* $2 < n < m \leq \omega$.

1. *If* $\mathfrak{A} \in \mathsf{CA}_n$ *is finite and* $\mathfrak{A}$ *has an* m*- square representation, then* $\exists$ *has a winning strategy in* $G^m(\mathsf{At}\mathfrak{A})$
2. *If* $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$*, then* $\forall$ *has a winning strategy in* $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$.

In our proof we use a variation on a *rainbow constructions*; in this we follow [6, 8]. Fix $2 < n < \omega$. Given relational structures G (the greens) and R (the reds) the rainbow atom structure of a CA_n consists of equivalence classes of surjective maps $a : n \to \Delta$, where Δ is a coloured graph. A *coloured graph* is a complete graph labelled by the rainbow colours, the greens $\mathsf{g} \in \mathsf{G}$, reds $\mathsf{r} \in \mathsf{R}$, and whites; and some $n-1$ tuples are labelled by 'shades of yellow'. In coloured graphs certain triangles are not allowed for example all green triangles are forbidden. A red triple $(\mathsf{r}_{ij}, \mathsf{r}_{j'k'}, \mathsf{r}_{i^*k^*})$ $i, j, j', k', i^*, k^* \in \mathsf{R}$ is not allowed, unless $i = i^*$, $j = j'$ and $k' = k^*$, in which case we say that the red indices match, cf.[6, 4.3.3]. The equivalence relation relates two such maps $\iff$ they essentially define the same graph [6, 4.3.4]. We let $[a]$ denote the equivalence class containing a. For $2 < n < \omega$, we use the graph version of the usual atomic ω–rounded game $G^m_\omega(\alpha)$ with m nodes, played on atomic networks of the CA_n atom structure α. The game $\mathbf{G}^m(\beta)$ where β is a CA_n atom structure is like $G^m_\omega(\mathsf{At}\mathfrak{A})$ except that $\forall$ has the option to reuse the m nodes in play. We use the 'graph versions' of these games as defined [6, 4.3.3]. The (complex) rainbow algebra based on G and R is denoted by $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$. The dimension n will always be clear from context.

3 Degrees of representability

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Recall that $\mathbf{S}_c$ denotes the operation of forming complete subalgebras and $\mathbf{S}_d$ is the operation of forming dense subalgebras. We let $\mathbf{I}$ denote the operation of forming isomorphic images. For any class K of BAOs, it is easy to check that $\mathbf{I}\mathsf{K} \subseteq \mathbf{S_d}\mathsf{K} \subseteq \mathbf{S_c}\mathsf{K}$. (It is not hard to show that if K is the class of Boolean algebras, that is to say, without extra operations, then the above two inclusions are proper.)

Definition 3.1. Let $2 < n \leq l < m \leq \omega$. Let $\mathbf{O} \in \{\mathbf{S}, \mathbf{S}_d, \mathbf{S}_c, \mathbf{I}\}$

(1) An algebra $\mathfrak{A} \in \mathsf{CA}_n$ *has the* $\mathbf{O}$ *neat embedding property up to* m if $\mathfrak{A} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_m$. If $m = \omega$ and $\mathbf{O} = \mathbf{S}$, we say that $\mathfrak{A}$ has the *neat embedding property.* Observe that the last condtion is equivalent to that $\mathfrak{A} \in \mathsf{RCA}_n$

(2) An atomic $\mathfrak{A} \in \mathsf{CA}_n$ *has the complex* $\mathbf{O}$ *neat embeddng property up to* m if $\mathfrak{CmAt}\mathfrak{A} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_m$. The word 'complex' in the present context referes to involving the Dedekind-MacNeille completion obtained by forming the complex algebra of the atom structure–in the definition at hand.

(3) An atomic algebra $\mathfrak{A} \in \mathsf{RCA}_n$ is it strongly representable up l and m if $\mathfrak{A} \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$ and $\mathfrak{CmAt}\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$. If $l = n$ and $m = \omega$, we say that $\mathfrak{A}$ is strongly representable.

In out first two main theorems, cf. Theorem 3.2, 3.5, we use a so-called *blow up and blur construction.* We find it useful to give the gist of the idea to make it easier for the reader–for the idea in essence is really simple and subtle, but may be overshadowed by the details of the specific otherwise possibly complicated construction at hand.

General Idea: The idea of a blow up and blur construction in (more than in) a nut shell is the following. Let $2 < n < \omega$.

- Assume that $\mathsf{RCA}_n \subseteq \mathsf{K} \subseteq \mathsf{CA}_n$, and that $\mathbf{S}\mathsf{K} = \mathsf{K}$, that is K is closed under forming subalgebras. The purpose is to show that K is not closed under Dedekind-MacNeille completions also known as Monk-minimal completions.
- One starts with an atomic algebra $\mathfrak{C} \in \mathsf{CA}_n$ (usually finite) outside K. Then one *blows up and blur* $\mathfrak{C}$, by splitting some of its atoms each to infinitely many, getting a new infinite atom structure $\mathbf{At}$. In this process a (finite) set of 'blurs' are involved in a way to be clarified in a moment. These blurs *do not blur* the complex algebra $\mathfrak{Cm}\mathbf{At}$, in the sense that $\mathfrak{C}$ is 'there on this global level', $\mathfrak{C}$ *embeds into* $\mathfrak{Cm}\mathbf{At}$.
- Thus the algebra $\mathfrak{Cm}\mathsf{At}$ will not be in K because $\mathfrak{C} \notin \mathsf{K}$, $\mathfrak{C} \subseteq \mathfrak{Cm}\mathbf{At}$ and $\mathbf{S}\mathsf{K} = \mathsf{K}$. The *completeness (existence of arbitray joins) of the complex algebra* plays a major role, because every splitted atom of $\mathfrak{C}$, is mapped to *the join* of its splitted copies which exist in $\mathfrak{Cm}\mathbf{At}$, because it is complete; the other atoms are mapped to themselves. These precarious joins prohibiting membership in K *do not* exist in the term algebra $\mathfrak{Tm}\mathbf{At}$, the subalgebra of $\mathfrak{Cm}\mathbf{At}$ generated by the atoms, because it is not complete; only joins of finite or cofinite subsets of the atoms do, so that now *'blurs' blur* $\mathfrak{C}$ on the level of the *term algebra*; more succintly, $\mathfrak{C}$ *does not embed* in $\mathfrak{Tm}\mathbf{At}$.

- In fact, it can (and will be) be arranged that $\mathfrak{Tm}\mathbf{At}$ will not only be in K, but actually it will be in (the possibly smaller) class RCA_n. This is where the blurs play another crucial role. Basically including essentially non-principal ultrafilters, the blurs, together with the principal ultrafilters generated by the atoms in $\mathbf{At}$ will be used as colours to represent $\mathfrak{Tm}\mathbf{At}\mathfrak{A}$ as an algebra of genuine n-ary relations with concrete set theoreic operations. In the process of representation, one cannot use *only* principal ultrafilters, because $\mathfrak{Tm}\mathbf{At}$ cannot be completely representable; for else this induces a representation of $\mathfrak{Cm}\mathbf{At}\mathfrak{A}$.

- But using the blurs one can actually *completely represent* $(\mathfrak{Tm}\mathbf{At})^+$ the *canonical extension of* $\mathfrak{Tm}\mathbf{At}$. Concluding we get an atom structure $\mathbf{At}$ that is only weakly representable, that is to say, $\mathfrak{Tm}\mathbf{At} \in \mathsf{RCA}_n$, but not strongly representable, that is to say, $\mathfrak{Cm}\mathbf{At} \notin \mathsf{RCA}_n$.

Let us get more concrete giving some specific examples to this subtle construction that proves highly efficient in proving non–atom canonicity.

Theorem 3.2. *Let $2 < n < \omega$. There exists an atomic, countable and simple $\mathfrak{A} \in \mathsf{RCA}_n$ (i.e. $\mathfrak{A}$ has the neat embedding property), but $\mathfrak{A}$ does not have the complex* $\mathbf{S}$ *neat embedding property up to m for any $m \geq n+3$.*

Proof. The proof is divided into four parts:

1: **Blowing up and blurring a finite rainbow algebra forming a weakly representable atom structure $\mathbf{At}$**: Take the finite rainbow CA_n, $\mathfrak{A}_{n+1,n}$ where the reds R is the complete irreflexive graph n, and the greens are $\{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}_0^i : 1 \leq i \leq n+1\}$. We will show $\mathfrak{A}_{n+1,n}$ detects that RCA_n is not atom-canonical with respect to $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+3}$. Denote the finite atom structure of $\mathfrak{A}_{n+1,n}$ by $\mathbf{At}_f$; so that $\mathbf{At}_f = \mathsf{At}(\mathfrak{A}_{n+1,n})$. One then replaces the red colours of the finite rainbow algebra of $\mathfrak{A}_{n+1,n}$ each by infinitely many reds (getting their superscripts from ω), obtaining this way a weakly representable atom structure $\mathbf{At}$. The cylindric reduct of the resulting atom structure after 'splitting the reds', namely, $\mathbf{At}$, is like the weakly (but not strongly) representable atom structure of the atomic, countable and simple algebra $\mathfrak{A}$ as defined in [10, Definition 4.1]; the sole difference is that we have $n+1$ greens and not ω–many as is the case in [10]. One then defines a larger the class of coloured graphs like in [10, Definition 2.5]. Let $2 < n < \omega$. Then the colours used are like above except that each red is 'split' into ω many having 'copies' the form r_{ij}^l with $i < j < n$ and $l \in \omega$, with an additional shade of red ρ such that the consistency conditions for the new reds (in addition to the usual rainbow consistency conditions) are as follows:

- $(r_{jk}^i, r_{j'k'}^i, r_{j^*k^*}^{i^*})$ unless $i = i' = i^*$ and $|\{(j,k),(j',k'),(j^*,k^*)\}| = 3$
- (r, ρ, ρ) and $(\mathsf{r}, \mathsf{r}^*, \rho)$, where r, r^* are any reds.

The consistency conditions can be coded in an $L_{\omega,\omega}$ theory T having signture the reds with ρ together with all other colours like in [8, Definition 3.6.9]. The theory T is only a first order theory (not an $L_{\omega_1,\omega}$ theory) because the number of greens is finite which is not the case with [8] where the number of available greens are countably infinite coded by an infinite disjunction. One construct an n-homogeneous model M is as a countable limit of finite models of T using a game played between $\exists$ and $\forall$like in [10, Theorem 2.16]. In the rainbow game $\forall$ challenges $\exists$ with *cones* having green *tints* (g_0^i), and $\exists$ wins if she can respond to such moves. This is the only way that $\forall$ can force a win. $\exists$ has to respond by labelling *appexes* of two succesive cones, having the *same base* played by $\forall$. By the rules of the game, she has to use a red label. She resorts to ρ whenever she is forced a red while using the rainbow reds will lead to an inconsistent triangle of reds; [10, Proposition 2.6, Lemma 2.7]. The number of greens make [10, Lemma 3.10] work with the same proof using only finitely many green and not infinitely many. The winning strategy implemented by $\exists$ using the red label ρ that comes to her rescue whenever she runs out of 'rainbow reds', so she can always and consistently respond with an extended coloured graph.

We denote the resulting term CA_n, $\mathfrak{TmAt}$ by $\mathfrak{Bb}(\mathfrak{A}_{n+1,n}, \mathsf{r}, \omega)$ short hand for blowing up and blurring $\mathfrak{A}_{n+1,n}$ by splitting each *red graph (atom)* into ω many. It can be shown exactly like in [10] that $\exists$ can win the rainbow ω–rounded game and build an n–homogeneous model M by using a shade of red ρ *outside* the rainbow signature, when she is forced a red; [10, Proposition 2.6, Lemma 2.7]. The n-homogeniuty entails that any subgraph (substructure) of M of size $\leq n$, is independent of its location in M; it is uniquely determined by its isomorphism type.

In the present context, after the splitting 'the finitely many red colours' replacing each such red colour r_{kl}, $k < l < n$ by ω many r^i_{kl}, $i \in \omega$, the rainbow signature for the resulting rainbow theory as defined in [7, Definition 3.6.9] call this theory T_{ra}, consists of $\mathsf{g}_i : 1 \leq i < n-1$, $\mathsf{g}^i_0 : 1 \leq i \leq n+1$, $\mathsf{w}_i : i < n-1$, $\mathsf{r}^t_{kl} : k < l < n$, $t \in \omega$, binary relations, and $n-1$ ary relations y_S, $S \subseteq_\omega n+k-2$ or $S = n+1$. The set algebra $\mathfrak{Bb}(\mathfrak{A}_{n+1,n}, \mathsf{r}, \omega)$ of dimension n has base an n–homogeneous model M of another theory T whose signature expands that of T_{ra} by an additional binary relation (a shade of red) ρ. In this new signature T is obtained from T_{ra} by some axioms (consistency conditions) extending T_{ra}. Such axioms (consistency conditions) specify consistent triples involving ρ. We call the models of T *extended* coloured graphs. In particular, M is an extended coloured graph.

To build M, the class of coloured graphs is considered in the signature $L \cup \{\rho\}$ like in uual rainbow constructions as given above with the two additional forbidden triples (r, ρ, ρ) and $(\mathsf{r}, \mathsf{r}^*, \rho)$, where r, r^* are any reds. Let $\mathfrak{GG}$ be the class of all models of this *extended rainbow first order theory.* The extra shade of red ρ will be used as a label. This model M is constructed as a countable limit of finite models of T using a game played between $\exists$ and $\forall$. Here, unlike the extended $L_{\omega_1,\omega}$ theory dealt with in [10], T is a *first order one* because the number of greens used are finite.

In the rainbow game [6, 7] $\forall$ challenges $\exists$ with *cones* having green *tints* (g^i_0), and $\exists$ wins if she can respond to such moves. This is the only way that $\forall$ can force a win. $\exists$ has to respond by labelling *appexes* of two succesive cones, having the *same base* played by $\forall$. By the rules of the game, she has to use a red label. She resorts to ρ whenever she is forced a red while using the rainbow reds will lead to an inconsitent triangle of reds; [10, Proposition 2.6, Lemma 2.7]. The winning strategy is implemented by $\exists$ using the red label ρ that comes to her rescue whenever she runs out of 'rainbow reds', so she can always and consistently respond with an extended coloured graphs.

2. **Representing a term algebra (and its completion) as (generalized) set algebras:** Having M at hand, one constructs two atomic n–dimensional set algebras based on M, sharing the same atom structure and having the same top element. The atoms of each will be the set of coloured graphs, seeing as how, quoting Hodkinson [10] such coloured graphs are 'literally indivisible'. Now L_n and $L^n_{\infty,\omega}$ are taken in the rainbow signature (without ρ). Continuing like in *op.cit*, deleting the one available red shade, set $W = \{\bar{a} \in {}^n\mathsf{M} : \mathsf{M} \models (\bigwedge_{i<j<n} \neg\rho(x_i, x_j))(\bar{a})\}$, and for $\phi \in L^n_{\infty,\omega}$, let $\phi^W = \{s \in W : \mathsf{M} \models_W \phi[s]\}$. Here W is the set of all n–ary assignments in ${}^n\mathsf{M}$, that have no edge labelled by ρ and $\models_W$ is first order emantics with quantifiers relativized to W, cf. [10, §3.2 and Definition 4.1]. We note that ρ

is used by $\exists$ infinitely many times during the game forming a 'red clique' in M [10].

Let $\mathfrak{A}$ be the relativized set algebra with domain $\{\varphi^W : \varphi \text{ a first-order } L_n - \text{formula}\}$ and unit W, endowed with the usual concrete cylindric operations read off the connectives. Classical semantics for L_n *rainbow formulas* and their semantics by relativizing to W coincide [10, Proposition 3.13] *but not with respect to* $L^n_{\infty,\omega}$ *rainbow formulas.* Hence the set algebra $\mathfrak{A}$ is isomorphic to a cylinric set algebra of dimension n having top element ${}^n\mathsf{M}$, so $\mathfrak{A}$ is simple, in fact its Df reduct is simple.

Let $\mathfrak{E} = \{\phi^W : \phi \in L^n_{\infty,\omega}\}$ [10, Definition 4.1] with the operations defined like on $\mathfrak{A}$ the usual way. $\mathfrak{CmAt}$ is a complete CA_n and, so like in [10, Lemma 5.3] we have an isomorphism from $\mathfrak{CmAt}$ to $\mathfrak{E}$ defined via $X \mapsto \bigcup X$. Since $\mathsf{At}\mathfrak{A} = \mathsf{At}\mathfrak{Tm}(\mathsf{At}\mathfrak{A})$, which we refer to only by $\mathbf{At}$, and $\mathfrak{Tm}\mathsf{At}\mathfrak{A} \subseteq \mathfrak{A}$, hence $\mathfrak{Tm}\mathsf{At}\mathfrak{A} = \mathfrak{Tm}\mathbf{At}$ is representable. The atoms of $\mathfrak{A}$, $\mathfrak{Tm}\mathsf{At}\mathfrak{A}$ and $\mathfrak{Cm}\mathsf{At}\mathfrak{A} = \mathfrak{Cm}\mathbf{At}$ are the coloured graphs whose edges are *not labelled* by ρ. These atoms are uniquely determined by the interpretion in M of so-called MCA formulas in the rainbow signature of $\mathbf{At}$ as in [10, Definition 4.3].

3. **Embedding $\mathfrak{A}_{n+1,n}$ into $\mathfrak{Cm}(\mathbf{At})$:** Let CRG_f be the class of coloured graphs on $\mathbf{At}_f$ and CRG be the class of coloured graph on $\mathbf{At}$. We can (and will) assume that $\mathsf{CRG}_f \subseteq \mathsf{CRG}$. Write M_a for the atom that is the (equivalence class of the) surjection $a : n \to M$, $M \in \mathsf{CGR}$. Here we identify a with $[a]$; no harm will ensue. We define the (equivalence) relation $\sim$ on $\mathbf{At}$ by $M_b \sim N_a$, $(M, N \in \mathsf{CGR})$:

- $a(i) = a(j) \iff b(i) = b(j)$,
- $M_a(a(i), a(j)) = \mathsf{r}^l \iff N_b(b(i), b(j)) = \mathsf{r}^k$, for some $l, k \in \omega$,
- $M_a(a(i), a(j)) = N_b(b(i), b(j))$, if they are not red,
- $M_a(a(k_0), \ldots, a(k_{n-2})) = N_b(b(k_0), \ldots, b(k_{n-2}))$, whenever defined.

We say that M_a is a *copy of* N_b if $M_a \sim N_b$ (by symmetry N_b is a copy of M_a.) Indeed, the relation 'copy of' is an equivalence relation on $\mathbf{At}$. An atom M_a is called a *red atom*, if M_a has at least one red edge. Any red atom has ω many copies, that are *cylindrically equivalent*, in the sense that, if $N_a \sim M_b$ with one (equivalently both) red, with $a : n \to N$ and $b : n \to M$, then we can assume that $\mathsf{nodes}(N) = \mathsf{nodes}(M)$ and that for all $i < n$, $a \upharpoonright n \sim \{i\} = b \upharpoonright n \sim \{i\}$. In $\mathfrak{Cm}\mathbf{At}$, we write M_a for $\{M_a\}$ and we denote suprema taken in $\mathfrak{Cm}\mathbf{At}$, possibly finite, by $\sum$. Define the map Θ from $\mathfrak{A}_{n+1,n} = \mathfrak{Cm}\mathbf{At}_f$ to $\mathfrak{Cm}\mathbf{At}$, by specifing first its values on $\mathbf{At}_f$, via $M_a \mapsto \sum_j M_a^{(j)}$ where $M_a^{(j)}$ is a copy of M_a. So each atom maps to the suprema of its copies.

This map is well-defined because $\mathfrak{Cm}\mathbf{At}$ is complete. We check that Θ is an injective homomorphim. Injectivity is easy.. We check preservation of all the CA_n extra Boolean operations.

- Diagonal elements. Let $l < k < n$. Then:

$$\begin{aligned}
M_x \leq \Theta(\mathsf{d}_{lk}^{\mathfrak{Cm}\mathbf{At}_f}) &\iff M_x \leq \sum_j \bigcup_{a_l = a_k} M_a^{(j)} \\
&\iff M_x \leq \bigcup_{a_l = a_k} \sum_j M_a^{(j)} \\
&\iff M_x = M_a^{(j)} \text{ for some } a : n \to M \text{ such that } a(l) = a(k) \\
&\iff M_x \in \mathsf{d}_{lk}^{\mathfrak{Cm}\mathbf{At}}.
\end{aligned}$$

- Cylindrifiers. Let $i < n$. By additivity of cylindrifiers, we restrict our attention to atoms $M_a \in \mathbf{At}_f$ with $a : n \to M$, and $M \in \mathsf{CRG}_f \subseteq \mathsf{CRG}$. Then:

$$\Theta(\mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}_f} M_a) = f(\bigcup_{[c]\equiv_i[a]} M_c) = \bigcup_{[c]\equiv_i[a]} \Theta(M_c)$$

$$= \bigcup_{[c]\equiv_i[a]} \sum_j M_c^{(j)} = \sum_j \bigcup_{[c]\equiv_i[a]} M_c^{(j)} = \sum_j \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}} M_a^{(j)}$$

$$= \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}}(\sum_j M_a^{(j)}) = \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}} \Theta(M_a).$$

4. $\forall$ **has a winning strategy in** $G_{n+3}\mathsf{At}(\mathfrak{A}_{n+1,n})$**; and the required result:** It is straightforward to show that $\forall$ has winning strategy first in the Ehrenfeucht–Fraïssé forth private game played between $\exists$ and $\forall$ on the complete irreflexive graphs $n(n-1)/2+2)$ and n in $n(n-1)/2+2$ rounds $\mathsf{EF}_{n(n-1)+2}^{n(n-1)2+2}(n+1,n)$ [8, Definition 16.2] since $n(n-1)/2+2$ is 'longer' than n. Using (any) $p > n$ many pairs of pebbles avalable on the board $\forall$ can win this game in $n+1$ many rounds. For brevity le $\mathfrak{D} = \mathfrak{A}_{n+1,n}$. Now $\forall$ lifts his winning strategy from the last private Ehrenfeucht–Fraïssé forth game to the graph game on $\mathbf{At}_f = \mathsf{At}(\mathfrak{D})$ [6, pp. 841] forcing a win using $n+3$ nodes, i.e in the grapg gmae $\forall$ need two exra nodes by the rainbow theorem [7]. By Lemma 2.4, $\mathfrak{D} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$ when $2 < n < \omega$). Since $\mathfrak{D}$ is finite, then $\mathfrak{D} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+3}$, because $\mathfrak{D}$ coincides with its canonical extension and for any $\mathfrak{D} \in \mathsf{CA}_n$, $\mathfrak{D} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m \implies \mathfrak{D}^+ \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$. But $\mathfrak{D}$ embeds into $\mathfrak{CmAt}\mathfrak{A}$, hence $\mathfrak{CmAt}\mathfrak{A}$ is outside the variety $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+3}$, as well. □

The following definition to be used in the sequel is taken from [2]:

Definition 3.3. [2, Definition 3.1] Let $\mathfrak{R}$ be a relation algebra, with non–identity atoms I and $2 < n < \omega$. Assume that $J \subseteq \wp(I)$ and $E \subseteq {}^3\omega$. We say that (J, E) is a *strong n–blur* for $\mathfrak{R}$ if it (J, E) is an n–blur of R in the sense of [2, Definition 3.1], that is to say J is a complex n blur and E is an index blur such that the complex n–blur satisfies:

$$(\forall V_1, \ldots V_n, W_2, \ldots W_n \in J)(\forall T \in J)(\forall 2 \leq i \leq n)\mathsf{safe}(V_i, W_i, T).$$

The following defintion will be used frequently. Its first encounter is in the second item of theorem 4.4. We use the notation in [2].

Definition 3.4. [2, Definition 3.1] Let $\mathfrak{R}$ be a relation algebra, with non–identity atoms I. Assume that $J \subseteq \wp(I)$ and $E \subseteq {}^3\omega$.

1. We say that (J, E) is an *n–blur* for $\mathfrak{R}$, if J is a *complex n–blur* defined as follows:
   (1) Each element of J is non empty,
   (2) $\bigcup J = I$,
   (3) $(\forall P \in I)(\forall W \in J)(I \subseteq P; W)$,
   (4) $(\forall V_1, \ldots V_n, W_2, \ldots W_n \in J)(\exists T \in J)(\forall 2 \leq i \leq n)\mathsf{safe}(V_i, W_i, T)$, that is there is for $v \in V_i$, $w \in W_i$ and $t \in T$, we have $v; w \leq t$.
   (5) $(\forall P_2, \ldots P_n, Q_2, \ldots Q_n \in I)(\forall W \in J)W \cap P_2; Q_n \cap \ldots P_n; Q_n \neq \emptyset$,

   and the tenary relation E is an *index blur* defined as in item (ii) of [2, Definition 3.1].

2. We say that (J, E) is a *strong n–blur*, if it (J, E) is an n–blur, such that the complex n–blur satisfies $(4)_s$:

$$(\forall V_1, \ldots V_n, W_2, \ldots W_n \in J)(\forall T \in J)(\forall 2 \leq i \leq n)\mathsf{safe}(V_i, W_i, T)$$

Theorem 3.5. *For any $2 < n < l < \omega$, there is an atomic algebra $\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$, but $\mathfrak{B}$ is not strongly representable up to l and ω. In particular, $\mathfrak{CmAt}\mathfrak{B} \notin \mathsf{RCA}_n$, $\mathfrak{B}$ is not completely representable a fortiori $\mathfrak{B}$ is not strongly representable.*

Proof. Let $2 < n < m \leq \omega$. First we prove the conditionally the non–atom canonicity of $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$ depending on the existence of certain finite relation algebras $\mathfrak{R}$ with strong m blur-satisfying a condition that we highlight as we go along. We use the flexible blow up and blur construction used in [2]. The idea is to use $\mathfrak{R}$ in place of the finite Maddux algebras denoted by $\mathfrak{E}_k(2,3)$ on [2, p.83]. Here $k(<\omega)$ is the number of non–identity atoms and then take it from there to reach the conditions, we move backwards if you like. The required algebra witnessing non–atom canonicity will be obtained by blowing up and blurring $\mathfrak{R}$ in place of the relation algebra $\mathfrak{E}_k(2,3)$ [2].

Our exposition addresses an (abstract) finite relation algebra $\mathfrak{R}$ having an l–blur in the sense of definition [2, Definition 3.1], with $3 \leq l \leq k < \omega$ and k depending on l. Occasionally we use the concrete Maddux algebra $\mathfrak{E}_k(2,3)$ to make certain concepts more tangible. We use the notation in [2]. Let $2 < n \leq l < \omega$. One starts with a finite relation algebra $\mathfrak{R}$ that has only representations, if any, on finite sets (bases), having an l–blur (J, E) as in [2, Definition 3.1] recalled in definition 3.4. After *blowing up and bluring* $\mathfrak{R}$, by splitting each of its atoms into infinitely many, one gets an infinite atomic representable relation algebra $\mathsf{Bb}(\mathfrak{R}, J, E)$ [2, p.73], whose atom structure $\mathbf{At}$ is weakly but not strongly representable. The atom structure $\mathbf{At}$ is not strongly representable, because $\mathfrak{R}$ is *not blurred* in $\mathsf{Cm}\mathbf{At}$. The finite relation algebra $\mathfrak{R}$ embeds into $\mathfrak{Cm}\mathbf{At}$, so that a representation of $\mathfrak{Cm}\mathbf{At}$, necessarily on an infinite base, induces one of $\mathfrak{R}$ on the same base, which is impossible. The representability of $\mathsf{Bb}(\mathfrak{R}, J, E)$ depend on the properties of the l–blur, which *blurs* $\mathfrak{R}$ *in* $\mathsf{Bb}(\mathfrak{R}, J, E)$. The set of blurs here, namely, J is finite. In the case of $\mathfrak{E}_k(2,3)$ used in [2], the set of blurs is the set of all subsets of non–identity atoms having the same size $l < \omega$, where $k = f(l) \geq l$ for some recursive function f from $\omega \to \omega$, so that k depends recursively on l.

One (but not the only) way to define the *index blur* $E \subseteq {}^3\omega$ is as follows [14, Theorem 3.1.1]: $E(i,j,k) \iff (\exists p,q,r)(\{p,q,r\} = \{i,j,k\}$ and $r - q = q - p$. This is a concrete instance of an index blur as defined in [2, Definition 3.1(iii)] (recalled in definition 3.4 above), but defined uniformly, it does not depends on the blurs. The underlying set of $\mathbf{At}$, the atom structure of $\mathsf{Bb}(\mathfrak{R}, J, E)$ is the following set consisting of triplets: $At = \{(i, P, W) : i \in \omega, P \in \mathsf{At}\mathfrak{R} \sim \{\mathsf{Id}\}, W \in J\} \cup \{\mathsf{Id}\}$. When $\mathfrak{R} = \mathfrak{E}_k(2,3)$ (some finite $k > 0$), composition is defined by singling out the following (together with their Peircian transforms), as the consistent triples: (a,b,c) is consistent $\iff$ one of a,b,c is Id and the other two are equal, or if $a = (i,P,S), b = (j,Q,Z), c = (k,R,W)$

$$S \cap Z \cap W \neq \emptyset \implies E(i,j,k)\&|\{P,Q,R\}| \neq 1.$$

(We are avoiding mononchromatic triangles). That is if for $W \in J$, $E^W = \{(i,P,W) : i \in \omega, P \in W\}$, then

$$(i,P,S);(j,Q,Z) = \bigcup\{E^W : S \cap Z \cap W = \emptyset\}$$

$$\bigcup\{(k,R,W) : E(i,j,k), |\{P,Q,R\}| \neq 1\}.$$

More generally, for the $\mathfrak{R}$ as postulated in the hypothesis, composition in $\mathbf{At}$ is defined as follow. First the index blur E can be taken to be like above. Now the triple $((i,P,S),(j,Q,Z),(k,R,W))$ in which no two entries are equal, is consistent if either S,Z,W are *safe*, briefly $\mathsf{safe}(S,Z,W)$, witness item (4) in definition 3.4 (which vacuously hold oif $S\cap Z\cap W=\emptyset$), or $E(i,j,k)$ and $P;Q\leq R$ in $\mathfrak{R}$. This generalizes the above definition of composition, because in $\mathfrak{E}_k(2,3)$, the triple of non–identity atoms (P,Q,R) is consistent $\iff$ they do not have the same colour $\iff$ $|\{P,Q,R\}|\neq 1$. Having specified its atom structure, its timely to specfiy the relation algebra $\mathsf{Bb}(\mathfrak{R},J,E)\subseteq\mathfrak{Cm}\mathbf{At}$. The relation algebra $\mathsf{Bb}(\mathfrak{R},J,E)$ is $\mathfrak{Tm}\mathbf{At}$ (the term algebra). Its universe is the set $\{X\subseteq H\cup\{\mathsf{Id}\}: X\cap E^W\in \mathsf{Cof}(E^W),\text{ for all }W\in J\}$, where $\mathsf{Cof}(E^W)$ denotes the set of co–finite subsets of E^W, that is subsets of E^W whose complement is infinite, with E^W as defined above. The relation algebra operations lifted from $\mathbf{At}$ the usual way. The algebra $\mathsf{Bb}(\mathfrak{R},J,E)$ is proved to be representable [2].

For brevity, denote $\mathsf{Bb}(\mathfrak{R},J,E)$ by $\mathcal{R}$, and its domain by R. For $a\in\mathbf{At}$, and $W\in J$, set $U^a=\{X\in R: a\in X\}$ and $U^W=\{X\in R:|X\cap E^W|\geq\omega\}$. Then the principal ultrafilters of $\mathcal{R}$ are exactly U^a, $a\in H$ and U^W are non-principal ultrafilters for $W\in J$ when E^W is infinite. Let $J'=\{W\in J:|E^W|\geq\omega\}$, and let $\mathsf{Uf}=\{U^a: a\in F\}\cup\{U^W: W\in J'\}$. Uf is the set of ultrafilters of $\mathcal{R}$ which is used as colours to represent $\mathcal{R}$, cf. [2, pp. 75-77]. The representation is built from coloured graphs whose edges are labelled by elements in Uf in a fairly standard step–by–step construction. The step–by–step construction builds in the way coloured graphs, which are basically networks whose edges are labelled by ultrafilters, with non–principal ultrafilters allowed. So such coloured graphs are networks that are *not atomic* because not only principal ultrafilters are allowed as labels. Furthermore, we *cannot restrict our attension to only atomic networks* because we do not want $\mathsf{Bb}(\mathfrak{R},J,E)$ to be strongly representable, least completely representable. The 'limit' of a sequence of atomic networks constructed in a step-by–step manner, or obtained via winning strategy strategy for $\exists$ in an ω–rounded atomic game, will necessarily produce a complete representation of $\mathsf{Bb}(\mathfrak{R},J,E)$. But the required representation will be extracted from *a complete representation* of the *canonical extension of* $\mathsf{Bb}(\mathfrak{R},J,E)$. Nothing wrong with that. A relation algebra $\mathcal{R}$ is representable $\iff$ its canonical extension is representable. A complete representation of the canonical extension of $\mathcal{R}$ induces a representation of $\mathcal{R}$, because $\mathcal{R}$ embeds into its a canonical extension, but the converse is not necessarily true. So here we are proving more than the mere representablity of $\mathfrak{Bb}(\mathfrak{R},J,E)$, because we are constructing a complete representation of its canonical extension, namely, the algebra $\mathfrak{Cm}\mathbf{Uf}$, where $\mathbf{Uf}$ is the atom structure having domain Uf, with Uf as defined above.

Now we show why the Dedekind-MacNeille completion $\mathfrak{Cm}\mathbf{At}$ is *not* representable. For $P\in I$, let $H^P=\{(i,P,W): i\in\omega, W\in J, P\in W\}$. Let $P_1=\{H^P: P\in I\}$ and $P_2=\{E^W: W\in J\}$. These are two partitions of At. The partition P_2 was used to *represent*, $\mathsf{Bb}(\mathfrak{R},J,E)$, in the sense that the tenary relation corresponding to composition was defined on $\mathbf{At}$, in a such a way so that the singletons generate the partition $(E^W: W\in J)$ up to "finite deviations." The partition P_1 will now be used to show that $\mathfrak{Cm}(\mathsf{Bb}(\mathfrak{R},J,E))=\mathfrak{Cm}(\mathbf{At})$ is *not* representable. This follows by observing that omposition restricted to P_1 satisfies: $H^P;H^Q=\bigcup\{H^Z: Z;P\leq Q\text{ in }\mathfrak{R}\}$ which means that $\mathfrak{R}$ embeds into the complex algebra $\mathfrak{Cm}\mathbf{At}$ prohibiting its representability, because $\mathfrak{R}$ allows only representations having a finite base.

The construction lifts to higher dimensions expressed in CA_ns, $2<n<\omega$. Because (J,E) is an l–blur, then by [2, Theorem 3.2 9(iii)], $\mathbf{At}_{ca}=\mathsf{Mat}_l(\mathsf{AtBb}(\mathfrak{R},J,E))$, the set of l by l basic matrices on $\mathbf{At}$ is an l–dimensional cylindric basis, giving an algebra $\mathfrak{B}_l=\mathsf{Bb}_l(\mathfrak{R},J,E)\in$

RCA_l. Again $\mathbf{At}_{ca}$ is not strongly representable, for had it been then a representation of $\mathfrak{Cm}\mathbf{At}_{ca}$, induces a representation of $\mathfrak{R}$ on an infinite base, because $\mathsf{Ra}\mathfrak{Cm}\mathbf{At}_{ca} \supseteq \mathfrak{Cm}\mathbf{At} \supseteq \mathfrak{R}$, and the representability of $\mathfrak{Cm}\mathbf{At}_{ca}$ induces one of $\mathsf{Ra}\mathfrak{Cm}\mathbf{At}_{ca}$, necessarily having an infinite base. For $2 < n \leq l < \omega$, denote by $\mathfrak{C}_l$ the non-representable Dedekind-MacNeille completion of the algebra $\mathsf{Bb}_l(\mathfrak{R}, J, E) \in \mathsf{RCA}_l$, that is $\mathfrak{C}_l = \mathfrak{Cm}\mathsf{At}(\mathsf{Bb}_l(\mathfrak{R}, J, E)) = \mathfrak{Cm}\mathsf{Mat}_l(\mathbf{At})$. If the l–blur happens to be *strong*, in the sense of definition 3.4 and $n \leq m \leq l$, then we get by [2, item (3) pp.80], that $\mathsf{Bb}_m(\mathfrak{R}, J, E) \cong \mathsf{Nr}_m\mathsf{Bb}_l(\mathfrak{R}, J, E)$. This is proved by defining an embedding $h : \mathfrak{Rd}_m\mathfrak{C}_l \to \mathfrak{C}_m$ via $x \mapsto \{M \upharpoonright m : M \in x\}$ and showing that $h \upharpoonright \mathsf{Nr}_m\mathfrak{C}_l$ is an isomorphism onto $\mathfrak{C}_m$ [2, p.80]. Surjectiveness uses the condition $(J5)_l$ formulated in the second item of definition 3.4 of strong l-blurness. Without this condition, that is if the l–blur (J, E) is not strong, then still $\mathfrak{C}_m$ and $\mathfrak{C}_l$ can be defined because by definition (J, E) is an t–blur for all $m \leq t \leq l$, so $\mathsf{Mat_t}(\mathbf{At})$ is a cylindric basis and for $t < l$ $\mathfrak{C}_t$ embeds into $\mathsf{Nr}_m\mathfrak{C}_l$ using the same above map, but this embedding might not be surjective. So for every l, now replacing $\mathfrak{R}$ by the Maddux algebra $\mathfrak{E}_{f(l)}(2, 3)$, the algebra $\mathfrak{A}_l = \mathsf{Nr}_n\mathsf{Bb}_l(\mathfrak{E}_{f(l)}(2, 3)), J, E)$– with $f(l)$ depending recursively on l, having strong l-blur due to the properties of the Maddux algebra $\mathfrak{E}_{f(l)}(2, 3)$, is as required. In other words, and more concisely, we have $\mathfrak{A}_l \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$, but $\mathfrak{Cm}\mathsf{At}\mathfrak{A}_l \notin \mathsf{RCA}_n$. □

The flexibility of the construction in *op.cit* allows one to refine the main result in [2] by varying the relation algebra $\mathfrak{R}$. All we need for the construction to work is that $\mathfrak{R}$ is finite having a (strong) l–blur with $n \leq l < \omega$. So one can get sharper results if one requires for example that $\mathfrak{R}$ has no infinite k–dimensional hyperbasis with $n \leq l < k \leq \omega$, k possibly finite, equivalently, $\mathfrak{R}$ does not have a k–flat infinite representation. The equivalence here is due to the fact that $\mathfrak{R}$ is finite. It cannot be the case that $l \geq k$ $(k \in \omega)$, for else $\mathfrak{A} = \mathsf{Bb}_n(\mathfrak{R}, J, E) \cong \mathsf{Nr}_n\mathsf{Bb}_l(\mathfrak{R}, J, E)$, and $\mathsf{Bb}_l(\mathfrak{R}, J, E)$ is atomic (and finite dimensional), so by lemma 4.2, $\mathfrak{A}$ will have a complete l, hence a complete k–flat representation, which is impossible because $\mathfrak{R}$ does not have an infinite k-flat represenation. Such requirements lead to negative results on atom–canonicity completely analogous to the result proved in the previous two subitems (a) and (b) of the present item, and possibly more of a kind.

Theorem 3.6. *There is an algebra $\mathfrak{E} \in \mathsf{RCA}_n$ that has the complex $\mathbf{I}$ neat embedding property up to m for any $m \geq n$ but does not have the $\mathbf{I}$ neat embedding property up to $n+1$, a fortior the atomic algebra $\mathfrak{E}$ has the complex neat embeding property up to $m \geq n+1$, but does bot have the $\mathbf{I}$ neat embedding property for any $m \geq n+1$.*

Let α be any ordinal and let $\mathfrak{F}$ is field of characteristic 0. Let $V = \{s \in {}^\alpha\mathfrak{F} : |\{i \in \alpha : s_i \neq 0\}| < \omega\}$. Note that $V$is a vector space over the field $\mathfrak{F}$. Let

$$\mathfrak{C} = \langle \wp(V), \cup, \cap, \setminus, \emptyset, V, \mathsf{c}_i, \mathsf{d}_{ij} \rangle_{i,j \in \alpha}.$$

Let y denote the following α-ary relation:

$$y = \{s \in V : s_0 + 1 = \sum_{i>0} s_i\}.$$

and

$$w = \{s \in V : s_1 + 1 = \sum_{i \neq 1} s_i\}.$$

For each $s \in y$ we let y_s be the singleton containing s, i.e. $y_s = \{s\}$. Let

$$\mathfrak{E} = \mathfrak{Sg}^{\mathfrak{C}}(\{y, y_s : s \in y\})$$

$$\mathfrak{A} = \mathfrak{Sg}^{\mathfrak{C}}(\{y, w, y_s : s \in y\}).$$

Clearly $\mathfrak{C}$ and $\mathfrak{A}$ are in RCA_α. We first show that $w \notin E$, then we show that if $\mathfrak{C} \in \mathsf{Nr}_n\mathsf{CA}_{\alpha+1}$ then $w \in E$ concluding that $\mathfrak{C} \notin \mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+1}$. Let

$$Pl = \{\{s \in {}^\alpha\mathfrak{F}^{(\mathbf{0})} : t + \sum(r_i s_i) = 0\} : \{t, r_i : i < \alpha\} \subseteq \mathfrak{F}\}.$$

$$Pl^< = \{p \in Pl : \mathsf{c}_i p = p, \text{ for some } i < \alpha\}.$$

Note that for $p \in Pl$, $p = \{s \in {}^\alpha\mathfrak{F}^{(\mathbf{0})} : t + \sum_i r_i s_i = 0\}$ say, then $\mathsf{c}_i p = p$ (i.e. p is parallel to the i-th axis) iff $r_i = 0$. Note too, that

$$\{y, w, \mathsf{d}_{ij} : i, j \in \alpha\} \subseteq Pl.$$

$$y, w \notin Pl^<, 1 \in Pl^<$$

and

$$\{\mathsf{d}_{ij} : i \neq j, i, j \in \alpha\} \subseteq Pl^< \text{ iff } \alpha \geq 3.$$

Now let

$$G = \{y, -y,\ p, -p,\ \mathsf{c}_{(\Delta)}\{\mathbf{0}\}, -\mathsf{c}_{(\Delta)}\{\mathbf{0}\} : p \in Pl^< \cup \{\mathsf{d}_{01}\}\ \Delta \subseteq_\omega \alpha,\ 0 \in \Delta\}.$$

$$G^* = \{\bigcap_{i \in n} g_i : n \in \omega, g_i \in G\}.$$

and

$$G^{**} = \{\bigcup_{i \in n} g_i : n \in \omega, g_i \in G^*\}.$$

It is easy to see that $\{y, y_s : s \in y\} \subseteq G^{**}$, and G^{**} is a boolean field of sets. We prove that $w \notin G^{**}$ and that G^{**} is closed under cylindrifications. To this end, we set:

$$L = \{p \in Pl^< : \mathsf{c}_0 p \neq p\} \text{ and } P(0) = L \cup \{\mathsf{d}_{01}\}.$$

Next we define

$$G_1 = \{g \in G^* : g \subseteq y\}$$

and

$$G_2 = \{g \in G^* : g \not\subseteq y \text{ and } g \subseteq p, \text{ for some } p \in P(0)\}.$$

We have $G_1 \cap G_2 = \emptyset$. Now let

$$G_3 = \{p_1 \cap p_2 \ldots \cap p_k :\ k \in \omega,\ \{p_1, p_2, \ldots, p_k\} \subseteq G \smallsetminus (\{y\} \cup P(0))\}.$$

It is easy to see that $G^* = G_1 \cup G_2 \cup G_3$. To prove that $w \notin G^{**}$ we need: If $g \in G_3$ and $0 \neq g$, then $g \not\subset w$. But this follows from the following. Assume that $g = p_1 \cap p_2 \ldots \cap p_k$ say, with $p_i \in G$ and $p_i \notin (\{y\} \cup P(0))$ for $1 \leq i \leq k$, and let $z \in g$. Let [] be the function from Pl into $\mathfrak{F}$ defined as follows:

$$[p] = \{1/r_0(-t - \sum r_i z_i)\} \text{ if } p = -\{s \in^\alpha \mathfrak{F}^{(\mathbf{0})} : t + \sum r_i s_i = 0\}, r_0 \neq 0$$

and else

$$[p] = 0.$$

Let

$$r \in \mathfrak{F} \smallsetminus ((\bigcup_{1\leq i\leq k} [p_i]) \cup [-w])$$

be arbitrary, and let

$$z_r^0 = z \smallsetminus \{(0, z_0)\} \cup \{(0, r)\}.$$

Then

$$z_r^0 \in g \smallsetminus w \text{ i.e. } g \not\subseteq w.$$

(Here we are using that when $\mathsf{c}_{(\Delta)}\{\mathbf{0}\} \in G$, then $0 \in \Delta$.) We now proceed to show that $w \notin G^{**}$. Assume that

$$x = \bigcup\{g_i^1 : i < n_1\} \cup \bigcup\{g_i^2 : i < n_2\} \cup \bigcup\{g_i^3 : i < n_3\}$$

where

$$\{g_i^j : i < n_j\} \subseteq G_j \text{ and } g_i^j \subseteq w \text{ for all } j \in \{1, 2, 3\}.$$

We show that $x \neq w$. By the above, we have $x \subseteq \bigcup_{i<n} p_i$ for some $\{p_i : i < n\} \subseteq P(0)$. Note that if $\alpha > 2$ then $P(0) = L$ and $P(0) = L \cup \{\mathsf{d}_{01}\}$ otherwise. If $\alpha = 2$ then $w \subseteq -\mathsf{d}_{01}$ otherwise $P(0) = L$. Now it is enough to show that w is not contained in $\bigcup E$ for any finite $E \subseteq L$. But it can be seen by implementing easy linear algebraic arguments that, for every $n \in \omega$, and for every system

$$t_0 + \sum(r_{0i}x_i) = 0$$

$$.$$

$$.$$

$$t_n + \sum(r_{ni}x_i) = 0,$$

of equations, such that for all $j \leq n$, there exists $i < \alpha$, such that

$$r_{ji} = 0 \text{ and } r_{j0} \neq 0,$$

the equation

$$\sum_{i<\alpha} x_i = 2x_1 + 1$$

has a solution s in the weak space ${}^{\alpha}\mathfrak{F}^{(\mathbf{0})}$, such that s is not a solution of

$$t_j + \sum_{i<\alpha}(r_{ji}x_i) = 0,$$

for every $j \leq n$. We have proved that $w \notin G^{**}$. To show that $w \notin A$, we will show that G^{**} is closed under the cylindric operations (i.e it is the universe of a CA_α. It is enough to show that (since the c_i's are additive), that for $j \in \alpha$ and $g \in G^*$ arbitrary, we have $\mathsf{c}_j g \in G^{**}$. For this purpose, put for every $p \in Pl$

$$p(j|0) = \mathsf{c}_j\{s \in p : s_j = 0\} \text{ and } (-p)(j|0) = -p(j|0).$$

Then it is not hard to see that

$$p(j|0) = \{s \in {}^{\alpha}\mathfrak{F}^{(\mathbf{0})} : t + \sum_{i\neq j}(r_i s_i) = 0\},$$

if

$$p = \{s \in {}^{\alpha}\mathfrak{F}^{\mathbf{0})} : t + \sum_{i<\alpha}(r_i s_i) = 0\},$$

and so

$$p(j|0) \in Pl^{<} \text{ for every } p \in Pl.$$

Let j and g be as indicated above. We can assume that

$$g = e \cap p_1 \cap \ldots \cap p_n \cap -P_1 \ldots \cap -P_m \cap z$$

$$\cap - \mathsf{c}_{(\Delta_1)}\{\mathbf{0}\} \ldots \cap -\mathsf{c}_{(\Delta_N)}\{\mathbf{0}\},$$

where

$$e \in \{y, -y, 1\}$$

$$n, m, N \in \omega \smallsetminus \{0\}, p_i, P_i \in Pl^{<} \cup \{\mathsf{d}_{01}\},$$

$$\mathsf{c}_j p_i \neq p_i,\ \ \mathsf{c}_j P_i \neq P_i,$$

$$z \in \{\mathsf{c}_{(\Delta)}\{\mathbf{0}\}, 1 : \Delta \in \wp_\omega \alpha,\ \ 0 \in \Delta,\ \ j \notin \Delta\},$$

and

$$\{\Delta_1, \ldots, \Delta_n\} \subseteq \{x \in \wp_\omega \alpha :\ \ j \notin x, 0 \in x\}.$$

We distinguish between 2 cases:

Case 1.

$$z = \mathsf{c}_{(\Delta)}\{\mathbf{0}\} \text{ and } j \notin \Delta.$$

Then

$$\mathsf{c}_j(e \cap p_1 \ldots \cap p_n \cap -P_1 \ldots \cap -P_m$$

$$\cap \mathsf{c}_{(\Delta)}\{\mathbf{0}\} \cap -\mathsf{c}_{(\Delta_1)}\{\mathbf{0}\} \ldots \cap -\mathsf{c}_{(\Delta_N)}\{\mathbf{0}\})$$

$$p_1(j|0) \cap \ldots p_n(j|0) \cap -P_1(j|0) \ldots \cap -P_m(j|0)$$

$$\cap \mathsf{c}_j \mathsf{c}_{(\Delta)}\{\mathbf{0}\} \cap -\mathsf{c}_j \mathsf{c}_{(\Delta_1)}\{\mathbf{0}\} \cap \cap - \mathsf{c}_j \mathsf{c}_{(\Delta_N)}\{\mathbf{0}\}.$$

Case 2.

$$z = 1$$

Then

$$\mathsf{c}_j(e.p_1 \cap \ldots \cap p_n \cap -P_1 \ldots \cap -P_m$$

$$\cap - \mathsf{c}_{(\Delta_1)}\{\mathbf{0}\} \ldots \cap -\mathsf{c}_{(\Delta_N)}\{\mathbf{0}\})$$

$$= f(e) \cap_{k \leq n} ((\cap_{i \leq n} \mathsf{c}_j(p_k \cap p_i) \cap \cap_{i \leq m} \mathsf{c}_j(p_k - P_i)$$

$$\cap_{i \leq N} \mathsf{c}_j(p_k - \mathsf{c}_{(\Delta_i)}\{\mathbf{0}\})).$$

where

$$f(y) = ((\cap_{i \leq n} \mathsf{c}_j(y \cap p_i) \cap \cap_{i \leq m} \mathsf{c}_j(y - P_i)$$

$$\cap_{i \leq N} \mathsf{c}_j(y - \mathsf{c}_{(\Delta_i)}\{\mathbf{0}\})).$$

$$f(-y) = \cap_{k \leq n} \mathsf{c}_j(p_k - y)$$

$$f(1) = 1.$$

Now for every $p, q \in Pl$, there are p', q', p'' and $q'' \in Pl^{<}$ such that

$$\mathsf{c}_j(p \cap q) = p' \cap q',$$

$$\mathsf{c}_j(p \smallsetminus q) = p'' \smallsetminus q''$$

and if $j \in \Delta p \smallsetminus \Gamma$, then

$$\mathsf{c}_j(p \setminus \mathsf{c}_{(\Gamma)}\{\mathbf{0}\}) = {}^{\alpha}\mathfrak{F}^{(\mathbf{0})} \smallsetminus p(j|0) \cup (p(j|0) \smallsetminus \mathsf{c}_j\mathsf{c}_{(\Gamma)}\{\mathbf{0}\}).$$

We have proved that $w \notin \mathfrak{C}$. Now we restrict α to be finite > 1 and according to the widespread custom of naming ordinals, we call it n. Let $\mathfrak{B}$ be the full set algebra with unit $V = {}^n\mathbb{Q}$. It is straightforwrad to show that $\mathsf{At}\mathfrak{B} = \mathsf{At}\mathfrak{C} = \{\{s\} : s \in V\}$, that is to say, the atoms of both algebras coincide with the singletons. Clearly $\mathfrak{Cm}\mathsf{At}\mathfrak{C} = \mathfrak{B}$, so that infcat $\mathfrak{B}$ is the Dedekind-MacNeille completion of $\mathfrak{C}$. Since $\mathfrak{B}$ is a full set algebra having top ement ${}^n\mathbb{Q}$ and universe $\wp({}^n\mathbb{Q})$, then $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$. So $\mathfrak{C}$ is an algebra that *has the complex* $\mathbf{I}$ *neat embedding propery up to* ω, but $\mathfrak{C}$ does not have the neat embeding property up to $n + 1$, since $\mathfrak{C} \notin \mathsf{Nr}_n\mathsf{CA}_{n+1}$.

Let $2 < n \leq l \leq m \leq \omega$. Denote the class of CA_ns having the complex $\mathbf{O}$ neat embedding property up to m by $\mathsf{CNPCA}^{\mathbf{O}}_{n,m}$, and let $\mathsf{RCA}^{\mathbf{O}}_{n,m} := \mathsf{CNPCA}^{\mathbf{O}}_{n,m} \cap \mathsf{RCA}_n$. Denote the class of strongly representable CA_ns up to l and m by $\mathsf{RCA}^{l,m}_n$. Call an algebra $\mathfrak{A} \in \mathsf{CA}_n$ *strongly representable* if $\mathfrak{A}$ is atomic and $\mathsf{At}\mathfrak{A}$ is strongly representable; that is $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathsf{RCA}_n$. Observe that $\mathsf{RCA}^{n,m}_n = \mathsf{RCA}^{\mathbf{S}}_{n,m}$ and that when $m = \omega$ both classes coincide with the class of strongly representable CA_ns. For a class $\mathbf{K}$ of BAOs, $\mathbf{K} \cap \mathsf{Count}$ denotes the class of countable algebras in $\mathbf{K}$ and (recall that) $\mathbf{K} \cap \mathbf{At}$ denotes the class of atomic algebras in $\mathbf{K}$.

Theorem 3.7. *Let* $2 < n \leq l < m \leq \omega$ *and* $\mathbf{O} \in \{\mathbf{S}, \mathbf{S_c}, \mathbf{S_d}, \mathbf{I}\}$. *Then the following hold:*

1. $\mathsf{RCA}^{\mathbf{O}}_{n,m} \subseteq \mathsf{RCA}^{\mathbf{O}}_{n,l}$ *and* $\mathsf{RCA}^{\mathbf{I}}_{n,l} \subseteq \mathsf{RCA}^{\mathbf{S}_d}_{n,l} \subseteq \mathsf{RCA}^{\mathbf{S}_c}_{n,l} \subseteq \mathsf{RCA}^{\mathbf{S}}_{n,l}$. *The last inclusion is proper for* $l \geq n + 3$,
2. *For* $\mathbf{O} \in \{\mathbf{S}, \mathbf{S_c}, \mathbf{S_d}\}$, $\mathsf{CNPCA}^{\mathbf{O}}_{n,l} \subseteq \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l$ *(that is the complex* $\mathbf{O}$ *neat embedding property is stronger than the* $\mathbf{O}$ *neat embedding property), and for* $\mathbf{O} = \mathbf{S}$, *the inclusion is proper for* $l \geq n + 3$. *But for* $\mathbf{O} = \mathbf{I}$, $\mathsf{CNPCA}^{\mathbf{I}}_{n,l} \nsubseteq \mathsf{Nr}_n\mathsf{CA}_l$ *(so the complex* $\mathbf{I}$ *neat embedding property does not imply the* $\mathbf{I}$ *neat embedding property),*
3. *If* $\mathfrak{A}$ *is finite, then* $\mathfrak{A} \in \mathsf{CNPCA}^{\mathbf{O}}_{n,l} \iff \mathfrak{A} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l$ *and* $\mathfrak{A} \in \mathsf{RCA}^{\mathbf{O}}_{n,l} \iff \mathfrak{A} \in \mathsf{RCA}_n \cap \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l$. *Furthermore, for any positive* k, $\mathsf{CNPCA}^{\mathbf{O}}_{n,n+k+1} \subsetneq \mathsf{CNPCA}^{\mathbf{O}}_{n,n+k}$, *and finally* $\mathsf{CNPCA}^{\mathbf{O}}_{n,\omega} \subsetneq \mathsf{RCA}_n$,
4. $(\exists \mathfrak{A} \in \mathsf{RCA}_n \cap \mathbf{At} \sim \mathsf{CNPCA}^{\mathbf{S}}_{n,l}) \implies \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_k$ *is not atom–canonical for all* $k \geq l$. *In particular,* $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_k$ *is not atom–canonical for all* $k \geq n + 3$,
5. *If* $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$ *is atom–canonical, then* $\mathsf{RCA}^{\mathbf{S}}_{n,l}$ *is first order definable. There exists a finite* $k > n + 1$, *such that* $\mathsf{RCA}^{\mathbf{S}}_{n,k}$ *is not first order definable.*
6. *Let* $2 < n < l \leq \omega$. *Then* $\mathsf{RCA}^{l,\omega}_n \cap \mathsf{Count} \neq \emptyset \iff l < \omega$.

Proof. 1. The inclusions in in the first item is by definiton. To show the strictness of the last inclusion, use (**1**) of Lemma 5.4.

2. Let $\mathbf{O} \in \{\mathbf{S}, \mathbf{S_c}, \mathbf{S_d}\}$. If $\mathfrak{CmAt}\mathfrak{A} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l$, then $\mathfrak{A} \subseteq_d \mathfrak{CmAt}\mathfrak{A}$, so $\mathfrak{A} \in \mathbf{S}_d\mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l \subseteq \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l$. This proves the first part. The strictness of the last inclusion follows Theorem 3.2, since the atomic countable algebra $\mathfrak{A}$ constructed in *op.cit* is in RCA_n, but $\mathfrak{CmAt}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$ for any $l \geq n+3$. For the last non–inclusion in item (2), we use the set algebras $\mathfrak{A}$ and $\mathfrak{E}$ in Theorem 3.6.

3. Follows by definition observing that if $\mathfrak{A}$ is finite then $\mathfrak{A} = \mathfrak{CmAt}\mathfrak{A}$. The strictness of the first inclusion follows from the construction in [9] where it shown that for any positive k, there is a *finite algebra* $\mathfrak{A}$ in $\mathsf{Nr}_n\mathsf{CA}_{n+k} \sim \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k+1}$. The inclusion $\mathsf{CNPCA}^{\mathbf{O}}_{n,\omega} \subseteq \mathsf{RCA}_n$ holds because if $\mathfrak{B} \in \mathsf{CNPCA}^{\mathbf{O}}_{n,\omega}$, then $\mathfrak{B} \subseteq \mathfrak{CmAt}\mathfrak{B} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathsf{RCA}_n$. The $\mathfrak{A}$ used in the last item of theorem 3.2 witnesses the strictness of the last inclusion proving the last required in this item.

4. Follows from the definition and the construction used in Theorem 3.2.

5. Follows from that $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$ is canonical. So if it is atom–canonical too, then $\mathsf{At}(\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l) = \{\mathfrak{F} : \mathfrak{Cm}\mathfrak{F} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l\}$, the former class is elementary [7, Theorem 2.84], and the last class is elementray $\iff$ $\mathsf{RCA}^{\mathbf{S}}_{n,l}$ is elementary. Non–elementarity follows from [8, Corollary 3.7.2] where it is proved that $\mathsf{RCA}^{\mathbf{S}}_{n,\omega}$ is not elementary, together with the fact that $\bigcap_{n<k<\omega} \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_k = \mathsf{RCA}_n$. In more detail, let $\mathfrak{A}_i$ be the sequence of strongly representable CA_ns with $\mathfrak{CmAt}\mathfrak{A}_i = \mathfrak{A}_i$ and $\mathfrak{A} = \Pi_{i/U}\mathfrak{A}_i$ is not strongly representable. Hence $\mathfrak{CmAt}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_\omega = \bigcap_{i\in\omega} \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+i}$, so $\mathfrak{CmAt}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{K}_l$ for all $l > k$, for some $k \in \omega$, $k > n$. But for each such l, $\mathfrak{A}_i \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l(\supseteq \mathsf{RCA}_n)$, so $\mathfrak{A}_i$ is a sequence of algebras such that $\mathfrak{CmAt}\mathfrak{A}_i = \mathfrak{A}_i \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$, but $\mathfrak{Cm}(\mathsf{At}(\Pi_{i/U}\mathfrak{A}_i)) = \mathfrak{CmAt}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$, for all $l \geq k$. That k has to be strictly greater than $n+1$, follows because $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+1}$ is atom–canonical.

6. $\Longleftarrow$: Let $l < \omega$. Then the required follows from the second part of theorem 3.2 proving $\Psi(l,\omega)$; namely, there exists a countable $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ such that $\mathfrak{CmAt}\mathfrak{A} \notin \mathsf{RCA}_n$. Now we prove $\Longrightarrow$: Assume for contradiction that there is an $\mathfrak{A} \in \mathsf{RCA}_n^{\omega,\omega} \cap \mathsf{Count}$. Then by definition $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$ so $\mathfrak{A} \in \mathsf{CRCA}_n$. But this complete representation, induces a(n ordinary) representation of $\mathfrak{CmAt}\mathfrak{A}$ which is a contradiction.

□

4 Clique-guarded semantics

Fix $2 < n < \omega$. The reader is referred to [7, Definitions 13.4, 13.6] for the notions of m–flat and m–square representations for relation algebras ($m > 2$) to be generalized next to CA_ns.

Definition 4.1. [16, §5, p.14] Assume that $2 < n < m < \omega$. Let M be the base of a relativized representation of $\mathfrak{A} \in \mathsf{CA}_n$ witnessed by an injective homomorphism $f : \mathfrak{A} \to \wp(V)$, where $V \subseteq {}^n\mathsf{M}$ and $\bigcup_{s\in V} \mathsf{rng}(s) = \mathsf{M}$. We write $\mathsf{M} \models a(s)$ for $s \in f(a)$. Let $\mathfrak{L}(\mathfrak{A})^m$ be the first order signature using m variables and one n-ary relation symbol for each element in A. Let $\mathfrak{L}(\mathfrak{A})^m_{\infty,\omega}$ be the infinitary extension of $\mathfrak{L}(\mathfrak{A})^m$ allowing infinite conjunctions. Then *an n-clique* is a set $C \subseteq \mathsf{M}$ such that $(a_1, \ldots, a_{n-1}) \in V = 1^{\mathsf{M}}$ for distinct $a_1, \ldots, a_n \in C$. Let $\mathsf{C}^m(\mathsf{M}) = \{s \in {}^m\mathsf{M} : \mathsf{rng}(s) \text{ is an } n\text{-clique}\}$. $\mathsf{C}^m(\mathsf{M})$ is called the *n-Gaifman hypergraph of* M, with the n–hyperedge relation 1^{M}. The *clique guarded semantics* $\models_c$ are defined inductively. We give only existential quantifiers (cylindrifiers): for $\bar{s} \in {}^m\mathsf{M}$, $i < m$, $\mathsf{M}, \bar{s} \models_c \exists x_i\phi \iff$ there is a $\bar{t} \in \mathsf{C}^m(\mathsf{M})$, $\bar{t} \equiv_i \bar{s}$ such that $\mathsf{M}, \bar{t} \models \phi$.

We say that M is *an m–square representation* of $\mathfrak{A}$, if for all $\bar{s} \in \mathsf{C}^m(\mathsf{M}), a \in \mathfrak{A}$, $i < n$, and injective map $l : n \to m$, whenever $\mathsf{M} \models \mathsf{c}_i a(s_{l(0)}, \ldots, s_{l(n-1)})$, then there is a $\bar{t} \in \mathsf{C}^m(\mathsf{M})$ with $\bar{t} \equiv_i \bar{s}$, and $\mathsf{M} \models a(t_{l(0)}, \ldots, t_{l(n-1)})$; M is an *(infinitary) m–flat representation* if it is m–square and for all $\bar{s} \in \mathsf{C}^m(\mathsf{M})$, for all distinct $i, j < m$, $\mathsf{M} \models_c [\exists x_i \exists x_j \phi \longleftrightarrow \exists x_j \exists x_i \phi](\bar{s})$, where $\phi \in (\mathfrak{L}(\mathfrak{A})^m_{\infty,\omega})\mathfrak{L}(\mathfrak{A})^m$. Complete representability for m-squareness and m-flatness is defined like the classical case.

The main ideas used in the next Theorem can be found in [7, Definitions 12.1, 12.9, 12.10, 12.25, Propositions 12.25, 12.27] adapted to the CA case. In all cases, the m–dimensional dilation stipulated in the statement of the Theorem, will have top element $\mathsf{C}^m(\mathsf{M})$, where M is the m–relativized representation of the given algebra, and the operations of the dilation are induced by the n-clique–guarded semantics.

Theorem 4.2. *[7, Theorems 13.45, 13.36]. Assume that $2 < n < m < \omega$ and let $\mathfrak{A} \in \mathsf{CA}_n$. Then $\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m \iff \mathfrak{A}$ has an infinitary m–flat representation $\iff \mathfrak{A}$ has an m–flat representation. Furthermore, if $\mathfrak{A}$ is atomic, then $\mathfrak{A}$ has a complete infinitary m–flat representation $\iff \mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n(\mathsf{CA}_m \cap \mathbf{At})$.*

Proof. We give a sketchy sample. We start from representations to dilations. Let M be an m–flat representation of $\mathfrak{A}$. For $\phi \in \mathfrak{L}(\mathfrak{A})^m$, let $\phi^{\mathsf{M}} = \{\bar{a} \in \mathsf{C}^m(\mathsf{M}) : \mathsf{M} \models_c \phi(\bar{a})\}$, where $\mathsf{C}^m(\mathsf{M})$ is the n–Gaifman hypergraph. Let $\mathfrak{D}$ be the algebra with universe $\{\phi^M : \phi \in \mathfrak{L}(\mathfrak{A})^m\}$ and with cylindric operations induced by the n-clique–guarded (flat) semantics. For $r \in \mathfrak{A}$, and $\bar{x} \in \mathsf{C}^m(\mathsf{M})$, we identify r with the formula it defines in $\mathfrak{L}(\mathfrak{A})^m$, and we write $r(\bar{x})^{\mathsf{M}} \iff \mathsf{M}, \bar{x} \models_c r$.

Then $\mathfrak{D}$ is a set algebra with domain $\wp(\mathsf{C}^m(\mathsf{M}))$ and with unit $1^{\mathfrak{D}} = \mathsf{C}^m(\mathsf{M})$. Since M is m–flat, then cylindrifiers in $\mathfrak{D}$ commute, and so $\mathfrak{D} \in \mathsf{CA}_m$. Now define $\theta : \mathfrak{A} \to \mathfrak{D}$, via $r \mapsto r(\bar{x})^{\mathsf{M}}$. Then exactly like in the proof of [7, Theorem 13.20], θ is an injective neat embedding, that is, $\theta(\mathfrak{A}) \subseteq \mathfrak{Nr}_n\mathfrak{D}$. The relativized model M itself might not be infinitary m–flat, but one can build an infinitary m–flat representation of $\mathfrak{A}$, whose base is an ω–saturated model of the consistent first order theory, stipulating the existence of an m–flat representation, cf. [7, Proposition 13.17, Theorem 13.46 items (6) and (7)]. The inverse implication from dilations to representations harder. One constructs from the given m–dilation, an m–dimensional hyperbasis (that can be defined similarly to the RA case, cf. [7, Definition 12.11]) from which the required m-relativized representation is built. This can be done in a step–by step manner treating the hyperbasis as a 'saturated set of mosaics', cf. [7, Proposition 13.37].

For results on *complete* m–flat representations, one works in $L^m_{\infty,\omega}$ instead of first order logic. With $\mathfrak{D}$ formed like above from (the complete m–flat representation) M, using $\mathfrak{L}(\mathfrak{A})^m_{\infty,\omega}$ instead of L_n, let ϕ^{M} be a non–zero element in $\mathfrak{D}$. Choose $\bar{a} \in \phi^{\mathsf{M}}$, and let $\tau = \bigwedge\{\psi \in \mathfrak{L}(\mathfrak{A})^m_{\infty,\omega} : \mathsf{M} \models_c \psi(\bar{a})\}$. Then $\tau \in \mathfrak{L}(\mathfrak{A})^m_{\infty,\omega}$, and τ^{M} is an atom below ϕ^{M}. The rest is entirely analogous, cf. [7, p.411].

□

4.1 Omitting types OTT_r for the clique guarded fragments

Fix $2 < n \leq l < m \leq \omega$. Consider the statement $\Psi(l, m)$: There exists a countable, complete and atomic L_n theory T (meaning that the Tarski-Lindenbuam qoutient algebra $\mathfrak{Fm}_T$ is atomic), such that the type Γ consisting of co-atoms of $\mathfrak{Fm}_T$ is realizable in every m-square model of T (m-representation of $\mathfrak{Fm}_T$) but cannot be be isolated using l variables.

Let $\mathsf{OTT}_r(l, m)$ be by definition $\neg\Psi(l, m)$, short for *a restricted version of the Omitting Types Theorem holds at the parameters l and m*, where by definition, we stipulate that

$\mathsf{OTT}_r(\omega,\omega)$ is just the consquene of the Omitting types Theorem, for $L_{\omega,\omega}$, that says that a countable atomic theory T has a countbale atomic (prime) model. This atomic (unique up to isomorphism) model of T is the model resulting by omitting the countably many non-principal types $(X_i : i \in \omega)$, where X_i is the set of co-atoms of $\mathsf{Nr}_i\mathfrak{Fm}_T$. These are indeed non-principal because by definition $]Nr_i\mathfrak{Fm}_T$ is an atomic Boolean algebra, since T is an atomic theory. Furthermore if T is complete, then $\mathsf{Nr}_i\mathfrak{Fm}_T$ is also a simple CA_i for each $i < \omega$; i.e. $\mathsf{Nr}_i\mathfrak{Fm}_T$ has no proper ideals (congruences).

For $2 < n \leq l < m \leq \omega$ and $l = m = \omega$, we investigate the plausability of the following statement which we abbreviate by (**):

$$\mathsf{OTT}_r(l,m) \iff l = m = \omega.$$

In other words: OTT_r *holds only in the limiting case when* $l \to \infty$ *and* $m = \omega$ *and not 'before'.* This will be proved on the 'paths' (l,ω), $n \leq l < \omega$ (x axis) and $(n, n+k)$, $k \geq n+3$ (y axis) using two different blow up and blur constructions, given in Theorem 3.2 and 3.5.

Let $n < \omega$. Then $\mathsf{D}_n(\mathsf{G}_n)$ is a class of (non–commutative) set algebras having the same signature as CA_n. If $\mathfrak{A} \in \mathsf{D}_n(\mathsf{G}_n)$, then the top element of $\mathfrak{A}$ is a set $V \subseteq {}^nU$ (some non–empty set U), such that if $s \in V$, and $i < j < n$ ($\tau : n \to n$), then $s \circ [i|j](s \circ \tau) \in V$. It is known that both D_n and G_n are finitely axiomatizable varieties [16], such that $\mathsf{Gs}_n \subseteq \mathsf{G}_n \subseteq \mathsf{D}_n$. It can be proved similarly to Theorem 4.2, that if $\mathfrak{A}$ satisfies all the CA_n axioms with the possible exception of commutativity of cylindrifiers, then for any $2 < n < m < \omega$, $\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_n\mathsf{D}_m \iff \mathfrak{A} \in \mathbf{S}\mathsf{Nr}_n\mathsf{G}_m \iff \mathfrak{A}$ has an m-square representation.

In the next Theorem several conditions are given implying $\Psi(l,m)_f$ for various values of the parameters l and m where $\Psi(l,m)_f$ is the formula obtained from $\Psi(l,m)$ replacing square by flat. In the first item of the next theorem by no infinite ω–dimensional hyperbasis (basis), we understand no representation on an infinite base. By ω–flat (square) representation, we mean an ordinary representation, and by complete ω–flat (square) representation, we mean a complete representation. [2]

We need a lemma before embarking on the Theorem.

Lemma 4.3. *Let* $\mathfrak{R}$ *be a relation algebra and* $3 < n < \omega$. *Then* $\mathfrak{R}^+$ *has an* n*–dimensional infinite relational (hyper)basis* $\iff$ $\mathfrak{R}$ *has an infinite* n*–square (flat) representation.* $\mathfrak{R}^+$ *has an* n*–dimensional infinite hyperbasis* $\iff$ $\mathfrak{R}$ *has an infinite* n*–flat representation.*

Proof. [7, Theorem 13.46, the equivalence (1) $\iff$ (5) for relational basis, and the equivalence (7) $\iff$ (11) for hyperbasis]. □

Theorem 4.4. *Let* $2 < n \leq l < m \leq \omega$. *Then every item implies the immediately following one.*

1. *There exists a finite relation algebra* $\mathfrak{R}$ *with a strong* l*–blur and no infinite* m*–dimensional hyperbasis,*

2. *There is a countable atomic* $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ *such that* $\mathfrak{CmAt}\mathfrak{A}$ *does not have an* m*–flat representation,*

[2] Here we deviate from [7] in the treatment of κ-square representations for κ an infinite cardinal, by identifying a complete ω-square representation with a complete representation for an atomic algebra $\mathfrak{A} \in \mathsf{CA}_n$. This is true in case $\mathfrak{A}$ has countably many atoms, but may not true in general according to [7, Definition 17.22]. If $\omega \leq \kappa < \lambda$, an algebra having a complete λ–square representation, may not have a complete κ–square one. The rainbow algebra of dimension n, for any $2 < n < \omega$, $\mathfrak{A} = \mathfrak{A}_{\lambda,\kappa}$ witnesses this. Any complete κ–square representation of $\mathfrak{A}$ will force a 'κ red clique' indexed by the λ greens which is impossible because the indices of reds must match within the red clique.

3. *There is a countable atomic* $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ *such that* $\mathfrak{CmAt}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$,

4. *There is a countable atomic* $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ *such that* $\mathfrak{A}$ *has no complete infinitary* m*–flat representation,*

5. $\Psi(l', m')_f$ *is true for any* $l' \leq l$ *and* $m' \geq m$.

The same implications hold upon replacing infinite m*–dimensional hyperbasis by* m*–dimensional relational basis (not necessarily infinite),* m*–flat by* m*–square and* $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$ *by* $\mathbf{S}\mathsf{Nr}_n\mathsf{D}_m$*. Furthermore, in the new chain of implications every item implies the corresponding item in Theorem 4.4. In particular,* $\Psi(l, m) \implies \Psi(l, m)_f$.

Proof. (1) $\implies$ (2): We proceed similarly to Theorem 3.5. Let $\mathfrak{R}$ be as in the hypothesis with strong l–blur (J, E). The idea is to 'blow up and blur' $\mathfrak{R}$ in place of the Maddux algebra $\mathfrak{E}_k(2, 3)$ blown up and blurred in [2, Lemma 5.1], where $k < \omega$ is the number of non–identity atoms and k depends recursively on l, giving the desired strong l–blurness, cf. [2, Lemmata 4.2, 4.3]. Let $2 < n \leq l < \omega$. The relation algebra $\mathfrak{R}$ is blown up by splitting all of the atoms each to infinitely many giving a new infinite atom structure $\mathbf{At}$ denoted in [2, p.73] by At. One proves that the blown up and blurred atomic relation algebra $\mathfrak{Bb}(\mathfrak{R}, J, E)$ (as defined in [2]) with atom structure $\mathbf{At}$ is representable; in fact this representation is induced by a complete representation of its canonical extension, cf. [2, Item (1) of Theorem 3.2]. Because (J, E) is a strong l–blur, then, by its definition, it is a strong j–blur for all $n \leq j \leq l$, so the atom structure $\mathbf{At}$ has a j–dimensional cylindric basis for all $n \leq j \leq l$, namely, $\mathsf{Mat}_j(\mathbf{At})$. For all such j, there is an RCA_j denoted on [2, Top of p. 9] by $\mathfrak{Bb}_j(\mathfrak{R}, J, E)$ such that $\mathfrak{TmMat}_j(\mathbf{At}) \subseteq \mathfrak{Bb}_j(\mathfrak{R}, J, E) \subseteq \mathfrak{CmMat}_j(\mathbf{At})$ and $\mathsf{At}\mathfrak{Bb}_j(\mathfrak{R}, J, E)$ is a weakly representable atom structure of dimension j, cf. [2, Lemma 4.3]. Now take $\mathfrak{A} = \mathfrak{Bb}_n(\mathfrak{R}, J, E)$. We claim that $\mathfrak{A}$ is as required. Since $\mathfrak{R}$ has a strong j–blur (J, E) for all $n \leq j \leq l$, then $\mathfrak{A} \cong \mathfrak{Nr}_n\mathfrak{Bb}_j(\mathfrak{R}, J, E)$ for all $n \leq j \leq l$ as proved in [2, item (3) p.80]. In particular, taking $j = l$, $\mathfrak{A} \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$. We show that $\mathfrak{CmAt}\mathfrak{A}$ does not have an m–flat representation. Assume for contradicton that $\mathfrak{CmAt}\mathfrak{A}$ does have an m–flat representation M. Then M is infinite of course. Since $\mathfrak{R}$ embeds into $\mathfrak{Bb}(\mathfrak{R}, J, E)$ which in turn embeds into $\mathfrak{RaCmAt}\mathfrak{A}$, then $\mathfrak{R}$ has an m–flat representation with base M. But since $\mathfrak{R}$ is finite, $\mathfrak{R} = \mathfrak{R}^+$, so by Lemma 4.3, $\mathfrak{R}$ has an infinite m–dimensional hyperbasis, contradiction.

(2) $\implies$ (3): By item (1) of Theorem 4.2.

(3) $\implies$ (4): A complete m–flat representation of (any) $\mathfrak{B} \in \mathsf{CA}_n$ induces an m–flat representation of $\mathfrak{CmAt}\mathfrak{B}$ which implies by Theorem 4.2 that $\mathfrak{CmAt}\mathfrak{B} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$. To see why, assume that $\mathfrak{B}$ has an m–flat complete representation via $f : \mathfrak{B} \to \mathfrak{D}$, where $\mathfrak{D} = \wp(V)$ and the base of the representation $\mathsf{M} = \bigcup_{s \in V} \mathsf{rng}(s)$ is m–flat. Let $\mathfrak{C} = \mathfrak{CmAt}\mathfrak{B}$. For $c \in C$, let $c\downarrow = \{a \in \mathsf{At}\mathfrak{C} : a \leq c\} = \{a \in \mathsf{At}\mathfrak{B} : a \leq c\}$; the last equality holds because $\mathsf{At}\mathfrak{B} = \mathsf{At}\mathfrak{C}$. Define, representing $\mathfrak{C}$, $g : \mathfrak{C} \to \mathfrak{D}$ by $g(c) = \sum_{x \in c\downarrow} f(x)$, then g is a homomorphism into $\wp(V)$ having base M.

(4) $\implies$ (5): By [5, §4.3], we can (and will) assume that $\mathfrak{A} = \mathfrak{Fm}_T$ for a countable, simple and atomic theory L_n theory T. Let Γ be the n–type consisting of co–atoms of T. Then Γ is realizable in every m–flat model, for if M is an m–flat model omitting Γ, then M would be the base of a complete infinitary m–flat representation of $\mathfrak{A}$, and so $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$ which is impossible. But $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l$, so using the same (terminology and) argument in [2, Theorem 3.1] we get that any witness isolating Γ needs more than l–variables. In more detail, suppose for contradiction that ϕ is an l witness, so that $T \models \phi \to \alpha$, for all $\alpha \in \Gamma$, where recall that Γ is the set of coatoms. Then since $\mathfrak{A}$ is simple, we can assume without loss of generality, that

$\mathfrak{A}$ is a set algebra with base M say. Let $\mathsf{M} = (M, R_i)_{i\in\omega}$ be the corresponding model (in a relational signature) to this set algebra in the sense of [5, §4.3]. Let ϕ^{M} denote the set of all assignments satisfying ϕ in M. We have $\mathsf{M} \models T$ and $\phi^{\mathsf{M}} \in \mathfrak{A}$, because $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_{m-1}$. But $T \models \exists x\phi$, hence $\phi^{\mathsf{M}} \neq 0$, from which it follows that ϕ^{M} must intersect an atom $\alpha \in \mathfrak{A}$ (recall that the latter is atomic). Let ψ be the formula, such that $\psi^{\mathsf{M}} = \alpha$. Then it cannot be the case that $T \models \phi \to \neg\psi$, hence ϕ is not a witness, contradiction and we are done. We have proved $\Psi(l, m)$. The rest follows from the definitions.

For squareness the proofs are essentially the same undergoing the obvious modifications (e.g. using the part on squareness in Lemma 4.3 and repacing CA_n by D_n). In the first implication 'infinite' in the hypothesis is not needed because any finite relation algebra having an infinite m–dimensional relational basis has a finite one, cf. [7, Theorem 19.18] which is not the case with hyperbasis, cf. [7, Prop. 19.19]. □

Corollary 4.5. *For $2 < n < \omega$ and $n \leq l < \omega$, $\Psi(n, n+3)$ and $\Psi(l, \omega)$ hold.*

Proof. The first case, follows from Theorem 3.2 and 4.4 (by taking $l = n$ and $m = n+3$). For the second case, it suffices by Theorem 4.4 (by taking $m = \omega$) to find a countable algebra $\mathfrak{C} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ such that $\mathfrak{CmAtC} \notin \mathsf{RCA}_n$. This algebra is constructed in [2], cf. Thjeorem 3.5. □

It is timely that we tie a few threads together.

Definition 4.6. Let $2 < n < \omega$. We say that VT *fails for L_n almost everywhere* if there exist positive $l, m \geq n$ such that $\mathsf{V}(k, \omega)$ and $\mathsf{V}(n, t)$ are false for all finite $k \geq l$ and all $t \geq m$. We say that VT *fails for L_n everywhere* if for $3 \leq l < m \leq \omega$ and $l = m = \omega$, $\mathsf{V}(l, m)$ holds $\iff$ $l = m = \omega$, that is to say $(**)$ above holds.

From Corollary 4.5 and the implication (1) $\implies$ (6) in Theorem 4.4 (by taking $l = m-1$), we get:

Theorem 4.7. *Let $2 < n < \omega$. Then $\mathsf{OTT_r}$ fails for L_n almost everywhere. Furthermore, if for each $n < m < \omega$, there exists a finite relation algebra $\mathfrak{R}_m$ having $m-1$ strong blur and no m-dimensional relational basis, then VT fails for L_n everywhere.*

Now we formulate an algebraic result implying that VT fails for any finite first order definable expansion of L_n as defined in [3]. We deviate from the notation in [3] by writing RCA_n^+ for a first order definable expansion of RCA_n.

Theorem 4.8. *Let $2 < n < \omega$. Let RCA_n^+ be a first order definable expansion of RCA_n such that the non–cylindric operations are first order definable by formulas using only finitely many variables $l > n$. If RCA_n^+ is completely additive, then it is not atom–canonical.*

Proof. Let $\mathfrak{n}$ be the finite number of variables occuring in the first order formulas defining the new connectives and let $l = \mathfrak{n}+1$. Let $\mathfrak{A}$ be countable and atomic such that $\mathfrak{A} \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$ and $\mathfrak{A}$ has no complete representation; such an $\mathfrak{A}$ exists, cf. Theorem 3.5. Without loss, we can assume that we have only one extra operation f definable by a first order formula ϕ, say, using $n < k < \omega$ variables with at most n free variables. Now ϕ defines a CA_k term $\tau(\phi)$ which, in turn, defines the unary operation f on $\mathfrak{A}$, via $f(a) = \tau(\phi)^{\mathfrak{B}}(a)$. This is well defined, in the sense that $f(a) \in \mathfrak{A}$, because $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_{\mathfrak{n}+1}$ and the first order formula ϕ defining f, has at most n free variables. Call the expanded structure $\mathfrak{A}^*(\in \mathsf{RCA}_n^+)$. By complete additivity, $\mathfrak{CmAtA}^*$ is the Dedekind-MacNeille completion of $\mathfrak{A}^*$. But $\mathfrak{Rd}_{ca}\mathfrak{CmAtA}^* = \mathfrak{CmAtA} \notin \mathsf{RCA}_n$, *a fortiori*, $\mathfrak{Cm}(\mathsf{At}\mathfrak{A}^*) \notin \mathsf{RCA}_n^+$, and we are done. □

Let $2 < n \leq l < m \leq \omega$. In $\mathsf{VT}(l,m)$, while the parameter l measures how close we are to $L_{\omega,\omega}$, m measures the *'degree' of squareness* of permitted models. One can view $\mathsf{lim}_{l\to\infty}\mathsf{VT}(l,\omega) = \mathsf{VT}(\omega,\omega)$ algebraically using ultraproducts as follows. Fix $2 < n < \omega$. For each $2 < n \leq l < \omega$, let $\mathfrak{R}_l$ be the finite Maddux algebra $\mathfrak{E}_{f(l)}(2,3)$ with strong l–blur (J_l, E_l) and $f(l) \geq l$ as specified in [2, Lemma 5.1] (denoted by k therein). Let $\mathcal{R}_l = \mathfrak{Bb}(\mathfrak{R}_l, J_l, E_l) \in \mathsf{RRA}$ and let $\mathfrak{A}_l = \mathfrak{Nr}_n\mathfrak{Bb}_l(\mathfrak{R}_l, J_l, E_l) \in \mathsf{RCA}_n$. Then $(\mathsf{At}\mathcal{R}_l : l \in \omega \sim n)$, and $(\mathsf{At}\mathfrak{A}_l : l \in \omega \sim n)$ are sequences of weakly representable atom structures that are not strongly representable with a completely representable ultraproduct. We immediately get:

Corollary 4.9. *Assume that $2 < n < \omega$. Then the following hold:*

1. *The (elementary) class LCA_n of algebras satisfying the Lyndon conditions (which is $\mathbf{El}\mathsf{CRCA}_n$) is not finitely axiomatizable,*
2. *[3, 11] The set of equations using only one variable that holds in each of the varieties RCA_n and RRA, together with any finite first order definable expansion of each, cannot be derived from any finite set of equations valid in the variety.*

5 Non-elementary classes of algebras having a neat embedding property

We define an atomic k rounded (atomic) game $\mathbf{H}_k$ stronger than the usual k-rounded (atomic) game G_k [7, 8]. To define the game we need a few definitions:

Definition 5.1. A λ*–neat hypernetwork* is roughly a network endowed with labelled hyperdeges of length $\neq n$ allowed to get arbitrarily long but are of finite length, and such hyperedges get their labels from a non–empty set of labels Λ; such that all so–called *short hyperedges* are constantly labelled by $\lambda \in \Lambda$. The board of the game consists of λ-neat hypernetworks:

Definition 5.2. For an n–dimensional atomic network N on an atomic CA_n and for $x, y \in \mathsf{nodes}(N)$, set $x \sim y$ if there exists $\bar{z}$ such that $N(x, y, \bar{z}) \leq \mathsf{d}_{01}$. Define the equivalence relation $\sim$ over the set of all finite sequences over $\mathsf{nodes}(N)$ by $\bar{x} \sim \bar{y}$ iff $|\bar{x}| = |\bar{y}|$ and $x_i \sim y_i$ for all $i < |\bar{x}|$. (It can be easily checked that this indeed an equivalence relation).

A *hypernetwork* $N = (N^a, N^h)$ over an atomic CA_n consists of an n–dimensional network N^a together with a labelling function for hyperlabels $N^h : {}^{<\omega}\mathsf{nodes}(N) \to \Lambda$ (some arbitrary set of hyperlabels Λ) such that for $\bar{x}, \bar{y} \in {}^{<\omega}\mathsf{nodes}(N)$ if $\bar{x} \sim \bar{y} \Rightarrow N^h(\bar{x}) = N^h(\bar{y})$. If $|\bar{x}| = k \in \mathbb{N}$ and $N^h(\bar{x}) = \lambda$, then we say that λ is a k-ary hyperlabel. $\bar{x}$ is referred to as a k–ary hyperedge, or simply a hyperedge.

We may remove the superscripts a and h if no confusion is likely to ensue. A hyperedge $\bar{x} \in {}^{<\omega}\mathsf{nodes}(N)$ is *short*, if there are $y_0, \ldots, y_{n-1}$ that are nodes in N, such that $N(x_i, y_0, \bar{z}) \leq \mathsf{d}_{01}$ or $\ldots N(x_i, y_{n-1}, \bar{z}) \leq \mathsf{d}_{01}$ for all $i < |x|$, for some (equivalently for all) $\bar{z}$. Otherwise, it is called *long.*

This game involves, besides the standard cylindrifier move, two new amalgamation moves. This game has k rounds with $k \leq \omega$, call it $\mathbf{H}_k$. Concerning his moves, $\forall$ can play a cylindrifier move, like before but now played on λ— neat hypernetworks (λ a constant label). Also $\forall$ can play a *transformation move* by picking a previously played λ neat hypernetwork N and a partial, finite surjection $\theta : \omega \to \mathsf{nodes}(N)$, this move is denoted (N, θ). $\exists$'s response is mandatory. She must respond with $N\theta$. Finally, $\forall$ can play an *amalgamation move* by picking previously played λ neat hypernetworks M, N such that $M{\restriction}_{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)} = N{\restriction}_{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)}$, and

$\mathsf{nodes}(M) \cap \mathsf{nodes}(N) \neq \emptyset$. This move is denoted (M, N). To make a legal response, $\exists$ must play a λ–neat hypernetwork L extending M and N, where $\mathsf{nodes}(L) = \mathsf{nodes}(M) \cup \mathsf{nodes}(N)$.

Getting these prliminaries out of the way, we are now ready to start digging deeper.

Lemma 5.3. *Let α be a countable atom structure. If $\exists$ has a winning strategy in $\mathbf{H}_\omega(\alpha)$, then any algebra $\mathfrak{F}$ having atom structure α is completely representable and there exists a complete $\mathfrak{D} \in \mathsf{RCA}_\omega$ such that $\mathfrak{Cm}\alpha \cong \mathsf{Nr}_n\mathfrak{D}$*

Proof. Fix some $a \in \alpha$. The game $\mathbf{H}_\omega$ is designed so that using $\exists$ s winning strategy in the game $\mathbf{H}_\omega(\alpha)$ one can define a nested sequence $M_0 \subseteq M_1, \ldots$ of λ–neat hypernetworks where M_0 is $\exists$'s response to the initial $\forall$-move a, such that: If M_r is in the sequence and $M_r(\bar{x}) \leq \mathsf{c}_i a$ for an atom a and some $i < n$, then there is $s \geq r$ and $d \in \mathsf{nodes}(M_s)$ such that $M_s(\bar{y}) = a$, $\bar{y}_i = d$ and $\bar{y} \equiv_i \bar{x}$. In addition, if M_r is in the sequence and θ is any partial isomorphism of M_r, then there is $s \geq r$ and a partial isomorphism θ^+ of M_s extending θ such that $\mathsf{rng}(\theta^+) \supseteq \mathsf{nodes}(M_r)$ (This can be done using $\exists$'s responses to amalgamation moves). Now let M_a be the limit of this sequence, that is $\mathsf{M}_a = \bigcup M_i$, the labelling of $n-1$ tuples of nodes by atoms, and hyperedges by hyperlabels done in the obvious way using the fact that the M_is are nested. Let L be the signature with one n-ary relation for each $b \in \alpha$, and one k–ary predicate symbol for each k–ary hyperlabel λ. *Now we work in $L_{\infty,\omega}$.* For fixed $f_a \in {}^\omega\mathsf{nodes}(\mathsf{M}_a)$, let $\mathfrak{U}_a = \{f \in {}^\omega\mathsf{nodes}(\mathsf{M}_a) : \{i < \omega : g(i) \neq f_a(i)\}$ is finite$\}$. We make $\mathfrak{U}_a$ into the base of an L relativized structure $\mathcal{M}_a$. We allow a clause for infinitary disjunctions. In more detail, for $b \in \alpha$, $l_0, \ldots, l_{n-1}, i_0 \ldots, i_{k-1} < \omega$, k–ary hyperlabels λ, and all L-formulas ϕ, ϕ_i, ψ, and $f \in U_a$:

$$\begin{aligned}
\mathcal{M}_a, f \models b(x_{l_0} \ldots, x_{l_{n-1}}) &\iff \mathcal{M}_a(f(l_0), \ldots, f(l_{n-1})) = b,\\
\mathcal{M}_a, f \models \lambda(x_{i_0}, \ldots, x_{i_{k-1}}) &\iff \mathcal{M}_a(f(i_0), \ldots, f(i_{k-1})) = \lambda,\\
\mathcal{M}_a, f \models \neg\phi &\iff \mathcal{M}_a, f \not\models \phi,\\
\mathcal{M}_a, f \models (\bigvee_{i \in I} \phi_i) &\iff (\exists i \in I)(\mathcal{M}_a, f \models \phi_i),\\
\mathcal{M}_a, f \models \exists x_i \phi &\iff \mathcal{M}_a, f[i/m] \models \phi, \text{ some } m \in \mathsf{nodes}(\mathcal{M}_a).
\end{aligned}$$

We are now working with (weak) set algebras whose semantics is induced by $L_{\infty,\omega}$ formulas in the signature L, instead of first order ones. For any such L-formula ϕ, write $\phi^{\mathcal{M}_a}$ for $\{f \in \mathfrak{U}_a : \mathcal{M}_a, f \models \phi\}$. Let $D_a = \{\phi^{\mathcal{M}_a} : \phi$ is an L-formula$\}$ and $\mathfrak{D}_a$ be the weak set algebra with universe D_a. Let $\mathfrak{D} = \mathbf{P}_{a \in \alpha}\mathfrak{D}_a$. Then $\mathfrak{D}$ is a generalized *complete* weak set algebra [5, Definition 3.1.2 (iv)]. By complete we mean (the usua) infinite suprema exists. This is true because we chose to work with $L_{\infty,\omega}$ while forming the dilations $\mathfrak{D}_a$ $(a \in \alpha)$. Each $\mathfrak{D}_a$ is complete, hence so is their product $\mathfrak{D}$. Let $X \subseteq \mathfrak{Nr}_n\mathfrak{D}$. Then by completeness of $\mathfrak{D}$, we get that $d = \sum^{\mathfrak{D}} X$ exists. Assume that $i \notin n$, then $\mathsf{c}_i d = \mathsf{c}_i \sum X = \sum_{x \in X} \mathsf{c}_i x = \sum X = d$, because the c_is are completely additive and $\mathsf{c}_i x = x$, for all $i \notin n$, since $x \in \mathfrak{Nr}_n\mathfrak{D}$. We conclude that $d \in \mathfrak{Nr}_n\mathfrak{D}$, hence d is an upper bound of X in $\mathfrak{Nr}_n\mathfrak{D}$. Since $d = \sum_{x \in X}^{\mathfrak{D}} X$ there can be no $b \in \mathfrak{Nr}_n\mathfrak{D}$ $(\subseteq \mathfrak{D})$ with $b < d$ such that b is an upper bound of X for else it will be an upper bound of X in $\mathfrak{D}$. Thus $\sum_{x \in X}^{\mathfrak{Nr}_n\mathfrak{D}} X = d$ We have shown that $\mathfrak{Nr}_n\mathfrak{D}$ is complete. Making the legitimate identification $\mathfrak{Nr}_n\mathfrak{D} \subseteq_d \mathfrak{Cm}\alpha$ by density, we get that $\mathfrak{Nr}_n\mathfrak{D} = \mathfrak{Cm}\alpha$ (since $\mathfrak{Nr}_n\mathfrak{D}$ is complete), hence $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$. This does not mean that $\mathfrak{Tm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$, witness Theorem **??** below. To show that an atomic algebra with atom structure α is completely representable, we use that given two atomic algebras $\mathfrak{A}, \mathfrak{B} \in \mathsf{CA}_n$ such that $\mathsf{At}\mathfrak{A} \cong \mathsf{At}\mathfrak{B}$, then $\mathfrak{A} \in \mathsf{CRCA}_n \iff \mathfrak{B} \in \mathsf{CRCA}_n$. Now $\mathfrak{Cm}\alpha \in \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega (\subseteq \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega)$ and α is countable, so by [13, Theorem 5.3.6], $\mathfrak{Cm}\alpha$ is completely representable, hence so is any algebra sharing the atom structure α. Alternatively to prove the last part, one can use that $\mathbf{H}_\omega$ is plainly stronger than the usual ω–rounded atomic game G (in the sense that a winning strategy for $\exists$ in $\mathbf{H}_\omega \implies$ a winning strategy for $\exists$ in G), and then uses [8, Theorem 3.3.3] whose more difficult implication says that a winning strategy for $G(\beta)$ (hence in $\mathbf{H}(\beta)$), β a countable atom structure, implies that β is completely representable. (The converse, when β is uncountable, may not be true [17, Theorem 4.5]). □

Lemma 5.4. *Any class* K *between* $\mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ *and* $\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$ *is not elementary*

Proof. **(1)** $\forall$ **has a winning strategy in** $\mathbf{G}_{n+3}(\mathsf{At}\mathfrak{C})$ **for a rainbow-like algebra** $\mathfrak{C}$**:**

Take the a rainbow–like CA_n, call it $\mathfrak{C}$, based on the ordered structure $\mathbb{Z}$ and $\mathbb{N}$. The reds R is the set $\{\mathsf{r}_{ij} : i < j < \omega(= \mathbb{N})\}$ and the green colours used constitute the set $\{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}_0^i : i \in \mathbb{Z}\}$. In complete coloured graphs the forbidden triples are like the usual rainbow constructions based on $\mathbb{Z}$ and $\mathbb{N}$, but now the triple $(\mathsf{g}_0^i, \mathsf{g}_0^j, \mathsf{r}_{kl})$ is also forbidden if $\{(i,k),(j,l)\}$ is not an order preserving partial function from $\mathbb{Z} \to \mathbb{N}$. It can be shown that $\forall$ has a winning strategy in the graph version of the game $\mathbf{G}^{n+3}(\mathsf{At}\mathfrak{C})$ played on coloured graphs [6]. The rough idea here, is that, as is the case with winning strategy's of $\forall$ in rainbow constructions, $\forall$ bombards $\exists$ with cones having distinct green tints demanding a red label from $\exists$ to appexes of succesive cones. The number of nodes are limited but $\forall$ has the option to re-use them, so this process will not end after finitely many rounds. The added order preserving condition relating two greens and a red, forces $\exists$ to choose red labels, one of whose indices form a decreasing sequence in $\mathbb{N}$. In ω many rounds $\forall$ forces a win, so $\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$. More rigorously, $\forall$ plays as follows: In the initial round $\forall$ plays a graph M with nodes $0,1,\ldots,n-1$ such that $M(i,j) = \mathsf{w}_0$ for $i < j < n-1$ and $M(i,n-1) = \mathsf{g}_i$ $(i = 1,\ldots,n-2)$, $M(0,n-1) = \mathsf{g}_0^0$ and $M(0,1,\ldots,n-2) = \mathsf{y}_{\mathbb{Z}}$. This is a 0 cone. In the following move $\forall$ chooses the base of the cone $(0,\ldots,n-2)$ and demands a node n with $M_2(i,n) = \mathsf{g}_i$ $(i = 1,\ldots,n-2)$, and $M_2(0,n) = \mathsf{g}_0^{-1}$. $\exists$ must choose a label for the edge $(n+1,n)$ of M_2. It must be a red atom r_{mk}, $m,k \in \mathbb{N}$. Since $-1 < 0$, then by the 'order preserving' condition we have $m < k$. In the next move $\forall$ plays the face $(0,\ldots,n-2)$ and demands a node $n+1$, with $M_3(i,n) = \mathsf{g}_i$ $(i = 1,\ldots,n-2)$, such that $M_3(0,n+2) = \mathsf{g}_0^{-2}$. Then $M_3(n+1,n)$ and $M_3(n+1,n-1)$ both being red, the indices must match. $M_3(n+1,n) = r_{lk}$ and $M_3(n+1,r-1) = r_{km}$ with $l < m \in \mathbb{N}$. In the next round $\forall$ plays $(0,1,\ldots n-2)$ and re-uses the node 2 such that $M_4(0,2) = \mathsf{g}_0^{-3}$. This time we have $M_4(n,n-1) = \mathsf{r}_{jl}$ for some $j < l < m \in \mathbb{N}$. Continuing in this manner leads to a decreasing sequence in $\mathbb{N}$. We have proved the required. Since $\mathfrak{Cm}\mathsf{At}\mathfrak{C} = \mathfrak{C}$ and $\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$ we are done.

(2) $\exists$ **has a winning strategy in** $\mathbf{H}_k(\mathsf{At}\mathfrak{C})$ **for all** $k < \omega$**:**

In [16] it is shown that for $k < \omega$, $\exists$ has a winning strategy in $G_k(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathbb{N}})$ inspite of the newly forbidden triple consisting of two greens and one red, synchronized by an order preserving function. This plainly makes her choices more restricted. But we can go further. It can be shown with some more effort (but not much more) that, in fact, $\exists$ has a winning strategy in even the stronger game $\mathbf{H}_k(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathbb{N}})$ for all $k < \omega$.

(2a) Response of $\exists$ **in labelling** λ**-neat hypredges:** We describe $\exists$'s strategy in dealing with labelling hyperedges in λ–neat hypernetworks, where λ is a constant label kept on short hyperedges. In a play, $\exists$ is required to play λ–neat hypernetworks, so she has no choice about the the short edges, these are labelled by λ. In response to a cylindrifier move by $\forall$ extending the current hypernetwork providing a new node k, and a previously played coloured hypernetwork M all long hyperedges not incident with k necessarily keep the hyperlabel they had in M. All long hyperedges incident with k in M are given unique hyperlabels not occurring as the hyperlabel of any other hyperedge in M. In response to an amalgamation move, which involves two hypernetworks required to be amalgamated, say (M,N) all long hyperedges whose range is contained in $\mathsf{nodes}(M)$ have hyperlabel determined by M, and those whose range is contained in $\mathsf{nodes}(N)$ have hyperlabels determined by N. If $\bar{x}$ is a long hyperedge of $\exists$ s response L where $\mathsf{rng}(\bar{x}) \nsubseteq \mathsf{nodes}(M)$, $\mathsf{nodes}(N)$ then $\bar{x}$ is given a new hyperlabel, not used in any previously played hypernetwork and not used within L as the label of any hyperedge other than $\bar{x}$. This completes her strategy for labelling hyperedges.

(2b) Response of $\exists$ to cylindrficatioon moves: We show that $\exists$ has a winning strategy in $G_k(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathbb{N}})$ where $0<k<\omega$ is the number of rounds; the part proved in [16]. Let $0<k<\omega$. We proceed inductively. Let $M_0, M_1, \ldots, M_r$, $r<k$ be the coloured graphs at the start of a play of G_k just before round $r+1$. Assume inductively, that $\exists$ computes a partial function $\rho_s : \mathbb{Z} \to \mathbb{N}$, for $s \leq r$: Let $0<k<\omega$. We proceed inductively. Let $M_0, M_1, \ldots, M_r$, $r<k$ be the coloured graphs at the start of a play of G_k just before round $r+1$. Assume inductively, that $\exists$ computes a partial function $\rho_s : \mathbb{Z} \to \mathbb{N}$, for $s \leq r$:

(i) $\rho_0 \subseteq \ldots \rho_t \subseteq \ldots \subseteq \ldots \rho_s$ is (strict) order preserving; if $i<j \in \mathsf{dom}\rho_s$ then $\rho_s(i) - \rho_s(j) \geq 3^{k-r}$, where $k-r$ is the number of rounds remaining in the game, and

$$\mathsf{dom}(\rho_s) = \{i \in \mathbb{Z} : \exists t \leq s,\ M_t \text{ contains an } i\text{–cone as a subgraph}\},$$

(ii) for $u, v, x_0 \in \mathsf{nodes}(M_s)$, if $M_s(u,v) = \mathsf{r}_{\mu,k}$, $\mu, k \in \mathbb{N}$, $M_s(x_0,u) = \mathsf{g}_0^i$, $M_s(x_0,v) = \mathsf{g}_0^j$, where $i,j \in \mathbb{Z}$ are tints of two cones, with base F such that x_0 is the first element in F under the induced linear order, then $\rho_s(i) = \mu$ and $\rho_s(j) = k$.

For the base of the induction $\exists$ takes $M_0 = \rho_0 = \emptyset$. Assume that M_r, $r<k$ (k the number of rounds) is the current coloured graph and that $\exists$ has constructed $\rho_r : \mathbb{Z} \to \mathbb{N}$ to be a finite order preserving partial map such conditions (i) and (ii) hold. We show that (i) and (ii) can be maintained in a further round. We check the most difficult case. Assume that $\beta \in \mathsf{nodes}(M_r)$, $\delta \notin \mathsf{nodes}(M_r)$ is chosen by $\forall$ in his cylindrifier move, such that β and δ are apprexes of two cones having same base and green tints $p \neq q \in \mathbb{Z}$. Now $\exists$ adds q to $\mathsf{dom}(\rho_r)$ forming ρ_{r+1} by defining the value $\rho_{r+1}(p) \in \mathbb{N}$ in such a way to preserve the (natural) order on $\mathsf{dom}(\rho_r) \cup \{q\}$, that is maintaining property (i). Inductively, ρ_r is order preserving and 'widely spaced' meaning that the gap between its elements is at least 3^{k-r}, so this can be maintained in a further round. Now $\exists$ has to define a (complete) coloured graph M_{r+1} such that $\mathsf{nodes}(M_{r+1}) = \mathsf{nodes}(M_r) \cup \{\delta\}$. In particular, she has to find a suitable red label for the edge (β,δ). Having ρ_{r+1} at hand she proceeds as follows. Now that $p,q \in \mathsf{dom}(\rho_{r+1})$, she lets $\mu = \rho_{r+1}(p)$, $b = \rho_{r+1}(q)$. The red label she chooses for the edge (β,δ) is: (*) $M_{r+1}(\beta,\delta) = \mathsf{r}_{\mu,b}$. This way she maintains property (ii) for ρ_{r+1}. Next we show that this is a winning strategy for $\exists$.

We check consistency of newly created triangles proving that M_{r+1} is a coloured graph completing the induction. Since ρ_{r+1} is chosen to preserve order, no new forbidden triple (involving two greens and one red) will be created. Now we check red triangles only of the form (β, y, δ) in M_{r+1} ($y \in \mathsf{nodes}(M_r)$). We can assume that y is the apex of a cone with base F in M_r and green tint t, say, and that β is the appex of the p–cone having the same base. Then inductively by condition (ii), taking x_0 to be the first element of F, and taking the nodes β, y, and the tints p,t, for u,v,i,j, respectively, we have by observing that $\beta, y \in \mathsf{nodes}(M_r)$, $\beta, y \in \mathsf{dom}(\rho_r)$ and $\rho_r \subseteq \rho_{r+1}$, the following: $M_{r+1}(\beta,y) = M_r(\beta,y) = \mathsf{r}_{\rho_r(p),\rho_r(t)} = r_{\rho_{r+1}(p),\rho_{r+1}(t)}$. By her strategy, we have $M_{r+1}(y,\delta) = \mathsf{r}_{\rho_{r+1}(t),\rho_{r+1}(q)}$ and we know by (*) that $M_{r+1}(\beta,\delta) = \mathsf{r}_{\rho_{r+1}(p),\rho_{r+1}(q)}$. The triple $(\mathsf{r}_{\rho_{r+1}(p),\rho_{r+1}(t)}, \mathsf{r}_{\rho_{r+1}(t),\rho_{r+1}(q)}, \mathsf{r}_{\rho_{r+1}(p),\rho_{r+1}(q)})$ of reds is consistent and we are done with this case. All other edge labelling and colouring $n-1$ tuples in M_{r+1} by yellow shades are exactly like in [6]. But we can go further. We show that $\exists$ has a winning strategy in the stronger game $H_k(\mathsf{At}\mathfrak{C})$ for all $k \in \omega$. $\exists$'s strategy dealing with λ–neat hypernetworks, where λ is a constant label kept on short hyperedges.

(2c) Response of $\exists$ to amalgamation moves: Now we change the board of play but only formally. We play on λ*–neat hypergraphs*. Given a rainbow algebra $\mathfrak{A}$, there is a one to one correspondence between coloured graphs on $\mathsf{At}\mathfrak{A}$ and networks on $\mathsf{At}\mathfrak{A}$ [8, Half of p. 76] denote this correspondence, expressed by a bijection from coloured graphs to networks by (*):

$$\Gamma \mapsto N_\Gamma, \quad \mathsf{nodes}(\Gamma) = \mathsf{nodes}(N_\Gamma).$$

Now the game H can be re-formulated to be played on λ*–neat hypergraphs* on a rainbow algebra $\mathfrak{A}$; these are of the form (Δ, N^h), where Δ is a coloured graph on $\mathsf{At}\mathfrak{A}$, λ is a hyperlabel, and N^h is as before, $N^h :^{<\omega} \mathsf{nodes}(\Delta) \to \Lambda$, such that for $\bar{x}, \bar{y} \in {}^{<\omega}\mathsf{nodes}(\Delta)$, if $\bar{x} \sim \bar{y} \Rightarrow N^h(\bar{x}) = N^h(\bar{y})$. Here $\bar{x} \sim \bar{y}$, making the obvious translation, is the equivalence relation defined by: $x \sim y \iff |x| = |y|$ and $N_\Delta(x_i, y_i, \bar{z}) \leq \mathsf{d}_{01}$ for all $i < |x|$ and some $\bar{z} \in {}^{n-2}\mathsf{nodes}(\Delta)$.

All notions earlier defined for hypernetworks, in particular, λ–neat ones, translate to λ–neat hypergraphs, using (*), like short hyperdges, long hypedges, λ–neat hypergraphs, etc. The game is played now on λ–neat hypergraphs on which the constant label λ is kept on the short hyperedges in ${}^{<\omega}\mathsf{nodes}(\Delta)$. We have already dealt with the 'graph part' of the game. We turn to the remaining amalgamation moves. We need some notation and terminology. Every edge of any hypergraph (edge of its graph part) has an *owner* $\forall$ *or* $\exists$, namely, the one who coloured this edge. We call such edges $\forall$ edges or $\exists$ edges. Each long hyperedge $\bar{x}$ in N^h of a hypergraph N occurring in the play has *an envelope* $v_N(\bar{x})$ to be defined shortly.
In the initial round, $\forall$ plays $a \in \alpha$ and $\exists$ plays N_0 then all edges of N_0 belongs to $\forall$. There are no long hyperedges in N_0. If $\forall$ plays a cylindrifier move requiring a new node k and $\exists$ responds with M then the owner in M of an edge not incident with k is the same as it was in N and the envelope in M of a long hyperedge not incident with k is the same as that it was in N. All edges (l, k) for $l \in \mathsf{nodes}(N) \sim \{k\}$ belong to $\exists$ in M. if $\bar{x}$ is any long hyperedge of M with $k \in \mathsf{rng}(\bar{x})$, then $v_M(\bar{x}) = \mathsf{nodes}(M)$.
If $\forall$ plays the amalgamation move (M, N) (of two λ–neat hypergraphs) and $\exists$ responds with L then for $m \neq n \in \mathsf{nodes}(L)$ the owner in L of a edge (m, n) is $\forall$ if it belongs to $\forall$ in either M or N, in all other cases it belongs to $\exists$ in L. If $\bar{x}$ is a long hyperedge of L then $v_L(\bar{x}) = v_M(\bar{x})$ if $\mathsf{rng}(\bar{x}) \subseteq \mathsf{nodes}(M)$, $v_L(\bar{x}) = v_N(\bar{x})$ and $v_L(\bar{x}) = \mathsf{nodes}(M)$ otherwise. If in a later move, $\forall$ plays the transformation move (N, θ) and $\exists$ responds with $N\theta$, then owners and envelopes are inherited in the obvious way. This completes the definition of owners and envelopes. The next claim, basically, reduces amalgamation moves to cylindrifier moves. By induction on the number of rounds one can show:

Claim: Let M, N occur in a play of H_m, $0 < m \in \omega$. in which $\exists$ uses the above labelling for hyperedges. Let $\bar{x}$ be a long hyperedge of M and let $\bar{y}$ be a long hyperedge of N. Then for any hyperedge $\bar{x}'$ with $\mathsf{rng}(\bar{x}') \subseteq v_M(\bar{x})$, if $M(\bar{x}') = M(\bar{x})$ then $\bar{x}' = \bar{x}$. If $\bar{x}$ is a long hyperedge of M and $\bar{y}$ is a long hyperedge of N, and $M(\bar{x}) = N(\bar{y})$, then there is a local isomorphism $\theta : v_M(\bar{x}) \to v_N(\bar{y})$ such that $\theta(x_i) = y_i$ for all $i < |x|$. For any $x \in \mathsf{nodes}(M) \sim v_M(\bar{x})$ and $S \subseteq v_M(\bar{x})$, if (x, s) belong to $\forall$ in M for all $s \in S$, then $|S| \leq 2$. Next, we proceed inductively with the inductive hypothesis exactly as before, except that now each N_r is a λ–neat hypergraph. All what remains is the amalgamation move. With the above claim at hand, this turns out an easy task to implement guided by $\exists$ s winning strategy in the graph part.
We consider an amalgamation move at round $0 < r$, (N_s, N_t) chosen by $\forall$ in round $r + 1$, $\exists$

has to deliver an amalgam N_{r+1}. $\exists$ lets $\mathsf{nodes}(N_{r+1}) = \mathsf{nodes}(N_s) \cup \mathsf{nodes}(N_t)$, then she, for a start, has to choose a colour for each edge (i,j) where $i \in \mathsf{nodes}(N_s) \sim \mathsf{nodes}(N_t)$ and $j \in \mathsf{nodes}(N_t) \sim \mathsf{nodes}(N_s)$. Let $\bar{x}$ enumerate $\mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t)$. If $\bar{x}$ is short, then there are at most two nodes in the intersection and this case is identical to the cylindrifier move. If not, that is if $\bar{x}$ is long in N_s, then by the claim there is a partial isomorphism $\theta : v_{N_s}(\bar{x}) \to v_{N_t}(\bar{x})$ fixing $\bar{x}$. We can assume that $v_{N_s}(\bar{x}) = \mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t) = \mathsf{rng}(\bar{x}) = v_{N_t}(\bar{x})$. It remains to label the edges $(i,j) \in N_{r+1}$ where $i \in \mathsf{nodes}(N_s) \sim \mathsf{nodes}(N_t)$ and $j \in \mathsf{nodes}(N_t) \sim \mathsf{nodes}(N_s)$. Her strategy is now again similar to the cylindrifier move. If i and j are tints of the same cone she chooses a red using ρ_{r+1} (constructed inductively like in the above proof), if not she chooses a white. She never chooses a green. Concerning $n-1$ tuples she needs to label $n-1$ hyperedges by shades of yellow. For each tuple $\bar{a} = a_0, \ldots a_{n-2} \in N_{r+1}$, with no edge (a_i, a_j) coloured green (we have already labelled edges), then $\exists$ colours $\bar{a}$ by y_S, where

$$S = \{i \in \mathbb{Z} : \text{ there is an } i \text{ cone in } N_{r+1} \text{ with base } \bar{a}\}.$$

We have shown that $\exists$ has a winning strategy in $\mathbf{H}_k(\mathsf{At}\mathfrak{C})$ for each finite k.

(3) Finishing the proof: All games used are deterministic. For each $k < \omega$, let σ_k describe the winning strategy of $\mathbf{H}_k(\alpha)$. Let $\mathfrak{C} = \mathfrak{Tm}\alpha$. Let $\mathfrak{D}$ be a non–principal ultrapower of $\mathfrak{C}$. Then $\exists$ has a winning strategy σ in $\mathbf{H}_\omega(\mathsf{At}\mathfrak{D})$ — essentially she uses σ_k in the k'th component of the ultraproduct so that at each round of $\mathbf{H}_\omega(\mathsf{At}\mathfrak{D})$, $\exists$ is still winning in co–finitely many components, this suffices to show she has still not lost. Now one can use an elementary chain argument to construct countable elementary subalgebras $\mathfrak{C} = \mathfrak{A}_0 \preceq \mathfrak{A}_1 \preceq \ldots \preceq \ldots \mathfrak{D}$ in the following way. One defines $\mathfrak{A}_{i+1}$ to be a countable elementary subalgebra of $\mathfrak{D}$ containing $\mathfrak{A}_i$ and all elements of $\mathfrak{D}$ that σ selects in a play of $\mathbf{H}_\omega(\mathsf{At}\mathfrak{D})$ in which $\forall$ only chooses elements from $\mathfrak{A}_i$.

Now let $\mathfrak{B} = \bigcup_{i<\omega} \mathfrak{A}_i$. This is a countable elementary subalgebra of $\mathfrak{D}$, hence necessarily atomic, and $\exists$ has a winning strategy in $\mathbf{H}_\omega(\mathsf{At}\mathfrak{B})$. (Cf. [8, Theorem 3.3.4 and Corollary 3.3.5] for a similar argument). So by Lemma 5.3 (using $\mathsf{At}\mathfrak{B}$ in place of α), we get that $\mathfrak{CmAt}\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$. Since $\mathfrak{B} \subseteq_d \mathfrak{CmAt}\mathfrak{B}$, then $\mathfrak{B} \in \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega$ and by Lemma 5.3, we also have that $\mathfrak{B} \in \mathsf{CRCA}_n$. But $\forall$ has a winning strategy in $\mathbf{G}^m(\mathsf{At}\mathfrak{B})$, so by lemma 2.4, $\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$. To finalize the proof, let $\mathbf{K}$ be as given. Then $\mathfrak{B} \equiv \mathfrak{C}$, $\mathfrak{B} \in \mathbf{K}(\subseteq \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n)$, but $\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}(\supseteq \mathbf{K})$ giving that $\mathbf{K}$ is not elementary.

$\square$

Theorem 5.5. *1. There is a finite $k \geq 2$, such that for all $m \geq n+k$ the class of frames $\mathsf{Str}(\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m) = \{\mathfrak{F} : \mathfrak{Cm}\mathfrak{F} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m\}$ is not elementary. An entirely analogous result holds for* RA*s,*

2. Let $\mathbf{O} \in \{\mathbf{S}_c, \mathbf{S}_d, \mathbf{I}\}$ and $k \geq 3$. Then the class of frames $\mathsf{K}_k = \{\mathfrak{F} : \mathfrak{Cm}\mathfrak{F} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_{n+k}\}$ is not elementary.

Proof. 1: We show that $\mathsf{Str}(\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m)$ is not elementary for some finite $m \geq n+2$. . Let $(\mathfrak{A}_i : i \in \omega)$ be a sequence of (strongly) representable CA_ns with $\mathfrak{CmAt}\mathfrak{A}_i = \mathfrak{A}_i$ and $\mathfrak{A} = \Pi_{i/U}\mathfrak{A}_i$ is not strongly representable with respect to any non-principal ultrafilter U on ω. Such algebras exist [8]. Hence $\mathfrak{CmAt}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_\omega = \bigcap_{i\in\omega} \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+i}$, so $\mathfrak{CmAt}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$ for all $l > m$, for some $m \in \omega$, $m \geq n+2$. But for each such l, $\mathfrak{A}_i \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l(\subseteq \mathsf{RCA}_n)$, so $(\mathfrak{A}_i : i \in \omega)$ is a sequence of algebras such that $\mathfrak{CmAt}(\mathfrak{A}_i) \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$ $(i \in I)$, but $\mathfrak{Cm}(\mathsf{At}(\Pi_{i/U}\mathfrak{A}_i)) = \mathfrak{CmAt}(\mathfrak{A}) \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$, for all $l \geq m$.

2: We use the same construction (and notation) as above. It suffices to show that the class of algebras $\mathbf{K}_k = \{\mathfrak{A} \in \mathsf{CA}_n \cap \mathbf{At} : \mathfrak{CmAt}\mathfrak{A} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_k\}$ is not elementary. $\exists$ has a winning strategy in $\mathbf{H}_\omega(\alpha)$ for some countable atom structure α, $\mathfrak{Tm}\alpha \subseteq_d \mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$ and $\mathfrak{Tm}\alpha \in \mathsf{CRCA}_n$. Since $\mathfrak{C}_{\mathbb{Z},\mathbb{N}} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$, then $\mathfrak{C}_{\mathbb{Z},\mathbb{N}} = \mathfrak{Cm}\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathbb{N}} \notin \mathbf{K}_k$, $\mathfrak{C}_{\mathbb{Z},\mathbb{N}} \equiv \mathfrak{Tm}\alpha$ and $\mathfrak{Tm}\alpha \in \mathbf{K}_k$ because $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega$. We have shown that $\mathfrak{C}_{\mathbb{Z},\mathbb{N}} \in \mathbf{El}\mathbf{K}_k \sim \mathbf{K}_k$, proving the required. □

We state an easy lemma towards strengthening Lemma 5.4. If $\mathfrak{B}$ is a Boolean algebra and $b \in \mathfrak{B}$, then $\mathfrak{Rl}_b\mathfrak{B}$ denotes the Boolean algebra with domain $\{x \in B : x \leq b\}$, top element b, and other Boolean operations those of $\mathfrak{B}$ relativized to b.

Lemma 5.6. *In the following $\mathfrak{A}$ and $\mathfrak{D}$ are Boolean algebras.*

1. *If $\mathfrak{A}$ is atomic and $0 \neq a \in \mathfrak{A}$, then $\mathfrak{Rl}_a\mathfrak{A}$ is also atomic. If $\mathfrak{A} \subseteq_d \mathfrak{D}$, and $a \in A$, then $\mathfrak{Rl}_a\mathfrak{A} \subseteq_d \mathfrak{Rl}_a\mathfrak{D}$,*

2. *If $\mathfrak{A} \subseteq_d \mathfrak{D}$ then $\mathfrak{A} \subseteq_c \mathfrak{D}$. In particular, for any class K of BAOs, $\mathsf{K} \subseteq \mathbf{S}_d\mathsf{K} \subseteq \mathbf{S}_c\mathsf{K}$. If furthermore $\mathfrak{A}$ and $\mathfrak{D}$ are atomic, then $\mathsf{At}\mathfrak{D} \subseteq \mathsf{At}\mathfrak{A}$.*

Proof. (1): Let $b \in \mathfrak{Rl}_a\mathfrak{D}$ be non–zero. Then $b \leq a$ and b is non-zero in $\mathfrak{D}$. By atomicity of $\mathfrak{D}$ there is an atom c of $\mathfrak{D}$ such that $c \leq b$. So $c \leq b \leq a$, thus $c \in \mathfrak{Rl}_a\mathfrak{D}$. Also c is an atom in $\mathfrak{Rl}_a\mathfrak{D}$ because if not, then it will not be an atom in $\mathfrak{D}$. The second part is similar.

(2): Assume that $\sum^{\mathfrak{A}} S = 1$ and for contradiction that there exists $b' \in \mathfrak{D}$, $b' < 1$ such that $s \leq b'$ for all $s \in S$. Let $b = 1 - b'$ then $b \neq 0$, hence by assumption (density) there exists a non-zero $a \in \mathfrak{A}$ such that $a \leq b$, i.e $a \leq (1 - b')$. If $a \cdot s \neq 0$ for some $s \in S$, then a is not less than b' which is impossible. So $a \cdot s = 0$ for every $s \in S$, implying that $a = 0$, contradiction. Now we prove the second part. Assume that $\mathfrak{A} \subseteq_d \mathfrak{D}$ and $\mathfrak{D}$ is atomic. Let $b \in \mathfrak{D}$ be an atom. We show that $b \in \mathsf{At}\mathfrak{A}$. By density there is a non–zero $a' \in \mathfrak{A}$, such that $a' \leq b$ in $\mathfrak{D}$. Since $\mathfrak{A}$ is atomic, there is an atom $a \in \mathfrak{A}$ such that $a \leq a' \leq b$. But b is an atom of $\mathfrak{D}$, and a is non–zero in $\mathfrak{D}$, too, so it must be the case that $a = b \in \mathsf{At}\mathfrak{A}$. Thus $\mathsf{At}\mathfrak{B} \subseteq \mathsf{At}\mathfrak{A}$ and we are done. □

The next Lemma strengthens the main Theorem in [13], and will be used later.

Lemma 5.7. *Let $1 < n < \omega$. There are two atomic cylindric algebras $\mathfrak{A}$ and $\mathfrak{B}$ having uncountably many atoms, such that $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, $\mathfrak{A} \equiv_{\infty,\omega} \mathfrak{B}$ and $\mathfrak{B} \notin \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_{n+1}$.*

Proof. We first need to slighty modify the construction in [13, Lemma 5.1.3, Theorem 5.1.4] reformulating it as a 'splitting argument'. The algebras $\mathfrak{A}$ and $\mathfrak{B}$ constructed in *op.cit* satisfy that $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, $\mathfrak{B} \notin \mathsf{Nr}_n\mathsf{CA}_{n+1}$ and $\mathfrak{A} \equiv \mathfrak{B}$. As they stand, $\mathfrak{A}$ and $\mathfrak{B}$ are not atomic, but they it can be fixed that they are to be so giving the same result, by interpreting the uncountably many tenary relations in the signature of M defined in [13, Lemma 5.1.3], which is the base of $\mathfrak{A}$ and $\mathfrak{B}$ to be *disjoint* in M, not just distinct. The construction is presented this way in [12], where (the equivalent of) M is built in a more basic step-by–step fashon. We work with $2 < n < \omega$ instead of only $n = 3$. The proof presented in *op.cit* lift verbatim to any such n. Let $u \in {}^nn$. Write $\mathbf{1}_u$ for χ_u^{M} (denoted by 1_u (for $n = 3$) in [13, Theorem 5.1.4].) We denote by $\mathfrak{A}_u$ the Boolean algebra $\mathfrak{Rl}_{\mathbf{1}_u}\mathfrak{A} = \{x \in \mathfrak{A} : x \leq \mathbf{1}_u\}$ and similarly for $\mathfrak{B}$, writing $\mathfrak{B}_u$ short hand for the Boolean algebra $\mathfrak{Rl}_{\mathbf{1}_u}\mathfrak{B} = \{x \in \mathfrak{B} : x \leq \mathbf{1}_u\}$. We show that $\exists$ has a winning strategy in an Ehrenfeucht–Fraïssé-game over $(\mathfrak{A}, \mathfrak{B})$ concluding that $\mathfrak{A} \equiv_\infty \mathfrak{B}$. At any stage of the game, if $\forall$ places a pebble on one of $\mathfrak{A}$ or $\mathfrak{B}$, $\exists$ must place a matching pebble, on the other algebra. Let $\bar{a} = \langle a_0, a_1, \ldots, a_{n-1}\rangle$ be the position of the pebbles played so far (by either player) on $\mathfrak{A}$ and let $\bar{b} = \langle b_0, \ldots, b_{n-1}\rangle$ be the the position of the pebbles played on $\mathfrak{B}$. $\exists$ maintains the following properties throughout the game: For any atom x (of either algebra) with $x \cdot \mathbf{1}_{Id} = 0$ then $x \in a_i \iff x \in b_i$ and $\bar{a}$ induces a finite partion of $\mathbf{1}_{Id}$ in $\mathfrak{A}$ of 2^n (possibly empty) parts $p_i : i < 2^n$ and $\bar{b}$ induces a partion of $\mathbf{1}_{Id}$ in $\mathfrak{B}$ of parts $q_i : i < 2^n$. Furthermore, p_i is finite $\iff q_i$ is finite and, in this case, $|p_i| = |q_i|$. That such properties can be maintained is fairly easy to show. Using that M has quantifier elimination we get, using the same argument in *op.cit* that $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$. The property that $\mathfrak{B} \notin \mathsf{Nr}_n\mathsf{CA}_{n+1}$ is also still maintained. To see why consider the substitutin operator ${}_n\mathsf{s}(0,1)$ (using one spare dimension) as defined in the proof of [13, Theorem 5.1.4]. Assume for contradiction that $\mathfrak{B} = \mathsf{Nr}_n\mathfrak{C}$, with $\mathfrak{C} \in \mathsf{CA}_{n+1}$. Let $u = (1, 0, 2, \ldots n-1)$. Then $\mathfrak{A}_u = \mathfrak{B}_u$ and so $|\mathfrak{B}_u| > \omega$. The term ${}_n\mathsf{s}(0,1)$ acts like a substitution operator corresponding to the transposition $[0,1]$; it 'swaps' the first two co–ordinates. Now one can show that ${}_n\mathsf{s}(0,1)^{\mathfrak{C}}\mathfrak{B}_u \subseteq \mathfrak{B}_{[0,1]\circ u} = \mathfrak{B}_{Id}$, so $|{}_n\mathsf{s}(0,1)^{\mathfrak{C}}\mathfrak{B}_u|$ is countable because $\mathfrak{B}_{Id}$ was forced by construction to be countable. But ${}_n\mathsf{s}(0,1)$ is a Boolean automorpism with inverse ${}_n\mathsf{s}(1,0)$, so that $|\mathfrak{B}_u| = |{}_n\mathsf{s}(0,1)^{\mathfrak{C}}\mathfrak{B}_u| > \omega$, contradiction.

Now we show that the algebra $\mathfrak{B}$ outside $\mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \supseteq \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$. Take κ the signature of M to be 2^{2^ω} and assume for contradiction that $\mathfrak{B} \in \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$. Then $\mathfrak{B} \subseteq_d \mathfrak{Nr}_n\mathfrak{D}$, for some $\mathfrak{D} \in \mathsf{CA}_\omega$ and $\mathfrak{Nr}_n\mathfrak{D}$ is atomic. For brevity, let $\mathfrak{C} = \mathfrak{Nr}_n\mathfrak{D}$. Then by the first item of Lemma 5.6 $\mathfrak{Rl}_{Id}\mathfrak{B} \subseteq_d \mathfrak{Rl}_{Id}\mathfrak{C}$. Since $\mathfrak{C}$ is atomic, then by the following item of the same Lemma $\mathfrak{Rl}_{Id}\mathfrak{C}$ is also atomic. Using the same reasoning as above, we get that $|\mathfrak{Rl}_{Id}\mathfrak{C}| > 2^\omega$ (since $\mathfrak{C} \in \mathsf{Nr}_n\mathsf{CA}_\omega$.) By the choice of κ, we get that $|\mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{C}| > \omega$. By density using Lemma 5.6, $\mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{C} \subseteq \mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{B}$. But by the construction of $\mathfrak{B}$, we have $|\mathfrak{Rl}_{Id}\mathfrak{B}| = |\mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{B}| = \omega$, which is a contradiction and we are done. □

In the following $\mathbf{Up}$, $\mathbf{Ur}$, $\mathbf{P}$ and $\mathbf{H}$ denote the operations of forming ultraproducts, ultraroots, products and homomorphic images, respectively.

Theorem 5.8. *1. Any class $\mathbf{K}$ such that $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n \subseteq \mathbf{K} \subseteq \mathsf{CRCA}_n \cap \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ or any class $\mathbf{K}$ between $\mathsf{CRCA}_n \cap \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega$ and $S_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$, $\mathbf{K}$ is not elementary.*

2. Any class K such that $\mathsf{AtNr}_n\mathsf{CA}_\omega \subseteq \mathsf{K} \subseteq \mathsf{AtNr_n}\mathsf{CA}_\omega$ is not elementary.

Proof. 1. Two atomic algebras $\mathfrak{A}$ and $\mathfrak{B}$ are constructed in Lemma 5.7 such that, $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, $\mathfrak{B} \notin \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_{n+1}$ and $\mathfrak{A} \equiv \mathfrak{B}$. Thus $\mathfrak{B} \in \mathbf{El}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n) \sim \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega$. Since $\mathbf{El}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n) \nsubseteq \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$, there can be no elementary class between $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$. Having already eliminated elementary classes between $\mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$, we are done.

2. We prove the following: α be a countable atom structure. If $\exists$ has a winning strategy in $\mathbf{H}_\omega(\alpha)$, then any algebra $\mathfrak{F}$ having atom structure α is completely representable and there exists a complete $\mathfrak{D} \in \mathsf{RCA}_\omega$ such that $\alpha \cong \mathsf{AtNr}_n\mathfrak{D}$. In particular, $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$ and $\alpha \in \mathsf{AtNr}_n\mathsf{CA}_\omega$. Combined with the proof of theorem 5.4 we will be done. For this purpose, let $x \in \mathfrak{D}$ formed as above. Then $x = (x_a : a \in \alpha)$, where $x_a \in \mathfrak{D}_a$. For $b \in \alpha$ let $\pi_b : \mathfrak{D} \to \mathfrak{D}_b$ be the projection map defined by $\pi_b(x_a : a \in \alpha) = x_b$. Conversely, let $\iota_a : \mathfrak{D}_a \to \mathfrak{D}$ be the embedding defined by $\iota_a(y) = (x_b : b \in \alpha)$, where $x_a = y$ and $x_b = 0$ for $b \neq a$. Suppose $x \in \mathfrak{Nr}_n\mathfrak{D} \setminus \{0\}$. Since $x \neq 0$, then it has a non-zero component $\pi_a(x) \in \mathfrak{D}_a$, for some $a \in \alpha$. Assume that $\emptyset \neq \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a} = \pi_a(x)$, for some L-formula $\phi(x_{i_0}, \ldots, x_{i_{k-1}})$. We have $\phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a} \in \mathfrak{Nr}_n\mathfrak{D}_a$. Pick $f \in \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a}$ and assume that $\mathcal{M}_a, f \models b(x_0, \ldots x_{n-1})$ for some $b \in \alpha$. We show that $b(x_0, x_1, \ldots, x_{n-1})^{\mathfrak{D}_a} \subseteq \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a}$. Take any $g \in b(x_0, x_1 \ldots, x_{n-1})^{\mathfrak{D}_a}$, so that $\mathcal{M}_a, g \models b(x_0, \ldots x_{n-1})$. The map $\{(f(i), g(i)) : i < n\}$ is a partial isomorphism of $\mathcal{M}_a$. Here that short hyperedges are constantly labelled by λ is used. This map extends to a finite partial isomorphism θ of M_a whose domain includes

$f(i_0), \ldots, f(i_{k-1})$. Let $g' \in \mathcal{M}_a$ be defined by

$$g'(i) = \begin{cases} \theta(i) & \text{if } i \in \mathsf{dom}(\theta) \\ g(i) & \text{otherwise} \end{cases}$$

We have $\mathcal{M}_a, g' \models \phi(x_{i_0}, \ldots, x_{i_{k-1}})$. But $g'(0) = \theta(0) = g(0)$ and similarly $g'(n-1) = g(n-1)$, so g is identical to g' over n and it differs from g' on only a finite set. Since $\phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a} \in \mathfrak{Nr}_n\mathfrak{D}_a$, we get that $\mathcal{M}_a, g \models \phi(x_{i_0}, \ldots, x_{i_k})$, so $g \in \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a}$ (this can be proved by induction on quantifier depth of formulas). This proves that

$$b(x_0, x_1 \ldots x_{n-1})^{\mathfrak{D}_a} \subseteq \phi(x_{i_0}, \ldots, x_{i_k})^{\mathfrak{D}_a} = \pi_a(x),$$

and so

$$\iota_a(b(x_0, x_1, \ldots x_{n-1})^{\mathfrak{D}_a}) \leq \iota_a(\phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a}) \leq x \in \mathfrak{D}_a \setminus \{0\}.$$

Now every non–zero element x of $\mathfrak{Nr}_n\mathfrak{D}_a$ is above a non–zero element of the following form $\iota_a(b(x_0, x_1, \ldots, x_{n-1})^{\mathfrak{D}_a})$ (some $a, b \in \alpha$) and these are the atoms of $\mathfrak{Nr}_n\mathfrak{D}_a$. The map defined via $b \mapsto (b(x_0, x_1, \ldots, x_{n-1})^{\mathfrak{D}_a} : a \in \alpha)$ is an isomorphism of atom structures, so that $\alpha \in \mathsf{AtNr}_n\mathsf{CA}_\omega$. □

6 Other notions of representability

Theorem 6.1. *Let κ be an infinite cardinal. Then there exists an atomless $\mathfrak{C} \in \mathsf{CA}_\omega$ such that for all $2 < n < \omega$, $\mathfrak{Nr}_n\mathfrak{C}$ is atomic, with $|\mathsf{At}(\mathfrak{Nr}_n\mathfrak{C})| = 2^\kappa$, $\mathfrak{Nr}_n\mathfrak{C} \in \mathsf{LCA}_n$, but $\mathfrak{Nr}_n\mathfrak{C}$ is not completely representable.*

Proof. We use the following uncountable version of Ramsey's theorem due to Erdos and Rado: If $r \geq 2$ is finite, k an infinite cardinal, then $exp_r(k)^+ \to (k^+)^{r+1}_k$ where $exp_0(k) = k$ and inductively $exp_{r+1}(k) = 2^{exp_r(k)}$. The above partition symbol describes the following statement. If f is a coloring of the $r+1$ element subsets of a set of cardinality $exp_r(k)^+$ in k many colors, then there is a homogeneous set of cardinality k^+ (a set, all whose $r+1$ element subsets get the same f-value). Let κ be the given cardinal. We use a variation on the construction which is a simplified more basic version of a rainbow construction where only the two predominent colours, namely, the reds and blues are available. The algebra $\mathfrak{C}$ will be constructed from a relation algebra possesing an ω-dimensional cylindric basis. To define the relation algebra we specify its atoms and the forbidden triples of atoms. The atoms are Id, $\mathsf{g}_0^i : i < 2^\kappa$ and $\mathsf{r}_j : 1 \leq j < \kappa$, all symmetric. The forbidden triples of atoms are all permutations of (Id, x, y) for $x \neq y$, $(\mathsf{r}_j, \mathsf{r}_j, \mathsf{r}_j)$ for $1 \leq j < \kappa$ and $(\mathsf{g}_0^i, \mathsf{g}_0^{i'}, \mathsf{g}_0^{i^*})$ for $i, i', i^* < 2^\kappa$. Write g_0 for $\{\mathsf{g}_0^i : i < 2^\kappa\}$ and r_+ for $\{\mathsf{r}_j : 1 \leq j < \kappa\}$. Call this atom structure α. Consider the term algebra $\mathfrak{R}$ defined to be the subalgebra of the complex algebra of this atom structure generated by the atoms. We claim that $\mathfrak{R}$, as a relation algebra, has no complete representation, hence any algebra sharing this atom structure is not completely representable, too.

Assume for contradiction that $\mathfrak{R}$ has a complete representation M. Let x, y be points in the representation with $\mathsf{M} \models \mathsf{r}_1(x, y)$. For each $i < 2^\kappa$, there is a point $z_i \in \mathsf{M}$ such that $\mathsf{M} \models \mathsf{g}_0^i(x, z_i) \wedge \mathsf{r}_1(z_i, y)$. Let $Z = \{z_i : i < 2^\kappa\}$. Within Z, each edge is labelled by one of the κ atoms in r_+. The Erdos-Rado theorem forces the existence of three points $z^1, z^2, z^3 \in Z$ such that $\mathsf{M} \models \mathsf{r}_j(z^1, z^2) \wedge \mathsf{r}_j(z^2, z^3) \wedge \mathsf{r}_j(z^3, z_1)$, for some single $j < \kappa$. This contradicts the definition of composition in $\mathfrak{R}$ (since we avoided monochromatic triangles). Let S be the set of all atomic $\mathfrak{R}$-networks N with nodes ω such that $\{\mathsf{r}_i : 1 \leq i < \kappa : \mathsf{r}_i \text{ is the label of an edge in } N\}$ is finite.

Then it is straightforward to show S is an amalgamation class, that is for all $M, N \in S$ if $M \equiv_{ij} N$ then there is $L \in S$ with $M \equiv_i L \equiv_j N$, witness [7, Definition 12.8] for notation. Now let X be the set of finite $\mathfrak{R}$-networks N with nodes $\subseteq \kappa$ such that:

1. each edge of N is either (a) an atom of $\mathfrak{R}$ or (b) a cofinite subset of $\mathsf{r}_+ = \{\mathsf{r}_j : 1 \leq j < \kappa\}$ or (c) a cofinite subset of $\mathsf{g}_0 = \{\mathsf{g}_0^i : i < 2^\kappa\}$ and
2. N is 'triangle-closed', i.e. for all $l, m, n \in \mathsf{nodes}(N)$ we have $N(l,n) \leq N(l,m); N(m,n)$. That means if an edge (l,m) is labelled by Id then $N(l,n) = N(m,n)$ and if $N(l,m), N(m,n) \leq \mathsf{g}_0$ then $N(l,n) \cdot \mathsf{g}_0 = 0$ and if $N(l,m) = N(m,n) = \mathsf{r}_j$ (some $1 \leq j < \omega$) then $N(l,n) \cdot \mathsf{r}_j = 0$.

For $N \in X$ let $\widehat{N} \in \mathfrak{Ca}(S)$ be defined by

$$\{L \in S : L(m,n) \leq N(m,n) \text{ for } m, n \in \mathsf{nodes}(N)\}.$$

For $i \in \omega$, let $N\restriction_{-i}$ be the subgraph of N obtained by deleting the node i. Then if $N \in X$, $i < \omega$ then $\widehat{\mathsf{c}_i N} = \widehat{N\restriction_{-i}}$. The inclusion $\widehat{\mathsf{c}_i N} \subseteq (\widehat{N\restriction_{-i}})$ is clear. Conversely, let $L \in (\widehat{N\restriction_{-i}})$. We seek $M \equiv_i L$ with $M \in \widehat{N}$. This will prove that $L \in \widehat{\mathsf{c}_i N}$, as required. Since $L \in S$ the set $T = \{\mathsf{r}_i \notin L\}$ is infinite. Let T be the disjoint union of two infinite sets $Y \cup Y'$, say. To define the ω-network M we must define the labels of all edges involving the node i (other labels are given by $M \equiv_i L$). We define these labels by enumerating the edges and labeling them one at a time. So let $j \neq i < \kappa$. Suppose $j \in \mathsf{nodes}(N)$. We must choose $M(i,j) \leq N(i,j)$. If $N(i,j)$ is an atom then of course $M(i,j) = N(i,j)$. Since N is finite, this defines only finitely many labels of M. If $N(i,j)$ is a cofinite subset of g_0 then we let $M(i,j)$ be an arbitrary atom in $N(i,j)$. And if $N(i,j)$ is a cofinite subset of r_+ then let $M(i,j)$ be an element of $N(i,j) \cap Y$ which has not been used as the label of any edge of M which has already been chosen (possible, since at each stage only finitely many have been chosen so far). If $j \notin \mathsf{nodes}(N)$ then we can let $M(i,j) = \mathsf{r}_k \in Y$ some $1 \leq k < \kappa$ such that no edge of M has already been labelled by r_k. It is not hard to check that each triangle of M is consistent (we have avoided all monochromatic triangles) and clearly $M \in \widehat{N}$ and $M \equiv_i L$. The labeling avoided all but finitely many elements of Y', so $M \in S$. So $(\widehat{N\restriction_{-i}}) \subseteq \widehat{\mathsf{c}_i N}$.

Now let $\widehat{X} = \{\widehat{N} : N \in X\} \subseteq \mathfrak{Ca}(S)$. Then the subalgebra of $\mathfrak{Ca}(S)$ generated by $\widehat{X}$ is simply obtained from $\widehat{X}$ by closing under finite unions. Thus $\mathfrak{R}$ is relation algebra reduct of $\mathfrak{C} \in \mathsf{CA}_\omega$ but has no complete representation. Let $n > 2$. Let $\mathfrak{B} = \mathfrak{Nr}_n\mathfrak{C}$. Then $\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, is atomic, but has no complete representation for plainly a complete representation of $\mathfrak{B}$ induces one of $\mathfrak{R}$. In fact, because $\mathfrak{B}$ is generated by its two dimensional elements, and its dimension is at least three, its Df reduct is not completely representable. We show that the ω–dilation $\mathfrak{C}$ is atomless. For any $N \in X$, we can add an extra node extending N to M such that $\emptyset \subsetneq M' \subsetneq N'$, so that N' cannot be an atom in $\mathfrak{C}$. Then $\mathfrak{Nr}_n\mathfrak{C}$ $(2 < n < \omega)$ is atomic, but has no complete representation. By Lemma 2.4, $\exists$ has a winning strategy in $\mathbf{G}_\omega(\mathsf{At}\mathfrak{Nr}_n\mathfrak{C})$, hence she has a winning strategy in $G_\omega(\mathsf{At}\mathfrak{Nr}_n\mathfrak{C})$, *a fortiori* in $G_k(\mathsf{At}\mathfrak{Nr}_n\mathfrak{C})$ for all $k \in \omega$, hence by coding the winning strategy's of the G_k's in first order sentences, we get that $\mathfrak{Nr}_n\mathfrak{C}$ satisfies these first order sentences which are precisely (by definition) the Lyndon conditions. $\square$

By observing from the last part of the proof of the previous Theorem that $\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathsf{LCA}_n(= \mathbf{El}\mathsf{CRCA}_n)$ and similarly for RAs, we have $\mathsf{RaCA}_\omega \subseteq \mathsf{LRRA} = (\mathbf{El}\mathsf{CRRA})$, we immeadiately get:

Corollary 6.2. *(Hirsch and Hodkinson) For $2 < n < \omega$, the classes CRCA_n and CRRA are not elementary.*

Theorem 6.3. *For $2 < n < \omega$ the following hold:*

1. $\mathsf{CRCA}_n \subseteq \mathbf{S}_c\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At}$. *At least two of the last three claseses are distinct.*

2. *All reverse inclusions in the first item hold if atomic algebras considerered have countably many atoms.*

3. *All classes in the first item are closed under $\mathbf{S}_c$ a fortiori $\mathbf{S}_d$, $\mathbf{P}$ (products), but are not closed under $\mathbf{S}$ nor $\mathbf{H}$ (homomorphic images) nor $\mathbf{Ur}$ (ultraroots). Their elementary closure coincides wih LCA_n.*

4. $\mathsf{Nr}_n\mathsf{CA}_\omega \subsetneq \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \subsetneq \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathbf{ELS}_c\mathsf{Nr}_n\mathsf{CA}_\omega \subsetneq \mathsf{RCA}_n$. *Furthermote strictness of inclusions can be witnessed by atomic algebras.*

Proof. We prove more considering infinite dimensions. To adapt to the infinite dimensional case, we define complete representability with respect to (generalized) weak set algebras of the same infinite dimension in the sense of [5]. This coincides with the usual definition for finite dimensions for in the finite dimensional case obviously weak set algebras are just set algebra (of the same dimension). Suppose that $\mathfrak{A}$ has complete representation. Assume that M is the base of a complete representation of $\mathfrak{A}$, whose the unit is a weak generalized space, that is, $1^{\mathsf{M}} = \bigcup {}^\alpha U_i^{(p_i)}$ $p_i \in {}^\alpha U_i$, where ${}^\alpha U_i^{(p_i)} \cap {}^\alpha U_j^{(p_j)} = \emptyset$ for distinct i and j, in some index set I, that is, we have an isomorphism $t : \mathfrak{B} \to \mathfrak{C}$, where $\mathfrak{C} \in \mathsf{Gs}_\alpha$ has unit 1^{M}, and t preserves arbitrary meets carrying them to set–theoretic intersections. For $i \in I$, let $E_i = {}^\alpha U_i^{(p_i)}$. Take $f_i \in {}^{\alpha+\omega} U_i^{(q_i)}$ where $q_i \upharpoonright \alpha = p_i$ and let $W_i = \{f \in {}^{\alpha+\omega} U_i^{(q_i)} : |\{k \in \alpha + \omega : f(k) \neq f_i(k)\}| < \omega\}$. Let $\mathfrak{C}_i = \wp(W_i)$. Then $\mathfrak{C}_i$ is atomic; indeed the atoms are the singletons. Let $x \in \mathfrak{Nr}_\alpha\mathfrak{C}_i$, that is $\mathsf{c}_i x = x$ for all $\alpha \leq i < \alpha + \omega$. Now if $f \in x$ and $g \in W_i$ satisfy $g(k) = f(k)$ for all $k < \alpha$, then $g \in x$. Hence $\mathfrak{Nr}_\alpha\mathfrak{C}_i$ is atomic; its atoms are $\{g \in W_i : \{g(i) : i < \alpha\} \subseteq U_i\}$. Define $h_i : \mathfrak{A} \to \mathfrak{Nr}_\alpha\mathfrak{C}_i$ by $h_i(a) = \{f \in W_i : \exists a' \in \mathsf{At}\mathfrak{A}, a' \leq a; (f(i) : i < \alpha) \in t(a')\}$. Let $\mathfrak{D} = \mathbf{P}_i\mathfrak{C}_i$. Let $\pi_i : \mathfrak{D} \to \mathfrak{C}_i$ be the ith projection map. Now clearly $\mathfrak{D}$ is atomic, because it is a product of atomic algebras, and its atoms are $(\pi_i(\beta) : \beta \in \mathsf{At}(\mathfrak{C}_i))$. Now $\mathfrak{A}$ embeds into $\mathfrak{Nr}_\alpha\mathfrak{D}$ via $J : a \mapsto (\pi_i(a) : i \in I)$. If $x \in \mathfrak{Nr}_\alpha\mathfrak{D}$, then for each i, we have $\pi_i(x) \in \mathfrak{Nr}_\alpha\mathfrak{C}_i$, and if x is non–zero, then $\pi_i(x) \neq 0$. By atomicity of $\mathfrak{C}_i$, there is an α–ary tuple y, such that $\{g \in W_i : g(k) = y_k\} \subseteq \pi_i(x)$. It follows that there is an atom of $b \in \mathfrak{A}$, such that $x \cdot J(b) \neq 0$, and so the embedding is atomic, hence complete. We have shown that $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\omega}$ and we are done.

(2) By [13, Theorem 5.3.6] the class CRCA_n coincides wit the class $\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega$ on atomic algebras with countably many atoms. Then together with [8, Theorem 3.3.3] we are done.

(3) We start with CRCA_n. Closure under $\mathbf{P}$ is straightforward. We show that $\mathbf{S}_c\mathsf{CRCA}_n = \mathsf{CRCA}_n$. Assume that $\mathfrak{D} \in \mathsf{CRCA}_n$ and $\mathfrak{A} \subseteq_c \mathfrak{D}$. Identifying set algebras with their domain, let $f : \mathfrak{D} \to \wp(V)$ be a complete representation of $\mathfrak{A}$ where V is a Gs_n unit. We claim that $g = f \upharpoonright \mathfrak{A}$ establishes the required complete representation of $\mathfrak{A}$. Let $X \subseteq A$, then for $x \in X(\subseteq \mathfrak{A})$, we have $f(x) = g(x)$, so that $\bigcup_{x \in X} g(x) = \bigcup_{x \in X} f(x) = V$, since it is given that f is a complete representation and we are done. Let C be any of the two remaining classes. Closure under $\mathbf{S}_c$ follows from that $\mathbf{S}_c\mathbf{S}_c\mathsf{C} = \mathbf{S}_c\mathsf{C}$. Closure under $\mathbf{P}$ follows from from that $\mathbf{PS}_c\mathsf{C} \subseteq \mathbf{S}_c\mathbf{P}\mathsf{C}$, and that $\mathbf{P}\mathsf{Nr}_n\mathsf{CA}_\omega = \mathsf{Nr}_n\mathsf{CA}_\omega$. Non closure under $\mathbf{S}$ is trivial for a subalgebra of an atomic algebra may well be non atomic. We prove non closure under $\mathbf{H}$ for all three clases in one go. Take a family $(U_i i \in \mathbb{N})$ of pairwise disjoint non-empty sets. Let $V_i = {}^n U_i (i \in \mathbb{N})$. Take the full Gs_n $\mathfrak{A}$ with universe $\wp(V)$ where $V = \bigcup_{i \in N} V_i$. Then

$\mathfrak{A} \in CRCA_n \subseteq \mathsf{C}$. Let I be the ideal of elements of $\mathfrak{A}$ intersecting with only finitely many elements of the V_is. Then $\mathfrak{A}/I$ is non-atomic, and so is outside all three classes.

Now we approach closure under ultraroots (**Ur**). Let $\mathfrak{C} \in \mathsf{CA}_n \sim \mathsf{CRCA}_n$ be atomic having countable many atoms and elementary equivalent to a $\mathfrak{B} \in \mathsf{CRCA}_n$. Such algebras exist (as shown above, see e.g the algebra $\mathfrak{C}_{\mathbb{Z},N}$ used in the proof of Theorems 5.4 and 5.5, or [6]). Since all the given classes are closed under ultraproducts, it must be the case that $\mathfrak{B} \notin \mathbf{Ur}\mathsf{C}$ of any of the given three classes C, since by the Keisler- Shelah ultrapower Therorem $\mathbf{El}\mathsf{K} = \mathbf{UrUp}\mathsf{K}$. Now we show pseudo-elementarity of $\mathsf{Nr}_n\mathsf{CA}_m$ (which is known to be non elementray [12]). If m is finite, then the psuedo-elementary class $\mathsf{Nr}_n\mathsf{CA}_m$ can be defined in two sorted theory in a fairly straightforward manner. When $m = \omega$, things are slightly (but not much more) involved. One proceeds as follows defining $Nr_n\mathsf{CA}_\omega$ in a three sorted theory: Use a sort of the CA_n (c), the second sort is for the Boolean reduct of the CA_n (b), and the third sort is for the set od dimensions (δ). For any infinite ordinal μ the defining theory of for $\mathsf{Nr}_n\mathsf{CA}_\mu = \mathsf{Nr}_n\mathsf{CA}_\omega$ will include sentences requiring that the constants i^δ for $i < \omega$ are distinct, and the last two sorts defines a CA_ω. There is a function I^b from sort (c) to sort (b) and one stipulates sentences forcing that I^b is injective and respects the CA_n operations. For example for all x^c and $i < n$ $I^b(\mathsf{c}_i x^c)) = \mathsf{c}_i^b(I^b(x^c))$. One finally requires that that I^b maps *onto set of n-dimensional elements.* This can be expressed vai (*):

$$\forall y^b(\forall z^\delta(z^\delta \neq 0^\delta, \dots (n-1)^\delta \implies c^b(z^\delta, y^b) = y^b)) \iff \exists x^c(y^b = I^b(x^c))).$$

In all cases it is is clear that any algebra of the right type is the first sort of a model of this theory. Conversely, a model of this theory will consist of $\mathfrak{A} \in \mathsf{CA}_n$ (sort c) and a $\mathfrak{B} \in \mathsf{CA}_\omega$; the dimension of the last is the cardinality of the δ-sorted elementsd which is ω such that that by (*) $\mathfrak{A} = \mathsf{Nr}_n\mathfrak{B}$. Thus the three sorted theory defines the class of neat reducts. Furthermore it is clearly recursive. Recursive enumerability for both classes follows from [7, Theorem 9.37].

For the last required we show that $\mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n = \mathbf{ElS}_c\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$. Assume that $\mathfrak{A} \in \mathsf{LCA}_n$. Then, by definition, for all $k < \omega$, $\exists$ a winning strategy in the k-rounded atomic game $G_k(\mathsf{At}\mathfrak{A})$. Using ultrapowers, followed by an elementay chain argument, like in [8, Theorem 3.3.3], there exisis a countable atomic $\mathfrak{B}$, such that $\mathfrak{B} \equiv \mathfrak{A}$ and $\exists$ has a winning strategy in the ω-rounded atomic game $G_\omega(\mathsf{At}\mathfrak{B})$. So $\mathfrak{A} \in \mathbf{El}\mathsf{CRCA}_n$, because by [7, Theorem 3.3.3], $\mathfrak{B} \in \mathsf{CRCA}_n$. One next shows that $\mathbf{El}(\mathbf{S}_c)\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}) \subseteq \mathsf{LCA}_\mathbf{n}$ as follows. Assume that $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$. Then by Lemma 2.4, $\exists$ has a winning strategy in $\mathbf{G}^\omega(\mathsf{At}\mathfrak{A})$. Since we have infinitely many nodes, and infinitely many rounds, reusing the nodes in play, is superfluous, so $\exists$ has a winning strategy in the usual ω-rounded atomic game $G_\omega(\mathsf{At}\mathfrak{A})$. This obviously implies that $\exists$ has winning strategy in the k-rounded usual atomic game $G_k(\mathsf{At}\mathfrak{A})$ for all $k < \omega$. But this means that, by definition, that $\mathfrak{A}$ satifies the Lyndon conditions. We have shown that $\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \subseteq \mathsf{LCA}_\mathbf{n}$. Since LCA_n is elementary, it readily follows that $\mathbf{ELS}_c\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \subseteq \mathsf{LCA}_\mathbf{n}$.

For the last item: The algebra $\mathfrak{E}$ used in Theorem 3.6 witnesses that $\mathsf{Nr}_n\mathsf{CA}_\omega \subsetneq \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega$, because $\mathfrak{E} \notin \mathbf{El}\mathsf{Nr}_n\mathsf{CA}_\omega \supseteq \mathsf{Nr}_n\mathsf{CA}_\omega$ and $\mathfrak{E} \subseteq_d \mathfrak{B}$ where $\mathfrak{B}$ is the full CA_n with unit ${}^n\mathbb{Q}$ and universe $\wp({}^n\mathbb{Q})$. We have constructed algebras with countably many atoms in $\mathbf{ELS}_c\mathsf{Nr}_n\mathsf{CA}_\omega \sim S_c\mathsf{Nr}_n\mathsf{CA}_\omega$ like the rainbow-like algebra $\mathfrak{C}_{\mathbb{Z},N}$. Let $\mathfrak{A} \in \mathsf{RCA}_n$ be simple, countable and atomic such that $\mathfrak{CmAt}\mathfrak{A} \notin \mathsf{RCA}_n$. These algebra exist in [10] and even finer ones were constructed in Theorem 3.2. Then $\mathfrak{A} \notin \mathsf{LCA}_n$, because $\mathsf{At}\mathfrak{A}$ does not satisfy the Lyndon conditions, lest $\mathfrak{CmAt}\mathfrak{A} \in \mathsf{LCA}_n(\subseteq \mathsf{RCA}_n)$ which we know is not he case. Then $\mathfrak{A} \in \mathsf{RCA}_n \sim \mathbf{El}\S_c\mathsf{Nr}_n\mathsf{CA}_\omega$ proving the strictness of the last inclusion. Since all three algebra $\mathfrak{E}, \mathfrak{C}_{\mathbb{Z},\mathbb{N}}$, and $\mathfrak{A}$ are all atomic, we are done.

□

Fix $2 < n < \omega$. Call an atomic $\mathfrak{A} \in \mathsf{CA}_n$ *weakly(strongly) representable* $\iff$ $\mathsf{At}\mathfrak{A}$ *is weaky (strongly) representable. Let* $\mathsf{WRCA}_n(\mathsf{SRCA}_n)$*)* denote the class of all such CA_ns, respectively. Then the class SRCA_n is not elementary, and $\mathsf{LCA}_n \subsetneq \mathsf{SRCA}_n \subsetneq \mathsf{WRCA}_n$ [8]. Recall that for a class K of atomic BAOs, $\mathsf{K} \cap \mathsf{Count}$ denotes the class of algebras havng countably many atoms.

Theorem 6.4. *Let $2 < n < \omega$. Then the following hold:*

1. $\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \cap \mathsf{Count} = \mathsf{CRCA}_n \cap \mathsf{Count}$.
2. $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} = \mathsf{WRCA}_n$.
3. $\mathbf{El}\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} = \mathsf{LCA}_n$.
4. $\mathbf{P}\mathbf{El}\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \subseteq \mathsf{SRCA}_n$

Proof. The first item is already dealy with, cf. [13, Theorem 3.6.2]. Item (2) follows from the definition, upon noting that $\mathsf{RCA}_n = \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_\omega$, and the last two items follows from that $\mathsf{LCA}_n \subseteq \mathsf{SRCA}_n$, that $\mathbf{El}\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} = \mathsf{LCA}_n$, and that (it is straightforward to check that) SRCA_n is closed under $\mathbf{P}$. ☐

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