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Blow up and blur constructions in algebraic logic

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Dedicated to my mentors Andréka and Németi

Summary. Fix $2 < n < \omega$ and let CA_n denote the class of cyindric algebras of dimension n. Roughly CA_n is the algebraic counterpart of the proof theory of first order logic restricted to the first n variables which we denote by L_n. The variety RCA_n of representable CA_ns reflects algebraically the semantics of L_n. We show using a so-called blow up and blur construction that several varieties (in fact infinitely many) containing and including the variety RCA_n are not atom-canonical. A variety V of Boolean algebras with operators is atom-canonical, if whenever $\mathfrak{A} \in \mathsf{V}$ is atomic, then its Dedekind-MacNeille completion, sometimes referred to as its minimal completion, is also in V. From our hitherto obtained algebraic results we show, employing the powerful machinery of algebraic logic, that the celebrated Henkin-Orey omitting types theorem, fails dramatically for L_n even if we allow certain generalized models that are only locally classical.

1.1 Introduction

Persistence properties and omitting types: The technical notion of a modal logic corresponds to the one of a variety of Boolean algebras with operators (BAOs). which provides *algebraic semantics* for modal logic. We assume familiarity with the very basics of the well developed duality theory between BAOs and multimodal logic; the class of all BAOs corresponds to the minimal normal multimodal logic; this correspondence is established by forming quotient Lindenbaum-Tarski algebras. Prominent examples of BAOs are relation, cylindric and polyadic algebras. Relation algebras (RA) correspond to so–called arrow logic, while cylindric algebras of dimension n (CA_n) and the relativized versions of the representable CA_ns, correspond to L_n and its guarded and clique-guarded fragments [6] dealt with below. Atom-canonicity, a well known persistence property in modal logic, is concerned with closure under forming Dedekind-MacNeille completions (sometimes occuring in the literature under the name of *the minimal completions*) of atomic algebras in the variety V, because for an atomic $\mathfrak{A} \in \mathsf{V}$, $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ is its Dedekind-MacNeille completion. Canonicity is the most famous persistence property in modal logic. Though RCA_n is canonical [4], it is not atom-canonical for $2 < n < \omega$ [9]. We shall see that (non-) atom-canonicity of subvarieties of RCA_n is closely related to (the failure) of some version of the omitting types theorem (OTT) in modal fragments of L_n. While the classical Orey-Henkin OTT holds for $L_{\omega,\omega}$, it is known [2] that the OTT fails for L_n in the following (strong) sense. For every $2 < n \leq l < \omega$, there is a countable and complete L_n atomic theory T, and a single type, namely, the type consisting of co-atoms of T, that is realizable in every model of T, but cannot be isolated by a formula ϕ using l variables. Such ϕ will be referred to henceafter as a *witness*. Here we prove stronger negative OTTs for L_n when types are required to be omitted with respect to certain (much wider) generalized semantics, called *m-flat* and *m–square* with $2 < n < m < \omega$. Considering such *clique-guarded* semantics swiftly leads us to rich territory.

Blow up and blur constructions in connection to failure of OTTs: From now on fix $2 < n < \omega$. To violate a special case of OTT for the clique guarded fragments of L_n, which we refer to below as *Vaught's Theorem*, we use a blow up and blur construction applied to a finite extremely simple (so-called rainbow) CA_n (to be defined below) to prove *non*-atom canonicity of infinitely many varieties of CA_ns. This subtle construction may be applied to any two classes $\mathbf{L} \subseteq \mathbf{K}$ of completely additive BAOs. One takes an atomic $\mathfrak{A} \notin \mathbf{K}$ (usually but not always finite), blows it up, by splitting one or more of its atoms each to infinitely many subatoms, obtaining an (infinite) *countable* atomic $\mathfrak{B}b(\mathfrak{A}) \in \mathbf{L}$, such that $\mathfrak{A}$ *is blurred* in $\mathfrak{B}b(\mathfrak{A})$ meaning that $\mathfrak{A}$ *does not* embed in $\mathfrak{B}b(\mathfrak{A})$, but $\mathfrak{A}$ embeds in the Dedekind-MacNeille completion of $\mathfrak{B}b(\mathfrak{A})$, namely, $\mathfrak{Cm}\mathsf{At}\mathfrak{B}b(\mathfrak{A})$. Then any class $\mathbf{M}$ say, between $\mathbf{L}$ and $\mathbf{K}$, that is closed under forming subalgebras will not be atom–canonical, for $\mathfrak{B}b(\mathfrak{A}) \in \mathbf{L}(\subseteq \mathbf{M})$, but $\mathfrak{Cm}\mathsf{At}\mathfrak{B}b(\mathfrak{A}) \notin \mathbf{K}(\supseteq \mathbf{M})$ because $\mathfrak{A} \notin \mathbf{M}$ and $\mathbf{SM} = \mathbf{M}$. We say, in this case, that $\mathbf{L}$ *is not atom–canonical with respect to* $\mathbf{K}$. This method is applied to $\mathbf{K} = \mathbf{S}\mathsf{RaCA}_l$, $l \geq 5$ and $\mathbf{L} = \mathsf{RRA}$ in [6, §17.7] and to $\mathbf{K} = \mathsf{RRA}$ and $\mathbf{L} = \mathsf{RRA} \cap \mathsf{RaCA}_k$ for all $k \geq 3$ in [2], and will be applied below to $\mathbf{K} = \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$, with $t(n) = n(n+1)/2+1$, where Nr_n denotes the operation of forming the n-neat reduct [4, Defintion 2.6.38] (to be defined below) and $\mathbf{L} = \mathsf{RCA}_n$, Ra denotes the operator of forming relation algebra reducts (applied to classes) of CAs, respectively, [4, Definition 5.2.7]. The last example will be used to show that OTT fails for the so-called m-clique guarded fragments of L_n, sometimes referred to as its m-packed fragments, or simply packed fragments, where the class of models omitting non-principal types is substantially broadened to allow m-square models for any $2n \leq m \leq \omega$.

1.2 The algebras and some basic concepts

For a set V, $\mathscr{B}(V)$ denotes the Boolean set algebra $\langle \wp(V), \cup, \cap, \sim, \emptyset, V\rangle$. Let U be a set and α an ordinal; α will be the dimension of the algebra. For $s, t \in {}^{\alpha}U$ write $s \equiv_i t$ if $s(j) = t(j)$ for all $j \neq i$. For $X \subseteq {}^{\alpha}U$ and $i, j < \alpha$, let

$$\mathsf{C}_i X = \{s \in {}^{\alpha}U : (\exists t \in X)(t \equiv_i s)\}$$

and

$$\mathsf{D}_{ij} = \{s \in {}^{\alpha}U : s_i = s_j\}.$$

$\langle \mathscr{B}({}^{\alpha}U), \mathsf{C}_i, \mathsf{D}_{ij}\rangle_{i,j<\alpha}$ is called *the full cylindric set algebra of dimension* α with unit (or greatest element) ${}^{\alpha}U$. Any subalgebra of the latter is called a *set algebra of dimension* α. Following [4], Cs_α denotes the class of all subalgebras of full set algebras of dimension α. The (equationally defined) CA_α class is obtained from cylindric set algebras by a process of abstraction.

Definition 1. *Let α be an ordinal. By a cylindric algebra of dimension α, briefly a CA_α, we mean an algebra*

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{d}_{ij}\rangle_{\kappa,\lambda<\alpha}$$

where $\langle A,+,\cdot,-,0,1\rangle$ is a Boolean algebra such that $0,1$, and d_{ij} are distinguished elements of A (for all $j,i<\alpha$), $-$ and c_i are unary operations on A (for all $i<\alpha$), $+$ and . are binary operations on A, and such that the following equations are satisfied for any $x,y\in A$ and any $i,j,\mu<\alpha$:

(C_1) $\mathsf{c}_i 0=0$,
(C_2) $x\leq \mathsf{c}_i x$ $(i.e., x+\mathsf{c}_i x=\mathsf{c}_i x)$,
(C_3) $\mathsf{c}_i(x\cdot \mathsf{c}_i y)=\mathsf{c}_i x\cdot \mathsf{c}_i y$,
(C_4) $\mathsf{c}_i\mathsf{c}_j x=\mathsf{c}_j\mathsf{c}_i x$,
(C_5) $\mathsf{d}_{ii}=1$,
(C_6) *if* $i\neq j,\mu$, *then* $\mathsf{d}_{j\mu}=\mathsf{c}_i(\mathsf{d}_{ji}\cdot \mathsf{d}_{i\mu})$,
(C_7) *if* $i\neq j$, *then* $\mathsf{c}_i(\mathsf{d}_{ij}\cdot x)\cdot \mathsf{c}_i(\mathsf{d}_{ij}\cdot -x)=0$.

Our main results involve the central notion of neat reducts:

Definition 2. *Let $\alpha<\beta$ be ordinals and $\mathfrak{B}\in \mathsf{CA}_\beta$. Then the α–neat reduct of $\mathfrak{B}$, in symbols $\mathfrak{Nr}_\alpha\mathfrak{B}$, is the algebra obtained from $\mathfrak{B}$, by discarding cylindrifiers and diagonal elements whose indices are in $\beta\sim\alpha$, and restricting the universe to the set $Nr_\alpha B=\{x\in\mathfrak{B}:\{i\in\beta: \mathsf{c}_i x\neq x\}\subseteq\alpha\}$.*

Let α be any ordinal. If $\mathfrak{A}\in\mathsf{CA}_\alpha$ and $\mathfrak{A}\subseteq\mathfrak{Nr}_\alpha\mathfrak{B}$, with $\mathfrak{B}\in\mathsf{CA}_\beta$ $(\beta>\alpha)$, then we say that $\mathfrak{A}$ *neatly embeds* in $\mathfrak{B}$, and that $\mathfrak{B}$ is a β*–dilation of* $\mathfrak{A}$, or simply a *dilation* of $\mathfrak{A}$ if β is clear from context. For $\mathsf{K}\subseteq\mathsf{CA}_\beta$, and $\alpha<\beta$, $\mathsf{Nr}_\alpha\mathsf{K}=\{\mathfrak{Nr}_\alpha\mathfrak{B}:\mathfrak{B}\in\mathsf{K}\}\subseteq\mathsf{CA}_\alpha$. Let α be an ordinal. Then RCA_α is defined to be the subdirect product of set algebras of dimension α. A *cartesian square of dimension* α is a set of the form ${}^\alpha U$ (U some non-empty set); these appear as top elements of Cs_αs. We let Gs_α denote the class of *generalized set algebras of dimension* α; $\mathfrak{A}\in\mathsf{Gs}_\alpha\iff\mathfrak{A}$ has top element a disjoint union of cartesian squares of dimension α and the cylindric operations are defined like in set algebras. It is known that $\mathsf{RCA}_\alpha=\mathbf{I}\mathsf{Gs}_\alpha=\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\omega}$, and that for $2<\alpha$ $k\geq 1$, $\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+k+1}\subsetneq\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+k}$ [8]. The class of completely representable CA_αs is denoted by CRCA_α.

Definition 3. *Let α b an ordinal. Then $\mathfrak{A}\in\mathsf{CA}_\alpha$ is completely representable, if there exists $\mathfrak{B}\in\mathsf{Gs}_\alpha$ and an isomorphism $f:\mathfrak{A}\to\mathfrak{B}$ such for all $X\subseteq\mathfrak{A}$, $f(\prod X)=\bigcap_{x\in X}f(x)$ whenever $\prod X$ exists. In this case we may say that $\mathfrak{A}$ is cmpletel representable via f*

If $\mathfrak{A}$ is an atomic CA_α, then an isomorphism $f:\mathfrak{A}\to\mathfrak{B}$, where $\mathfrak{B}\in\mathsf{Gs}_\alpha$ with top element V, is *atomic*, if $\bigcup_{a\in\mathsf{At}\mathfrak{A}}f(a)=V$. It can be easily shown that $\mathfrak{A}$ is completely representable via $f\iff\mathfrak{A}$ is atomic and f is an atomic representation [5].

1.3 Non-atom canonicity of any V between $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{2n}$ and RCA_n

1.3.1 Clique guarded semantics

We study (locally well–behaved) relativized representations of RCA_n , in analogy to the relation algebra case dealt with in [6, Chapter 13]. Examples include m–flat and m–square representations, where $2<n<m<\omega$. It will always be the case, unless otherwise explicitly indicated, that $1<n<m<\omega$; n denotes the dimension. We identify notationally a set algebra with its universe. Let M be a *relativized representation* of $\mathfrak{A}\in\mathsf{CA}_n$, that is, there exists an injective homomorphism $f:\mathfrak{A}\to\wp(V)$ where $V\subseteq{}^n\mathsf{M}$ and $\bigcup_{s\in V}\mathsf{rng}(s)=\mathsf{M}$. For $s\in V$ and $a\in\mathfrak{A}$, we may write $a(s)$ for $s\in f(a)$. This notation does not refer to f, but whenever used then either f will be clear from context, or immaterial in the context. We may also write 1^{M} for V. Let $\mathfrak{L}(\mathfrak{A})^m$ be the first order signature using m variables and one n–ary relation symbol for each element of $\mathfrak{A}$. Allowing infinitary conjunctions, we denote the resulting signature taken in $L_{\infty,\omega}$ by $\mathfrak{L}(\mathfrak{A})^m_{\infty,\omega}$.

An n–clique, or simply a clique, is a set $C \subseteq \mathsf{M}$ such $(a_0,\dots,a_{n-1}) \in V = 1^{\mathsf{M}}$ for all distinct $a_0,\dots,a_{n-1} \in C$. Let

$$\mathsf{C}^m(\mathsf{M}) = \{s \in {}^m\mathsf{M} : \mathsf{rng}(s) \text{ is an } n \text{ clique}\}.$$

Then $\mathsf{C}^m(\mathsf{M})$ is called the *n–Gaifman hypergraph*, or simply Gaifman hypergraph of M, with the n–hyperedge relation 1^{M}. The *n-clique–guarded semantics*, or simply clique–guarded semantics, $\models_c$, are defined inductively. Let f be as above. For an atomic n–ary formula $a \in \mathfrak{A}$, $i \in {}^n m$, and $s \in {}^m\mathsf{M}$, $\mathsf{M}, s \models_c a(x_{i_0},\dots x_{i_{n-1}}) \iff (s_{i_0},\dots s_{i_{n-1}}) \in f(a)$. For equality, given $i < j < m$, $\mathsf{M}, s \models_c x_i = x_j \iff s_i = s_j$. Boolean connectives, and infinitary disjunctions, are defined as expected. Semantics for existential quantifiers (cylindrifiers) are defined inductively for $\phi \in \mathfrak{L}(A)^m_{\infty,\omega}$ as follows: For $i < m$ and $s \in {}^m\mathsf{M}$, $\mathsf{M}, s \models_c \exists x_i \phi \iff$ there is a $t \in \mathsf{C}^m(\mathsf{M})$, $t \equiv_i s$ such that $\mathsf{M}, t \models_c \phi$.

Definition 4. *Let $\mathfrak{A} \in \mathsf{CA}_n$, M a relativized representation of $\mathfrak{A}$ and $\mathfrak{L}(\mathfrak{A})^m$ be as above.*

1) Then M is said to be m–square if for all $\bar{s} \in \mathsf{C}^m(\mathsf{M}), a \in \mathfrak{A}$, $i < n$, and for any injective map $l : n \to m$, if $\mathsf{M} \models \mathsf{c}_i a(s_{l(0)}\dots, s_{l(n-1)})$, then there exists $\bar{t} \in \mathsf{C}^m(\mathsf{M})$ with $\bar{t} \equiv_i \bar{s}$, and $\mathsf{M} \models a(t_{l(0)},\dots,t_{l(n-1)})$.

2) M is said to be (infinitary) m–flat if it is m–square and for all $\phi \in (\mathfrak{L}(\mathfrak{A})^m_{\infty,\omega})\mathfrak{L}(\mathfrak{A})^m$, for all $\bar{s} \in \mathsf{C}^m(\mathsf{M})$, for all distinct $i, j < m$, we have $\mathsf{M} \models_c [\exists x_i \exists x_j \phi \longleftrightarrow \exists x_j \exists x_i \phi](\bar{s})$.

The proof of the following lemma can be distilled from its relation algebra (RA) analogue [6, Theorem 13.20]. A set V $(\subseteq {}^nU)$ is *diagonizable* if $s \in V \implies s \circ [i|j] \in V$. We write $\mathbf{S}_c$ for the operation of forming complete subalgebras. Complete m-square or m flat representations are defined lke the classical case. For two BAOs, $\mathfrak{A}$ and $\mathfrak{B}$ having the same signature, we write $\mathfrak{A} \subseteq_c \mathfrak{B}$, if $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{B}$.

Lemma 1. *[13, Section 5], [6, Theorems 13.45, 13.36]. Assume that $2 < n < m < \omega$ and let $\mathfrak{A}$ be a BAO having the same signature as CA_n and satisfying all the CA_n axioms except possibly for commutativity of cylindrifications. (1) Then $\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m \iff \mathfrak{A}$ has an infinitary m–flat representation $\iff \mathfrak{A}$ has an m–flat representation. Furthermore, if $\mathfrak{A}$ is atomic, then $\mathfrak{A}$ has a complete infinitary m–flat representation $\iff \mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n(\mathsf{CA}_m \cap \mathbf{At})$.*
(2) We can replace infinitary m-flat and CA_m by m-square and D_m, respectively, where D_m are set algebras having a diagonizable top element V with operations defined like Cs_m restricted to V.

Definition 5. *Let $\mathbf{M}$ be a variety of completely additive BAOs.*

(1) Let $\mathfrak{A} \in \mathbf{M}$ be a finite algebra. We say that $\mathfrak{D} \in \mathbf{M}$ is obtained by blowing up and blurring $\mathfrak{A}$ if $\mathfrak{D}$ is atomic, $\mathfrak{A}$ does not embed in $\mathfrak{D}$, but $\mathfrak{A}$ embeds into $\mathfrak{CmAtD}$.

(2) Assume that $\mathsf{K} \subseteq \mathsf{L} \subseteq \mathbf{M}$, such that $\mathbf{S}\mathsf{L} = \mathsf{L}$.

(a) We say that K is atom-canonical with respect to L if for every atomic $\mathfrak{D} \in \mathsf{K}$ $\mathfrak{Cm}\mathsf{At}\mathfrak{D} \notin \mathsf{L}$. In particular, K is atom–canonical $\iff$ K atom-canoincal with respect to itself.

(b) We say that a finite algebra $\mathfrak{A} \in \mathbf{M}$ detects that $\mathbf{K}$ is not atom–canonical with respect to L, if $\mathfrak{A} \notin \mathsf{L}$, and there is a(n atomic) $\mathfrak{D} \in \mathsf{K}$ obtained by blowing up and blurring $\mathfrak{A}$.

From now on, unless otherwise indicated, n is fixed to be a finite ordinal > 2.

Definition 6. *An n–dimensional atomic network on an atomic algebra $\mathfrak{A} \in \mathsf{CA}_n$ is a map $N : {}^n\Delta \to \mathsf{At}\mathfrak{A}$, where Δ is a non–empty finite set of nodes, denoted by $\mathsf{nodes}(N)$, satisfying the following consistency conditions for all $i < j < n$:*

(i) If $\bar{x} \in {}^n\mathsf{nodes}(N)$ then $N(\bar{x}) \leq \mathsf{d}_{ij} \iff \bar{x}_i = \bar{x}_j$,
(ii) If $\bar{x}, \bar{y} \in {}^n\mathsf{nodes}(N)$, $i < n$ and $\bar{x} \equiv_i \bar{y}$, then $N(\bar{x}) \leq \mathsf{c}_i N(\bar{y})$,

Definition 7. *(1) Assume that $\mathfrak{A} \in \mathsf{CA}_n$ is atomic and that $m, k \leq \omega$. The atomic game $G^m_k(\mathsf{At}\mathfrak{A})$, or simply G^m_k, is the game played on atomic networks of $\mathfrak{A}$ using m nodes and having k rounds [7, Definition 3.3.2], where $\forall$ is offered only one move, namely, a cylindrifier move: Suppose that we are at round $t > 0$. Then $\forall$ picks a previously played network N_t $(\mathsf{nodes}(N_t) \subseteq m)$, $i < n$, $a \in \mathsf{At}\mathfrak{A}$, $\bar{x} \in {}^n\mathsf{nodes}(N_t)$, such that $N_t(\bar{x}) \leq \mathsf{c}_i a$. For her response, $\exists$ has to deliver a network M such that $\mathsf{nodes}(M) \subseteq m$, $M \equiv_i N$, and there is $\bar{y} \in {}^n\mathsf{nodes}(M)$ that satisfies $\bar{y} \equiv_i \bar{x}$ and $M(\bar{y}) = a$.*
We write $G_k(\mathsf{At}\mathfrak{A})$, or simply G_k, for $G^m_k(\mathsf{At}\mathfrak{A})$ if $m \geq \omega$.
(2) The ω–rounded game $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$ or simply $\mathbf{G}^m$ is like the game $G^m_\omega(\mathsf{At}\mathfrak{A})$ except that $\forall$ has the option to reuse the m nodes in play.

The following Lemma is proved in [14, Lemma 4.3] and [13, Lemma 5.8]:

Lemma 2. *Let $2 < n < \omega$, and assume that $m > n$. If $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$ is atomic, then $\exists$ has a winning strategy in $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$. If $\mathfrak{A} \in \mathsf{CA_n}$, and $\mathfrak{A}$ has a complete m-square representation then $\exists$ has a winning strategy in $G^m_\omega(\mathsf{At}\mathfrak{A})$.*

1.3.2 Blowing up and blurring finite rainbow cylindric algebas

The most general exposition of CA rainbow constructions is given in [7, Section 6.2, Definition 3.6.9] in the context of constructing atom structures from classes of models. Our models are just coloured graphs [5]. Let G, R be two relational structures. Let $2 < n < \omega$. Then the colours used are:

- greens: g_i $(1 \leq i \leq n-2)$, g^i_0, $i \in \mathsf{G}$,
- whites : $\mathsf{w}_i : i \leq n-2$,
- reds: r_{ij} $(i, j \in \mathsf{R})$,
- shades of yellow : y_S : S a finite subset of ω or $S = \omega$.

A *coloured graph* is a graph such that each of its edges is labelled by the colours in the above first three items, greens, whites or reds, and some $n-1$ hyperedges are also labelled by the shades of yellow. Certain coloured graphs will deserve special attention.

Definition 8. *Let $i \in \mathsf{G}$, and let M be a coloured graph consisting of n nodes $x_0, \ldots, x_{n-2}, z$. We call M an i - cone if $M(x_0, z) = \mathsf{g}^i_0$ and for every $1 \leq j \leq n-2$, $M(x_j, z) = \mathsf{g}_j$, and no other edge of M is coloured green. $(x_0, \ldots, x_{n-2})$ is called the base of the cone, z the apex of the cone and i the tint of the cone.*

The rainbow algebra depending on G and R from the class $\mathbf{K}$ consisting of all coloured graphs M such that:

1. M is a complete graph and M contains no triangles (called forbidden triples) of the following types:

$$(\mathsf{g},\mathsf{g}^{'},\mathsf{g}^*),(\mathsf{g}_i,\mathsf{g}_i,\mathsf{w}_i) \quad \text{any } 1\leq i\leq n-2, \tag{1.1}$$
$$(\mathsf{g}_0^j,\mathsf{g}_0^k,\mathsf{w}_0) \quad \text{any } j,k\in \mathsf{G}, \tag{1.2}$$
$$(\mathsf{r}_{ij},\mathsf{r}_{j'k'},\mathsf{r}_{i^*k^*}) \quad \text{unless } |\{(j,k),(j',k'),(j^*,k^*)\}|=3 \tag{1.3}$$
$$\tag{1.4}$$

and no other triple of atoms is forbidden.

2. If $a_0,\ldots,a_{n-2}\in M$ are distinct, and no edge (a_i,a_j) $i<j<n$ is coloured green, then the sequence $(a_0,\ldots,a_{n-2})$ is coloured a unique shade of yellow. No other $(n-1)$ tuples are coloured shades of yellow. Finally, if $D=\{d_0,\ldots,d_{n-2},\delta\}\subseteq M$ and $M\restriction D$ is an i cone with apex δ, inducing the order $d_0,\ldots,d_{n-2}$ on its base, and the tuple $(d_0,\ldots,d_{n-2})$ is coloured by a unique shade y_S then $i\in S$.

Let G and R be relational structures as above. Take the set J consisting of all surjective maps $a:n\to\Delta$, where $\Delta\in\mathbf{K}$ and define an equivalence relation $\sim$ on this set relating two such maps iff they essentially define the same graph [5]; the nodes are possibly different but the graph structure is the same. Let At be the atom structure with underlying set $J\sim$. We denote the equivalence class of a by $[a]$. Then define, for $i<j<n$, the accessibility relations corresponding to ijth–diagonal element, and ith–cylindrifier, as follows:

(1) $[a]\in E_{ij}$ iff $a(i)=a(j)$,

(2) $[a]T_i[b]$ iff $a\restriction n\smallsetminus\{i\}=b\restriction n\smallsetminus\{i\}$,

This, as easily checked, defines a CA_n atom structure. The complex CA_n over this atom structure will be denoted by $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$. The dimension of $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$, always finite and >2, will be clear from context. For rainbow atom structures, there is a one to one correspondence between atomic networks and coloured graphs [5, Lemma 30], so for $2<n<m\leq\omega$, we use the graph versions of the games G^m_k, $k\leq\omega$, and $\mathbf{G}^m$ played on rainbow atom structures of dimension m [5, pp.841–842]. The the atomic k rounded game game G^m_k where the number of nodes are limited to n to games on coloured graphs [5, lemma 30]. The game $\mathbf{G}^m$ lifts to a game on coloured graphs, that is like the graph games G^m_ω [5], where the number of nodes of graphs played during the ω rounded game does not exceed m, but $\forall$ has the option to re-use nodes. The typical winning strategy for $\forall$ in the graph version of both atomic games is bombarding $\exists$ with cones having a common base and *green* tints until she runs out of (suitable) *reds*, that is to say, reds whose indicies do not match [5, 4.3].

Definition 9. *A* CA_n *atom structure* $\mathbf{At}$ *is weakly representable if there is an atomic* $\mathfrak{A}\in\mathsf{RCA}_n$ *such that* $\mathbf{At}=\mathsf{At}\mathfrak{A}$*; it is strongly representable if* $\mathfrak{Cm}\mathbf{At}\in\mathsf{RCA}_n$.

These two notions are distinct, cf. [9] and the following Theorem1; see also the forthcoming Theorem 2.

Theorem 1. *Let* $2<n<\omega$ *and* $t(n)=n(n+1)/2+1$. *The variety* RCA_n *is not-atom canonical with respect to* $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$. *In fact, there is a countable atomic simple* $\mathfrak{A}\in\mathsf{RCA}_n$ *such that* $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ *does not have an* $t(n)$*-square,a fortiori* $t(n)$*- flat, representation.*

Proof. The proof is long and uses many ideas in [9]. We will highlight only the differences in detail from the proof in [9] needed to make our result work. When parts of the proof coincide we will be more sketchy. The proof is divided into four parts:

1: **Blowing up and blurring** $\mathfrak{B}_f$ **forming a weakly representable atom structure** $\mathbf{At}$: Take the finite rainbow CA_n, $\mathfrak{B}_f$ where the reds R is the complete irreflexive graph n, and the greens are $\{\mathsf{g}_i : 1 \le i < n-1\} \cup \{\mathsf{g}_0^i : 1 \le i \le n(n-1)/2\}$, endowed with the cylindric operations. We will show $\mathfrak{B}$ detects that RCA_n is not atom-canonical with respect to $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$ with $t(n)$ as specified in the statement of the theorem. Denote the finite atom structure of $\mathfrak{B}_f$ by $\mathbf{At}_f$; so that $\mathbf{At}_f = \mathsf{At}(\mathfrak{B}_f)$. One then defines a larger the class of coloured graphs like in [9, Definition 2.5]. Let $2 < n < \omega$. Then the colours used are like above except that each red is 'split' into ω many having 'copies' the form r_{ij}^l with $i < j < n$ and $l \in \omega$, with an additional shade of red ρ such that the consistency conditions for the new reds (in addition to the usual rainbow consistency conditions) are as follows:

- $(r_{jk}^i, r_{j'k'}^{i}, r_{j^*k^*}^{i^*})$ unless $i = i' = i^*$ and $|\{(j,k),(j',k'),(j^*,k^*)\}| = 3$
- (r, ρ, ρ) and $(\mathsf{r}, \mathsf{r}^*, \rho)$, where r, r^* are any reds.

The consistency conditions can be coded in an $L_{\omega,\omega}$ theory T having signture the reds with ρ together with all other colours like in [7, Definitio 3.6.9]. The theory T is only a first order theory (not an $L_{\omega_1,\omega}$ theory) because the number of greens is finite which is not the case with [7] where the number of available greens are countably infinite coded by an infinite disjunction. One construct an n-homogeneous model M is as a countable limit of finite models of T using a game played between $\exists$ and $\forall$like in [9, Theorem 2.16]. In the rainbow game $\forall$ challenges $\exists$ with *cones* having green *tints* (g_0^i), and $\exists$ wins if she can respond to such moves. This is the only way that $\forall$ can force a win. $\exists$ has to respond by labelling *appexes* of two succesive cones, having the *same base* played by $\forall$. By the rules of the game, she has to use a red label. She resorts to ρ whenever she is forced a red while using the rainbow reds will lead to an inconsistent triangle of reds; [9, Proposition 2.6, Lemma 2.7]. The number of greens make [9, Lemma 3.10] work with the same proof. using only finitely many green and not infinitely many.

2. **Representing a term algebra (and its completion) as (generalized) set algebras:** Having M at hand, one constructs two atomic n–dimensional set algebras based on M, sharing the same atom structure and having the same top element. The atoms of each will be the set of coloured graphs, seeing as how, quoting Hodkinson [9] such coloured graphs are 'literally indivisible'. Now L_n and $L^n_{\infty,\omega}$ are taken in the rainbow signature (without ρ). Continuing like in *op.cit*, deleting the one available red shade, set $W = \{\bar{a} \in {}^n\mathsf{M} : \mathsf{M} \models (\bigwedge_{i<j<n} \neg\rho(x_i,x_j))(\bar{a})\}$, and for $\phi \in L^n_{\infty,\omega}$, let $\phi^W = \{s \in W : \mathsf{M} \models \phi[s]\}$. Here W is the set of all n–ary assignments in ${}^n\mathsf{M}$, that have no edge labelled by ρ. Let $\mathfrak{A}$ be the relativized set algebra with domain $\{\varphi^W : \varphi \text{ a first-order } L_n - \text{formula}\}$ and unit W, endowed with the usual concrete quasipolyadic operations read off the connectives. Classical semantics for L_n *rainbow formulas* and their semantics by relativizing to W coincide [9, Proposition 3.13] *but not with respect to $L^n_{\infty,\omega}$ rainbow formulas*. Hence the set algebra $\mathfrak{A}$ is isomorphic to a cylinric set algebra of dimension n having top element ${}^n\mathsf{M}$, so $\mathfrak{A}$ is simple, in fact its Df reduct is simple. Let $\mathfrak{E} = \{\phi^W : \phi \in L^n_{\infty,\omega}\}$ [9, Definition 4.1] with the operations defined like on $\mathfrak{A}$ the usual way. $\mathfrak{Cm}\mathbf{At}$ is a complete CA_n and, so like in [9, Lemma 5.3] we have an isomorphism from $\mathfrak{Cm}\mathbf{At}$ to $\mathfrak{E}$ defined via $X \mapsto \bigcup X$. Since $\mathsf{At}\mathfrak{A} = \mathsf{At}\mathfrak{Tm}(\mathsf{At}\mathfrak{A})$, which we refer to only by $\mathbf{At}$, and $\mathfrak{Tm}\mathsf{At}\mathfrak{A} \subseteq \mathfrak{A}$, hence $\mathfrak{Tm}\mathsf{At}\mathfrak{A} = Tm\mathbf{At}$ is representable. The atoms of $\mathfrak{A}$, $\mathfrak{Tm}\mathsf{At}\mathfrak{A}$ and $\mathfrak{Cm}\mathsf{At}\mathfrak{A} = \mathfrak{Cm}\mathbf{At}$ are the coloured graphs whose edges are *not labelled* by ρ. These atoms are uniquely determined by the interpretion in M of so-called MCA formulas in the rainbow signature of $\mathbf{At}$ as in [9, Definition 4.3].

3. **Embedding $\mathfrak{A}_{n+1,n}$ into $\mathfrak{Cm}(\mathsf{At}(\mathfrak{Bb}(\mathfrak{A}_{n+1,n},\mathsf{r},\omega)))$:** Let CRG_f be the class of coloured graphs on $\mathbf{At}_f$ and CRG be the class of coloured graph on $\mathbf{At}$. We can (and will) assume that $\mathsf{CRG}_f \subseteq \mathsf{CRG}$. Write M_a for the atom that is the (equivalence class of the) surjection $a : n \to M$, $M \in \mathsf{CGR}$. Here we identify a with $[a]$; no harm will ensue. We define the (equivalence) relation $\sim$ on $\mathbf{At}$ by $M_b \sim N_a$, $(M, N \in \mathsf{CGR})$:

- $a(i) = a(j) \iff b(i) = b(j)$,
- $M_a(a(i), a(j)) = \mathsf{r}^l \iff N_b(b(i), b(j)) = \mathsf{r}^k$, for some $l, k \in \omega$,
- $M_a(a(i), a(j)) = N_b(b(i), b(j))$, if they are not red,
- $M_a(a(k_0), \ldots, a(k_{n-2})) = N_b(b(k_0), \ldots, b(k_{n-2}))$, whenever defined.

We say that M_a is a *copy of* N_b if $M_a \sim N_b$ (by symmetry N_b is a copy of M_a.) Indeed, the relation 'copy of' is an equivalence relation on $\mathbf{At}$. An atom M_a is called a *red atom*, if M_a has at least one red edge. Any red atom has ω many copies, that are *cylindrically equivalent*, in the sense that, if $N_a \sim M_b$ with one (equivalently both) red, with $a : n \to N$ and $b : n \to M$, then we can assume that $\mathsf{nodes}(N) = \mathsf{nodes}(M)$ and that for all $i < n$, $a \restriction n \sim \{i\} = b \restriction n \sim \{i\}$. In $\mathfrak{Cm}\mathbf{At}$, we write M_a for $\{M_a\}$ and we denote suprema taken in $\mathfrak{Cm}\mathbf{At}$, possibly finite, by $\sum$. Define the map Θ from $\mathfrak{A}_{n+1,n} = \mathfrak{Cm}\mathbf{At}_f$ to $\mathfrak{Cm}\mathbf{At}$, by specifing first its values on $\mathbf{At}_f$, via $M_a \mapsto \sum_j M_a^{(j)}$ where $M_a^{(j)}$ is a copy of M_a. So each atom maps to the suprema of its copies. This map is well-defined because $\mathfrak{Cm}\mathbf{At}$ is complete. We check that Θ is an injective homomorphim. Injectivity is easy.. We check preservation of all the CA_n extra Boolean operations.

- Diagonal elements. Let $l < k < n$. Then:

$$\begin{aligned} M_x \leq \Theta(\mathsf{d}_{lk}^{\mathfrak{Cm}\mathbf{At}_f}) &\iff M_x \leq \sum_j \bigcup_{a_l = a_k} M_a^{(j)} \\ &\iff M_x \leq \bigcup_{a_l = a_k} \sum_j M_a^{(j)} \\ &\iff M_x = M_a^{(j)} \text{ for some } a : n \to M \text{ such that } a(l) = a(k) \\ &\iff M_x \in \mathsf{d}_{lk}^{\mathfrak{Cm}\mathbf{At}}. \end{aligned}$$

- Cylindrifiers. Let $i < n$. By additivity of cylindrifiers, we restrict our attention to atoms $M_a \in \mathbf{At}_f$ with $a : n \to M$, and $M \in \mathsf{CRG}_f \subseteq \mathsf{CRG}$. Then:

$$\begin{aligned} \Theta(\mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}_f} M_a) = f(\bigcup_{[c] \equiv_i [a]} M_c) = \bigcup_{[c] \equiv_i [a]} \Theta(M_c) \\ = \bigcup_{[c] \equiv_i [a]} \sum_j M_c^{(j)} = \sum_j \bigcup_{[c] \equiv_i [a]} M_c^{(j)} = \sum_j \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}} M_a^{(j)} \\ = \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}}(\sum_j M_a^{(j)}) = \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}} \Theta(M_a). \end{aligned}$$

4.: **$\forall$ has a winning strategy in $G^{t(n)}\mathsf{At}(\mathfrak{B}_f)$; and the required result:** It is straightforward to show that $\forall$ has winning strategy first in the Ehrenfeucht–Fraïssé forth private game played between $\exists$ and $\forall$ on the complete irreflexive graphs $n+1 (\leq n(n-1)/2+1)$ and n in $n+1$ rounds $\mathsf{EF}_{n+1}^{n+1}(n+1, n)$ [7, Definition 16.2] since $n+1$ is 'longer' than n. Using (any) $p > n$ many pairs of pebbles avalable on the board $\forall$ can win this game in $n+1$ many rounds. $\forall$ lifts his winning strategy from the lst private Ehrenfeucht–Fraïssé forth game to the graph game on $\mathbf{At}_f = \mathsf{At}(\mathfrak{B}_f)$ [5, pp. 841] forcing a win using $t(n)$ nodes. One uses the $n(n-1)/2+2$ green relations in the usual way to force a red clique C, say with $n(n-1)/2+2$. Pick any point $x \in C$. Then there are $> n(n-1)/2$ points y in $C\{x\}$. There are only $n(n-1)/2$ red relations. So there must be distinct $y, z \in C\{x\}$ such that (x, y) and (x, z) both have the same red label (it will be some r_{ij}^m for $i < j < n$). But (y, z) is also red, and this contradicts [9, Definition 2.5(2), 4th bullet point]. In more detail, $\forall$ bombards $\exists$ with cones having common base and distinct green tints until $\exists$ is forced to play an inconsistent red triangle (where indicies of reds do not match). He needs $n-1$ nodes as the base of cones, plus $|P|+2$ more nodes, where $P=\{(i,j): i<j<n\}$ forming a red clique, triangle with two edges satisfying the same r_p^m for $p\in P$. Calculating, we get $t(n)=n-1+n(n-1)/2+2=n(n+1)/2+1$ By Lemma 2, $\mathfrak{B}_f\notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$ when $2<n<\omega$). Since $\mathfrak{B}_f$ is finite, then $\mathfrak{B}_f\notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$, because $\mathfrak{B}_f$ coincides with its canonical extension and for any $\mathfrak{D}\in\mathsf{CA}_n$, $\mathfrak{D}\in\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{2n}\implies \mathfrak{D}^+\in\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{2n}$. But $\mathfrak{B}_f$ embeds into $\mathfrak{CmAt}\mathfrak{A}$, hence $\mathfrak{CmAt}\mathfrak{A}$ is outside the variety $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$, as well. By the second part of Lemma 2, the required follows.

In both cases of RAs addressed in [6, Lemmata 17.32, 17.34, 17.35, 17.36] and CAs adressed in Theorem 1 pproving non atom canoicity for infinitely many varieties of RAS and CA_ns, respectively, the relational structures G and R used satisfy $|\mathsf{G}|=|\mathsf{R}|+1$. For RA, $\mathsf{R}=3$ and for CA_ns, $\mathsf{R}=n$ (the dimension), where the finite ordinals 3 and n are viewed as complete irreflexive graphs. Using the rainbow algebras based on such graphs, we have proved that $\mathfrak{B}_f$ ($\mathbf{R}_{4,3}$ which is the rainbow relation algebra based on the complete irreflexive graphs with nodes m and n defined the obvious way, cf.[6, Definition 17.31]) detects that RCA_n (RRA) is not atom-canonical with repect to $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$ ($\mathbf{S}\mathsf{RaCA}_6$) with t(n) as defined in the statement of Theorem 1 Worthy of note, is that it is commonly accepted that relation algebras have dimension three being a natural habitat for three variable first order logic. Nevertheless, sometimes it is argued that the dimension should be three and a half in the somewhat loose sense that RAs lie 'halfway' between CA_3 and CA_4 manifesting behaviour of each; for example associativiy in RAs needs 4 variables to be proved From Hodkinson's construction in [9], we know that $\mathfrak{CmAt}\mathfrak{A}\notin\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$ for some finite $m>n$, but the (semantical) argument used in [9] does not give any information on the value of such m. By truncating the greens to be $n)n-1/2$ (instead of the 'overkill' of infinitely many in [9]), and using a syntactical blow up and blur construction, we could pin down such a value of m, namely, $m=t(n)$ as specified in the statement of Theorem 1, by showing that $\mathfrak{Cm}\mathbf{At}\notin\mathbf{S}\mathsf{Nr}_{\mathbf{n}}\mathsf{CA}_{\mathbf{t(n)}}$.

For a class $\mathbf{K}$ of BAOs, let $\mathbf{K}\cap\mathsf{Count}$ denote the class of atomic algebras in $\mathbf{K}$ having countably many atoms.

Proposition 1. *Let $2<n<\omega$.*

1. *For any ordinal $0\le j$, $\mathsf{RCA}_n\cap\mathsf{Nr}_n\mathsf{CA}_{n+j}\cap\mathsf{Count}$ is not atom-canonical with respect to $\mathsf{RCA}_n \iff j<\omega$,*
2. *For any ordinal j, $\mathsf{Nr}_n\mathsf{CA}_{n+j}\cap\mathsf{RCA}_n\cap\mathbf{At}\nsubseteq\mathsf{CRCA}_n$,*
3. *There exists an atomic RCA_n such that its Dedekind-MacNeille (minimal) completion does not embed into its canonical extension.*[1]

Proof. (1): One implication follows from [2] where for eah $2<n<l<\omega$ an algebra $\mathfrak{A}_l\in\mathsf{RCA}_n\cap\mathsf{Nr}_n\mathsf{CA}_l$ is constructed such that $\mathfrak{CmAt}\mathfrak{A}_l\notin\mathsf{RCA}_n$, so $\mathfrak{A}_l$ cannot be completely representable. Conversely, for any infinite ordinal j, $\mathsf{Nr}_n\mathsf{CA}_{n+j}=\mathsf{Nr}_n\mathsf{CA}_\omega$ and if $\mathfrak{A}\in\mathsf{Nr}_n\mathsf{CA}_\omega\cap\mathsf{Count}$, then by [12, Theorem 5.3.6], $\mathfrak{A}\in\mathsf{CRCA}_n$, so $\mathfrak{CmAt}\mathfrak{A}\in\mathsf{RCA}_n$.

(2): The case $j<\omega$, follows from the fact that the algebra $\mathfrak{A}_{n+j}$ used in the previous item is in $\mathsf{Nr}_n\mathsf{CA}_{n+j}\cap\mathsf{RCA}_n$ but has no complete representation. For infinite j one uses the construction in [14, Theorem 4.5].

[1] In the CA context, the terminology *minimal completion* is misleading because $\mathfrak{A}^+$ is *another completion* of $\mathfrak{A}$; so supposedly the minimal completion of $\mathfrak{A}$ should embed into $\mathfrak{A}^+$, which is not, as we have already seen in Theorem 1, always true. Conversely, for an atomic Boolean algebra $\mathfrak{B}$, $\mathfrak{CmAt}\mathfrak{B}$ always embeds into $\mathfrak{B}^+$ as it should.

(3): Let $\mathfrak{A} = \mathfrak{Tm}\mathbf{At}$ be the CA_n as defined in the proof of Theorem 1. Since $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathsf{RCA}_n$, it does not embed into $\mathfrak{A}^+$, because $\mathfrak{A}^+ \in \mathsf{RCA}_n$ since $\mathfrak{A} \in \mathsf{RCA}_n$ and RCA_n is a canonical variety.

1.3.3 An application on omitting types for the clique guarded fragment of L_n

Consider the following statement: *There exists a countable, complete and atomic L_n first order theory T in a signature L, meaning that the Tarski Lindenbuam quotient algebra $\mathfrak{Fm}_T$ is atomic, such that the type Γ consisting of co-atoms $\mathfrak{Fm}_T$ is realizable in every m–square model, but Γ cannot be isolated using $\leq l$ variables, where $n \leq l < m \leq \omega$.* A co-atom of $\mathfrak{Fm}_T$ is the negation of an atom in $\mathfrak{Fm}_T$. An m-square model of T is an m-square representation of $\mathfrak{Fm}_T$. The last statement denoted by $\Psi(l,m)$, is short for Vaught's Theorem (VT*) fails at (the parameters) l and m.* Let $\mathsf{VT}(l,m)$ stand for VT *holds at l and m*, so that by definition $\Psi(l,m) \iff \neg\mathsf{VT}(l,m)$. We also include $l = \omega$ in the equation by defining $\mathsf{VT}(\omega,\omega)$ as VT holds for $L_{\omega,\omega}$: Atomic countable first order theories have atomic countable models. It is well known that $\mathsf{VT}(\omega,\omega)$ is a direct consequence of the Orey-Henkin OTT. From Theorem 1 and the construction in [2] using essentially the argument in [12, Proof of Theorem 3.1.1] one obtains:

Theorem 2. *For $2 < n < \omega$ and $n \leq l < \omega$, $\Psi(n,t(n))$, with $t(n) = n(n+1)/2+1$ and $\Psi(l,\omega)$ hold.*

Let $2 < n \leq l < m \leq \omega$. In $\mathsf{VT}(l,m)$, while the parameter l measures how close we are to $L_{\omega,\omega}$, m measures the *'degree' of squareness* of permitted models. Using elementary calculas terminology one can view $\lim_{l\to\infty}\mathsf{VT}(l,\omega) = \mathsf{VT}(\omega,\omega)$ algebraically using ultraproducts as follows. Fix $2 < n < \omega$. For each $2 < n \leq l < \omega$, let $\mathfrak{R}_l$ be the finite Maddux algebra $\mathfrak{E}_{f(l)}(2,3)$, as defined on [2, p.83, §5, in the proof of Theorem 5.1] with l–blur (J_l, E_l) as defined in [2, Definition 3.1] and $f(l) \geq l$ as specified in [2, Lemma 5.1] (denoted by k therein). Let $\mathscr{R}_l = \mathfrak{Bb}(\mathfrak{R}_l, J_l, E_l) \in \mathsf{RRA}$ where $\mathscr{R}_l$ is the relation algebra having atom structure denoted At in [2, p. 73] when the blown up and blurred algebra denoted $\mathfrak{R}_l$ happens to be the finite Maddux algebra $\mathfrak{E}_{f(l)}(2,3)$ and let $\mathfrak{A}_l = \mathfrak{Nr}_n\mathfrak{Bb}_l(\mathfrak{R}_l, J_l, E_l) \in \mathsf{RCA}_n$ as defined in [2, Top of p.80] (with $\mathfrak{R}_l = \mathfrak{E}_{f(l)}(2,3)$). Then $(\mathsf{At}\mathscr{R}_l : l \in \omega \sim n)$, and $(\mathsf{At}\mathfrak{A}_l : l \in \omega \sim n)$ are sequences of weakly representable atom structures that are not strongly representable with a completely representable ultraproduct. Let LCA_n denote the class of CA_ns satisfying the Lyndon conditions [7], which is the elementary closure of the class of completely representable CA_ns. We immediately get:

Corollary 1. *(Monk, Maddux, Biro, Hirsch and Hodkinson) Let $2 < n < \omega$. Then the set of equations using only one variable that holds in each of the varieties RCA_n and RRA, together with any finite first order definable expansion of each, cannot be derived from any finite set of equations valid in the variety [3, 10]. Furthermore, LCA_n is not finitely axiomatizable.*

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