ATOMIC CANONICITY AND FIRST ORDER DEFINABILITY IN CLASSES OF ALGEBRAS OF RELATIONS

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Abstract

Fix $2 < n < \omega$ and let $\mathbf{CA}_n$ denote the class of cylindric algebras of dimension $n$. Roughly $\mathbf{CA}_n$ is the algebraic counterpart of the proof theory of first order logic restricted to the first $n$ variables which we denote by $L_n$. The variety $\mathbf{RCA}_n$ of representable $\mathbf{CA}_n$s reflects algebraically the semantics of $L_n$. Members of $\mathbf{RCA}_n$ are concrete algebras consisting of genuine $n$-ary relations, with set theoretic operations induced by the nature of relations, such as projections referred to as cylindrifications. Although $\mathbf{CA}_n$ has a finite equational axiomatization, $\mathbf{RCA}_n$ is not finitely axiomatizable, and it generally exhibits wild, often unpredictable and unruly behavior. This makes the theory of $\mathbf{CA}_n$ substantially richer than that of Boolean algebras, just as much as $L_{\omega, \omega}$ is richer than propositional logic. We show using a so-called blow up and blur construction that several varieties (in fact infinitely many) containing and including the variety $\mathbf{RCA}_n$ are not atom-canonical. A variety $\mathbf{V}$ of Boolean algebras with operators is atom canonical, if whenever $\mathbf{A} \in \mathbf{V}$ is atomic, then its Dedekind-MacNeille completion, sometimes referred to as its minimal completion, is also in $\mathbf{V}$. From our hitherto obtained algebraic results we show, employing the powerful machinery of algebraic logic, that the celebrated Henkin-Orey omitting types theorem, which is one of the classical first (historically) cornerstones of model theory of $L_{\omega, \omega}$, fails dramatically for $L_n$ even if we allow certain generalized models that are only locally classical. It is also shown that any class $\mathbf{K}$ such that $\operatorname{Nr}_n \mathbf{CA}_n \cap \mathbf{CRCA}_n \subseteq \mathbf{K} \subseteq \mathbf{S}_n \operatorname{Nr}_n \mathbf{CA}_{n+3}$, where $\mathbf{CRCA}_n$ is the class of completely representable $\mathbf{CA}_n$s, and $\mathbf{S}_n$ denotes the operation of forming dense (complete) subalgebras, is not elementary. Finally, we show that any class $\mathbf{K}$ such that $\mathbf{S}_5 \mathbf{RaCA}_5 \subseteq \mathbf{K} \subseteq \mathbf{S}_5 \mathbf{RaCA}_5$ is not elementary, where $\mathbf{S}_5$ denotes the operation of forming dense subalgebra.

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Persistence properties and omitting types: The technical notion of a modal logic corresponds to the one of a variety of \textit{Boolean algebras with operators} \textit{(BAOs)} which provides \textit{algebraic semantics} for modal logic. We assume familiarity with the very basics of the well developed duality theory between BAOs and multimodal logic; the class of all BAOs correspond to the minimal normal multimodal logic; this correspondence is established by forming quotient Tarski-Lindenbaum algebras. The starting point of this duality is that algebraic terms correspond to modal formulas. By that identification we get: $\mathcal{F} \models \phi \iff \mathcal{CmF} \models \phi = 1$, where $\mathcal{F}$ is a relational structure (Kripke frame) $(F, R_i)_{i \in I}$, $I$ a non-empty indexing set, and $\mathcal{CmF}$ (its complex algebra) is an algebra having signature $(f_i : i \in I)$ where each $f_i$ is a unary modality, in other words an operator. Prominent examples of BAOs are relation, cylindric and polyadic algebras. Relation algebras (RA) correspond to so-called arrow logic, while cylindric algebras of dimension $n$ (CA$_n$) and the relativized versions of the representable CA$_n$s, correspond to $L_n$ and its guarded and clique-guarded fragments, cf. [15, Chapter 13] dealt with below. The topic of this paper is typical of what happens in algebraic logic and that is: Using algebraic machinery to obtain fine results in first order logic via so called `bridge theorems’. The algebraic machinery we use, is in essence a disguised form of Andreka’s splitting, cf. [2] wrapped up in, a \textit{blow up and blur construction}. The last is an indicative term invented by Andreka and Németi in [5] and has several reincarnations in the literature though not under this name. One purpose of this paper is to gather such instances of blow up and blur constructions in algebraic logic. The metalogical result our investigations have impact on is the celebrated Henkin-Orey omitting types theorem. Let $\mathcal{L}$ be an extension or reduct or variant of first order logic, like first logic itself, $L_n$ as defined in the abstract with $2 < n < \omega$, $L_{\omega_1, \omega}$, $L_\omega$ as defined in [11, §4.3], etc. An omitting types theorem for $\mathcal{L}$, briefly an \textit{OTT}, is typically of the form ‘A countable family of non-isolated types in a countable $\mathcal{L}$ theory $T$ can be omitted in a countable model of $T$.’ From this it directly follows that if a type is realizable in every model of a countable theory $T$, then there should be a formula consistent with $T$ that isolates this type. A type is simply a set of formulas $\Gamma$ say. The type $\Gamma$ is realizable in a model if there is an assignment that satisfies (uniformly) all formulas in $\Gamma$. Finally, $\phi$ isolates $\Gamma$ means that $T \vdash \phi \rightarrow \psi$ for all $\psi \in \Gamma$. What Orey and Henkin proved is that the OTT holds for $L_{\omega, \omega}$ when such types are \textit{finitary}, meaning that they all consist of $n$-variable formulas for some $n < \omega$. OTT has an algebraic facet exhibited in the property of \textit{atom-canonicity}; which in turn reflects an important \textit{persistence} property in modal logic. Algebraically, so-called \textit{persistence properties} refer to closure of a variety $\mathcal{V}$ under passage from a given algebra $\mathfrak{A} \in \mathcal{V}$ to some `larger’ algebra $\mathfrak{A}^\ast$. Centricity, which is the most prominent persistence property in
modal logic, the ‘large algebra’ $A^*$ is the canonical embedding algebra (or perfect) extension of $A$, a complex algebra based on the ultrafilter frame of $A$ whose underlying set is the set of all Boolean ultrafilters of $A$.

A completely additive variety $V$ is *atom-canonical* if whenever $A \in V$ is atomic, then the complex algebra of its atom structure, in symbols $\text{CmAt}_A$, is also in $V$. More concisely, $V$ is such if $\text{CmAt}_V \subseteq V$. Atom-canonicity is concerned with closure under forming Dedekind-MacNeille completions (sometimes occurring in the literature under the name of the minimal completions) of atomic algebras in the variety $V$, because for an atomic $A \in V$, $\text{CmAt}_A$ is its Dedekind-MacNeille completion. Though $\text{RCA}_n$ is canonical, it is not atom-canonical for $2 < n < \omega$, cf. [19]. From non-atom-canonicity of $\text{RCA}_n$, it follows from [41] that $\text{RCA}_n$ cannot be axiomatized by Sahlqvist equations. We shall see that (non-) atom-canonicity of subvarieties of $\text{RCA}_n$ is closely related to (the failure) of some version of the OTT in modal fragments of $L_n$.

**Omitting type and Vaught’s Theorem:** We violate a special case of OTT for the clique guarded fragments of $L_n$, which we refer to below as *Vaught’s Theorem* (VT). The VT holds for $L_{\omega, \omega}$ and is a well known famous consequence of the (Orey-Henkin) Omitting Types Theorem (OTT). The OTT has been formalized in the paper [38] for even more general contexts allowing predicates of infinite arity but quantification is similar to first order logic. Algebraically VT can be formulated for first order logic as follows. For an algebra $A$ having a Boolean reduct, by $\text{At}_A$ we understand the set of all atoms of $A$. If $A$ is a locally finite dimensional simple cylindric algebra of dimension $\omega$ as defined in [11] (here and elsewhere by simple we mean has no proper congruences or equivalently ideals), such that $\text{Nr}_n A$, the $n$-reduct of $A$ as defined in [11, Definition 2.6.28] is atomic for every $n < \omega$, then for each $i < \omega$, $X_i = \{ \neg x : x \in \text{At}_\text{Nr}_n A \}$ would be a non-principal type meaning that $\prod X_i = 0$, sometimes, such an $X_i$ is called non-isolated. Any representation omitting these non principal types, known to exist by the OTT, will provide a unique up to isomorphism an atomic model of the theory $T$ represented algebraically by $A$; namely the base of the representation which is countable. Here by representation omitting the $X_i$s, we mean an isomorphism $f : A \rightarrow B$, where $B \in \text{Cs}_\omega$ (the last is the class of cylindric set algebras of dimension $\omega$) such that $\bigcap_{x \in X_i} f(x) = \emptyset$ for all $i < \omega$. Now this idea extends to $L_n$ first order logic restricted to the first $n$ variables. We formulate it algebraically baring in mind that $\text{RCA}_n$ is the algebraic counterpart of $L_n$, in the sense that if $T$ is an $L_n$ theory formulated in a relational signature using only $n$ variables, then the quotient Lindenbaum-Tarski cylindric algebra $\text{Fr}_T \in \text{RCA}_n$, cf. [11, §4.3]. Given an atomic $A \in \text{RCA}_n$, then conversely there exists an $L_n$ theory $T$ such that $A \cong \text{Fr}_T$, and since $A$ is atomic, $T$ is also an atomic theory in the usual first order sense. The question is: Does $A$ have an atomic representation meaning, meaning there exists $B \in \text{Cs}_n$ with base $M$ say, i.e top element $^n M$, and an isomorphism $f : A \rightarrow B$ such that $\bigcup_{x \in \text{At}_A} f(x) = ^n M$, or equivalently $f$ omits the non-principal type consist-
ing of co-atoms. A co-atom is simply the negation of an atom. From the modal point of view RCA\(_n\) can be viewed as the variety of modal algebras corresponding to the class of so-called square frames. A square frame \(\mathfrak{F}\) of dimension \(n\) is a triplet of the form \(\mathfrak{F} = \langle U, T_i, D_{ij}\rangle_{i,j<n}\), so that the set of worlds is the set of \(n\)-ary sequences in \(U\) and for \(i,j < n\), the accessibility relation \(T_i\) is defined via \((s,t) \in T_i \iff s(j) = t(j)\) for all \(j \neq i\) and \(D_{ij} = \{s \in U: s_i = s_j\}\). It can be shown that \(\text{RCA}_n = \text{SPc}_m\text{K}\) where \(\text{K}\) is the class of all square frames of dimension \(n\). Since \(L_n\) has a modal formalism provided by Venema in \([40]\), then \(\text{VT}\) can be generalized to multi-modal logics. In fact, \(\text{VT}\) will be adressed for the clique guarded fragments of \(L_n\).

While the classical Orey-Henkin OTT holds for \(L_{\omega,\omega}\), it is known, cf. \([5]\) that the OTT fails for \(L_n\) in the following (strong) sense. For every \(2 < n \leq l < \omega\), there is a countable and complete \(L_n\) atomic theory \(T\), and a single type, namely, the type consisting of co-atoms of \(T\), that is realizable in every model of \(T\), but cannot be isolated by a formula \(\phi\) using \(l\) variables. Such \(\phi\) will be referred to henceafter as a witness. Here we prove stronger negative OTTs for \(L_n\) when types are required to be omitted with respect to certain (much wider) generalized semantics, called \(m\)-flat and \(m\)-square with \(2 < n < m < \omega\). Roughly, if we zoom in by a movable window to an \(m\)-square model, there will come a point determined by \(m\) where we mistake the \(m\) square model for an ordinary model. Classical representations can be regarded as a limiting case; such representations are \(\omega\)-square. An \(m\)-flat model is \(m\)-square, but the converse may fail; in this respect \(m\)-flatness is a ‘better’ approximation to a representation, but still on a local level. The semantics provided by \(m\)-flatness respects commutativity of cylindrifications on’ \(m\) squares’. These locally relativized representations were invented by Hirsch and Hodkinson in the context of relation algebras, cf. \([15,\ \text{Chapter 13}]\) and were adapted to CAs in \([35, \S5]\) which is the form we stick to in this paper. Considering such clique-guarded semantics swiftly leads us to rich territory.

**Blow up and blur constructions in connection to failure of OTTs:**
To violate a special case of OTT for the clique guarded fragments of \(L_n\), which we refer to below as *Vaught’s Theorem*, we use a blow up and blur construction applied to a finite extremely simple rainbow CA\(_n\) (to be defined below) to prove non-atom canonicity of several varieties of relation and cylindric algebras. This subtle construction may be applied to any two classes \(L \subseteq K\) of completely additive BAOs. One takes an atomic \(A \notin K\) (usually but not always finite), blows it up, by splitting one or more of its atoms each to infinitely many subatoms, obtaining an (infinite) countable atomic \(\mathfrak{B}b(A) \in L\), such that \(A\) is blurred in \(\mathfrak{B}b(A)\) meaning that \(A\) does not embed in \(\mathfrak{B}b(A)\), but \(A\) embeds in the Dedekind-MacNeille completion of \(\mathfrak{B}b(A)\), namely, \(\mathfrak{CmAt}\mathfrak{B}b(A)\). Then any class \(M\) say, between \(L\) and \(K\) that is closed under forming subalgebras will not be atom-canonical, for \(\mathfrak{B}b(A) \in L(\subseteq M)\), but \(\mathfrak{CmAt}\mathfrak{B}b(A) \notin K(\supseteq M)\) because \(A \notin M\)
and $SM = M$. We say, in this case, that $L$ is not atom-canonical with respect to $K$. This method is applied to $K = S\text{R}CA_l$, $l \geq 5$ and $L = \text{RRA}$ in [15, §17.7] and to $K = \text{RRA}$ and $L = \text{RRA} \cap \text{RCA}_k$ for all $k \geq 3$ in [5], and will applied below to $K = \text{SNr}_n \text{CA}_{n+k}$, $k \geq 3$ where $\text{Nr}_n$ denotes the operation of forming the $n$-neat reduct in the sense of [11, Definition 2.6.38] (to be defined below) and $L = \text{RCA}_n$, where $\text{Ra}$ denotes the operator of forming relation algebra reducts (applied to classes) of CAs, respectively, cf. [11, Definition 5.2.7].

We obtain negative results of the form: There exists a countable, complete and atomic $L_n$ first order theory $T$ in a signature $L$, meaning that the Tarski Lindenbaum quotient algebra $\mathfrak{Fm}_T$ is atomic, such that the type $\Gamma$ consisting of co-atoms $\mathfrak{m}_T$ is realizable in every $m$-square model, but $\Gamma$ cannot be isolated using $\leq l$ variables, where $n \leq l < m \leq \omega$. A co-atom of $\mathfrak{m}_T$ is the negation of an atom in $\mathfrak{m}_T$, that is to say, is an element of the form $\Psi/\equiv_T$, where $\Psi/\equiv_T = (\neg \phi/\equiv_T) = (\phi/\equiv_T)$ and $\phi/\equiv_T$ is an atom in $\mathfrak{m}_T$ (for $L$-formulas, $\phi$ and $\psi$). Here the quotient algebra $\mathfrak{m}_T$ is formed relative to the congruence relation of semantical equivalence modulo $T$. An $m$-square model of $T$ is an $m$-square representation of $\mathfrak{m}_T$. The last statement denoted by $\Psi(l,m)$, short for Vaught’s Theorem ($\text{VT}$) fails at (the parameters) $l$ and $m$. Let $\text{VT}(l,m)$ stand for $\text{VT}$ holds at $l$ and $m$, so that by definition $\Psi(l,m) \iff \sim \text{VT}(l,m)$. We also include $l = \omega$ in the equation by defining $\text{VT}(\omega,\omega)$ as $\text{VT}$ holds for $L_{\omega,\omega}$. Atomic countable first order theories have atomic countable models. It is well known that $\text{VT}(\omega,\omega)$ is a direct consequence of the Orey-Henkin OTT. In this paper we provide strong evidence that $\text{VT}$ fails everywhere in the sense that for the permitted values $n \leq l, m \leq \omega$, namely, for $n \leq l < m \leq \omega$ and $l = m = \omega$, $\text{VT}(l,m) \iff l = m = \omega$. For example, from the non-atom canonicity of $\text{RCA}_n$ with respect to the variety of $\text{CA}_n$s having $t(n)$-square representations $(\supseteq \text{SNr}_n \text{CA}_{t(n)})$, where $t(n) = n(n+1)/2 + 2$ we prove $\Psi(n, t(n) + k)$ for $k \geq 0$ and from the non-atom canonicity of $\text{Nr}_n \text{CA}_{n+k} \cap \text{RCA}_n$ with respect to $\text{RCA}_n$ for all $k \in \omega$, we prove $\Psi(l, \omega)$ for all finite $l \geq n$. Both results are obtained by blowing up and blurring finite algebras; a rainbow $\text{CA}_n$ in the former case, and a finite $\text{Ra}$ (whose number of atoms depend on $k$) in the second case.

**Complete representations, omitting types and atom canonicity.** Lately, it has become fashionable to study representations that preserve infinitary meets and joins. This phenomenon is extensively discussed in [33], where it is shown that it has affinity with the algebraic notion of complete representations for cylindric like algebras and atom-canonicity in varieties of Boolean algebras with operators, a prominent persistence property studied in modal logic. The typical question is: given an algebra and a set of meets, is there a representation that carries this set of meets to set theoretic intersections? (assuming that our semantics is specified by set algebras, with the concrete Boolean operation of intersection among its basic operations.)
When the algebra in question is countable, and we have countably many meets; this is an algebraic version of an omitting types theorem; the representation omits the given set of possibly infinitary meets or non-principal types. When it is only one meet consisting of co-atoms, in an atomic algebra, this representation is a complete one. The correlation of atomicity to complete representations has caused a lot of confusion in the past. It was mistakenly thought for a while, among algebraic logicians, that atomic representable relation and cylindric algebras are completely representable, an error attributed to Lyndon and now referred to as Lyndon’s error. For Boolean algebras, however this is true. The class of completely representable algebras is simply the class of atomic ones. An analogous result holds for Polyadic algebras, cf. [1]. We recall that for BAOs \( \mathfrak{A} \) and \( \mathfrak{B} \) having the same signature, \( \mathfrak{B} \) is dense in \( \mathfrak{A} \), written \( \mathfrak{B} \subseteq d \mathfrak{A} \), if \( \mathfrak{B} \) is subalgebra of \( \mathfrak{A} \) such that for all non-zero \( a \in \mathfrak{A} \), there exists a non-zero \( b \in \mathfrak{B} \) with \( b \leq a \). It is known that for any \( K \) of BAOs \( K \subseteq S_d K \subseteq S_c K \); and the inclusion are proper for Boolean algebras (without operators). For a class \( K \) of BAO, we let \( S_c K = \{ \mathfrak{B} : (\exists \mathfrak{A} \in K)(\forall X \subseteq \mathfrak{A})[\sum^{\mathfrak{A}} X = 1 \implies \sum X \text{ exists in } \mathfrak{B} \text{ and } \sum^{\mathfrak{B}} X = 1] \} \).

It is known that for any \( K \) of BAOs, \( K \subseteq S_d K \subseteq S_c K \). If \( K \) happens to be the class of Boolean algebras (without operators) then these inclusions are proper In [28] it is proved that for any pair of ordinals \( \alpha < \beta \), the class \( \text{Nr}_\alpha \text{CA}_\beta \) is not elementary. A different model theoretic proof for finite \( \alpha \) is given in [34, Theorem 5.4.1]. This result is extended to many cylindric like algebras like Halmos’ polyadic algebras with and without equality, and Pinter’s substitution algebras in [29,30]. The class \( \text{CRCA}_n \) of completely representable \( \text{CA}_n \)s is proved not be elementary by Hirsch and Hodkinson in [14]. Neat embeddings and complete representations are linked in [33, Theorem 5.3.6] where it is shown that \( \text{CRCA}_n \) coincides with the class \( S_c \text{Nr}_n \text{CA}_\omega \) on atomic algebra having countably many atoms. Below it is proved that this characterization does not generalize to atomic algebras having uncountably many atoms. Completely analogous results are obtained for RAs, that is to say, \( S_c \text{RaCA}_\omega \) and \( \text{CRRA} \) coincide on atomic algebras with countably many atoms, and this characterization does not generalize to the case of atomic RAs having uncountably many atoms. In fact, we shall prove that there exists an atomless \( \mathcal{C} \in \text{CA}_\omega \), such that for all \( n < \omega \), \( \text{Nr}_n \mathfrak{A} \) and \( \text{Ra} \mathfrak{A} \) are atomic algebras having uncountably many atoms, but do not have a complete representation. We show that for any \( 2 < n < \omega \), any class \( K \) such that \( \text{Nr}_n \text{CA}_\omega \cap \text{CRCA}_n \subseteq K \subseteq S_d \text{Nr}_n \text{CA}_n+1 \), or \( S_d \text{Nr}_n \text{CA}_\omega \subseteq K \subseteq S_c \text{Nr}_n \text{CA}_n+3 \), \( K \) is not elementary. The notion of complete representations has been linked to Martin’s axiom, omitting types theorems and existence of atomic models in model theory, cf. [33].

**On the techniques used:** In this paper, we use combinatorial game theory combined with basic graph theory resorting to (as mentioned above) Rainbow construction which is extremely efficient and flexible in constructing subtle delicate counterexamples. Rainbow constructions are based on two
player deterministic games and as the name suggests they involve ‘colours’. Such games happen to be simple Ehrenfeucht–Fraïssé forth games where the two players ℓloise and ∀belard, between them, use pebble pairs outside the board, each player pebbling one of the two structures which she/he sticks to it during the whole play. In the number of rounds played (that can be transfinite), ∃ tries to show that two simple relational structures G (the greens) and R (the reds) have similar structures while ∀ tries to show that they are essentially distinct. Such structures may include ordered structures and complete irreflexive graphs, such as finite ordinals, ω₁, N, Z or R. A winning strategy for either player in the Ehrenfeucht–Fraïssé game can be lifted to winning strategy in a rainbow game played on so-called atomic networks on a rainbow atom structure (for both CAₖs and RAₖs ) based also on G and R. Once G and R are specified, the rainbow atom structure is uniquely defined. Though more (rainbow) colours (like whites and shades of yellow) are involved in the rainbow atom structure, the crucial thing here is that the number of rounds and nodes in networks used in the rainbow game, depend recursively on the number of rounds and pebble pairs in the simple Ehrenfeucht–Fraïssé forth two player game played on G and R. Due to the control on winning strategy’s in terms of the relational structures (G and R) chosen in advance, and the number of pebble pairs used outside the board, rainbow constructions have proved highly effective in providing subtle counterexamples to really bewildering ‘yes or no’ assertions for both CAₖs and RAₖs (relation algebras) cf. [14,15,17,35]

2. The algebras and some basic concepts

For a set V, \( \mathcal{B}(V) \) denotes the Boolean set algebra \( \langle \wp(V), \cup, \cap, \sim, \emptyset, V \rangle \). Let \( U \) be a set and \( \alpha \) an ordinal; \( \alpha \) will be the dimension of the algebra. For \( s, t \in ^\alpha U \) write \( s \equiv_i t \) if \( s(j) = t(j) \) for all \( j \neq i \). For \( X \subseteq ^\alpha U \) and \( i, j < \alpha \), let

\[
C_iX = \{ s \in ^\alpha U : (\exists t \in X)(t \equiv_i s) \}
\]

and

\[
D_{ij} = \{ s \in ^\alpha U : s_i = s_j \}.
\]

\( \langle \mathcal{B}(^\alpha U), C_i, D_{ij} \rangle_{i,j<\alpha} \) is called the full cylindric set algebra of dimension \( \alpha \) with unit (or greatest element) \( ^\alpha U \). Any subalgebra of the latter is called a set algebra of dimension \( \alpha \). Examples of subalgebras of such set algebras arise naturally from models of first order theories. Indeed, if \( M \) is a first order structure in a first order signature \( L \) with \( \alpha \) many variables, then one manufactures a cylindric set algebra based on \( M \) as follows, cf. [11, §4.3]. Let

\[
\phi^M = \{ s \in ^\alpha M : M \models \phi[s] \},
\]
(here $M \models \phi[s]$ means that $s$ satisfies $\phi$ in $M$), then the set $\{\phi^M : \phi \in Fm^L\}$ is a cylindric set algebra of dimension $\alpha$, where $Fm^L$ denotes the set of first order formulas taken in the signature $L$. To see why, we have:

\[
\phi^M \cap \psi^M = (\phi \land \psi)^M, \\
\alpha M \sim \phi^M = (-\phi)^M, \\
C_i(\phi^M) = (\exists v_i \phi)^M, \\
D_{ij} = (x_i = x_j)^M.
\]

Following [11], $C_{\alpha}$ denotes the class of all subalgebras of full set algebras of dimension $\alpha$. The (equationally defined) $CA_{\alpha}$ class is obtained from cylindric set algebras by a process of abstraction and is defined by a finite schema of equations given in [11, Definition 1.1.1] that holds of course in the more concrete set algebras.

**Definition 2.1.** Let $\alpha$ be an ordinal. By a cylindric algebra of dimension $\alpha$, briefly a $CA_{\alpha}$, we mean an algebra

\[
\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, d_{ij}\rangle_{\kappa, \lambda < \alpha},
\]

where $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra such that 0, 1, and $d_{ij}$ are distinguished elements of $A$ (for all $j, i < \alpha$), $-$ and $c_i$ are unary operations on $A$ (for all $i < \alpha$), $+$ and $\cdot$ are binary operations on $A$, and such that the following equations are satisfied for any $x, y \in A$ and any $i, j, \mu < \alpha$:

1. $c_i 0 = 0$,
2. $x \leq c_i x$ (i.e., $x + c_i x = c_i x$),
3. $c_i(x \cdot c_i y) = c_i x \cdot c_i y$,
4. $c_i c_j x = c_j c_i x$,
5. $d_{ii} = 1$,
6. if $i \neq j, \mu$, then $d_{j\mu} = c_i(d_{ji} \cdot d_{i\mu})$,
7. if $i \neq j$, then $c_i(d_{ij} \cdot x) \cdot c_i(d_{ij} \cdot -x) = 0$.

Our main results involve the central notion of neat reducts:

**Definition 2.2.** Let $\alpha < \beta$ be ordinals and $\mathfrak{B} \in CA_{\beta}$. Then the $\alpha$-neat reduct of $\mathfrak{B}$, in symbols $\mathfrak{N} \alpha \mathfrak{B}$, is the algebra obtained from $\mathfrak{B}$, by discarding cylindrifiers and diagonal elements whose indices are in $\beta \sim \alpha$, and restricting the universe to the set $\mathfrak{N} \alpha \mathfrak{B} = \{x \in \mathfrak{B} : \{i \in \beta : c_i x \neq x\} \subseteq \alpha\}$.

Let $\alpha$ be any ordinal. If $\mathfrak{A} \in CA_{\alpha}$ and $\mathfrak{A} \subseteq \mathfrak{N} \alpha \mathfrak{B}$, with $\mathfrak{B} \in CA_{\beta}$ ($\beta > \alpha$), then we say that $\mathfrak{A}$ neatly embeds in $\mathfrak{B}$, and that $\mathfrak{B}$ is a $\beta$-dilation of $\mathfrak{A}$, or simply a dilation of $\mathfrak{A}$ if $\beta$ is clear from context. For $K \subseteq CA_{\beta}$, and
For any $\alpha < \beta$, $\text{Nr}_\alpha \mathcal{B} = \{ \text{Nr}_\alpha \mathcal{B} : \mathcal{B} \in \mathcal{K} \} \subseteq \text{CA}_\alpha$. Let $\alpha$ be an ordinal and $\mathfrak{A} \in \text{CA}_\alpha$. For any $i,j,l < \alpha$, let $s^i_j x = x$ if $i = j$ and $s^i_j x = c_j(d_{ij} \cdot x)$ if $i \neq j$. Let $s(i,j) x = s^i_j s^j_i x$. In the next definition, in its first item we define the notion of forming $\alpha$-neat reducts of $\text{CA}_\beta$s with $\beta > \alpha$, in symbols $\text{Nr}_\alpha$, and in the second item we define relation algebras obtained from cylindric algebras using the operator $\text{Nr}_2$.

**Definition 2.3.** 1. Assume that $\alpha < \beta$ are ordinals and that $\mathfrak{B} \in \text{CA}_\beta$. Then the $\alpha$-neat reduct of $\mathfrak{B}$, in symbols $\text{Nr}_\alpha \mathfrak{B}$, is the algebra obtained from $\mathfrak{B}$, by discarding cylindrifiers and diagonal elements whose indices are in $\beta \setminus \alpha$, and restricting the universe to the set $\text{Nr}_\alpha B = \{ x \in \mathfrak{B} : \{ i \in \beta : c_i x \neq x \} \subseteq \alpha \}$.

2. Assume that $\alpha \geq 3$. Let $\mathfrak{A} \in \text{CA}_\alpha$. Then $\text{Ra} \mathfrak{A} = \langle \text{Nr}_2 \mathfrak{A} : +, -, :, d_{01} \rangle$ where for any $x, y \in \text{Nr}_2 \mathfrak{A}$, $x; y = c_2(s^2_1 x \cdot s^0_2 y)$ and $x =_2 s(0.1)x$.

If $\mathfrak{A} \in \text{CA}_3$, $\text{Ra} \mathfrak{A}$, having the same signature as RA may not be a relation algebra as associativity of the (abstract) composition operation may fail, but for $\alpha \geq 4$, $\text{Ra} \text{CA}_\beta \subseteq \text{RA}$. It is straightforward to check that $\text{Nr}_\alpha \mathfrak{B} \in \text{CA}_\alpha$. Let $\alpha < \beta$ be ordinals. If $\mathfrak{A} \in \text{CA}_\alpha$ and $\mathfrak{A} \subseteq \text{Nr}_\alpha \mathfrak{B}$, with $\mathcal{B} \in \text{CA}_\beta$, then we say that $\mathfrak{A}$ neatly embeds in $\mathfrak{B}$, and that $\mathcal{B}$ is a $\beta$-dilation of $\mathfrak{A}$, or simply a dilation of $\mathfrak{A}$ if $\beta$ is clear from context. For $\mathcal{K} \subseteq \text{CA}_\beta$, we write $\text{Nr}_\alpha \mathcal{K}$ for the class $\{ \text{Nr}_\alpha \mathfrak{B} : \mathfrak{B} \in \mathcal{K} \}$. Let $\alpha$ be an ordinal. Following [11], $\text{Gs}_\alpha$ denotes the class of generalized cylindric set algebra of dimension $\alpha$: $\mathfrak{C} \in \text{Gs}_\alpha$, if $\mathfrak{C}$ has top element $V$ a disjoint union of cartesian squares, that is $V = \bigcup_{i \in I}^\alpha V_i$, $I$ is a non-empty indexing set, $U_i \neq \emptyset$ and $U_i \cap U_j = \emptyset$ for all $i \neq j$. The operations of $\mathfrak{C}$ are defined like in cylindric set algebras of dimension $\alpha$ relativized to $V$. By the same token the variety of representable relation algebras is denoted by RRA. It is known that $\text{IGs}_\alpha = \text{RC} \text{A}_\alpha = \text{SNr}_\alpha \text{CA}_{\alpha + \omega} = \bigcap_{k \in \omega} \text{SNr}_\alpha \text{CA}_{\alpha + k}$ and that $\text{RRA} = \text{SRa} \text{CA}_\omega = \bigcap_{k \in \omega} \text{SRa} \text{CA}_{\omega + k}$.

We shall have the occasion to deal with (in addition to CAs), the following cylindric-like algebras [4]: Df short for diagonal free cylindric algebras, Sc short for Pinter’s substitution algebras, QA(QEA) short for quasi-polyadic (equality) algebras, PA(PEA) short for polyadic (equality) algebras. For $\mathcal{K}$ any of these classes and $\alpha$ any ordinal, we write $\mathfrak{K}_\alpha$ for variety of $\alpha$-dimensional $\mathcal{K}$ algebras which can be axiomatized by a finite schema of equations, and $\text{RK}_\alpha$ for the class of representable $\mathcal{K}_\alpha$s, which happens to be a variety too (that cannot be axiomatized by a finite schema of equations for $\alpha > 2$ unless $\mathcal{K} = \text{PA}$ and $\alpha \geq \omega$). The standard reference for all the classes of algebras mentioned previously is [11]. We recall the concrete versions of such algebras. Let $\tau : \alpha \to \alpha$ and $X \subseteq \alpha U$, then

$$S_\tau X = \{ s \in \alpha U : s \circ \tau \in X \}.$$
For $i, j \in \alpha$, $[i|j]$ is the replacement on $\alpha$ that sends $i$ to $j$ and is the identity map on $\alpha \sim \{i\}$ while $[i, j]$ is the transposition on $\alpha$ that interchanges $i$ and $j$.

- A diagonal free cylindric set algebra of dimension $\alpha$ is an algebra of the form $\langle \mathcal{B}(\alpha U), C_i \rangle_{i,j<\alpha}$.
- A Pinter’s substitution set algebra of dimension $\alpha$ is an algebra of the form $\langle \mathcal{B}(\alpha U), C_i, S_{[i|j]} \rangle_{i,j<\alpha}$.
- A quasi-polyadic set algebra of dimension $\alpha$ is an algebra of the form $\langle \mathcal{B}(\alpha U), C_i, S_{[i|j]}, S_{[i,j]} \rangle_{i,j<\alpha}$.
- A quasi-polyadic equality set algebra is an algebra of the form $\langle \mathcal{B}(\alpha U), C_i, S_{[i|j]}, S_{[i,j]}, D_{ij} \rangle_{i,j<\alpha}$.
- A polyadic set algebra of dimension $\alpha$ is an algebra of the form $\langle \mathcal{B}(\alpha U), C_i, S_{\tau} \rangle_{\tau: \alpha \to \alpha}$.

- A polyadic equality set algebra of dimension $\alpha$ is an algebra of the form $\langle \mathcal{B}(\alpha U), C_i, S_{\tau} \rangle_{\tau: \alpha \to \alpha, i,j<\alpha}$

The reader is kindly referred to [9, 24, 26, 36] for an extensive overview of polyadic algebras of infinite dimensions.

Let $\alpha$ be an ordinal. For any such abstract class of algebras $K_\alpha$ in the above table, $R K_\alpha$ is defined to be the subdirect product of set algebras of dimension $\alpha$. A cartesian square of dimension $\alpha$ is a set of the form $\langle \mathcal{B}(\alpha U), C_i, S_{\tau} \rangle_{\tau: \alpha \to \alpha, i,j<\alpha}$; these appear as top elements of $\mathcal{C}_\alpha$. We let $G_\alpha$ denote the class of generalized set algebras of dimension $\alpha$; $A \in G_\alpha$ if and only if $A$ has top element a disjoint union of cartesian squares of dimension $\alpha$ and the cylindric operations are defined like in set algebras. It is known that $R_{\alpha} = I_\alpha G_\alpha$.

For $\alpha < \omega$, $P_{\alpha}(PEA_\alpha)$ is definitionally equivalent to $Q_{\alpha}(QEA_\alpha)$ which is no longer the case for infinite $\alpha$ where the deviation is largely significant.

<table>
<thead>
<tr>
<th>class</th>
<th>extra non-Boolean operators</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_f_\alpha$</td>
<td>$c_i: i &lt; \alpha$</td>
</tr>
<tr>
<td>$S_{c_\alpha}$</td>
<td>$c_i, s_{i}^j: i, j &lt; \alpha$</td>
</tr>
<tr>
<td>$C_{A_\alpha}$</td>
<td>$c_i, d_{ij}: i, j &lt; \alpha$</td>
</tr>
<tr>
<td>$P_{\alpha}$</td>
<td>$c_i, s_t: i &lt; n, t \in ^\alpha \alpha$</td>
</tr>
<tr>
<td>$PEA_{\alpha}$</td>
<td>$c_i, d_{ij}, s_t: i, j &lt; n, t \in ^\alpha \alpha$</td>
</tr>
<tr>
<td>$Q_{\alpha}$</td>
<td>$c_i, s_{i}^j, s_{[i,j]}: i, j &lt; \alpha$</td>
</tr>
<tr>
<td>$QEA_{\alpha}$</td>
<td>$c_i, d_{ij}, s_{i}^j, s_{[i,j]}: i, j &lt; \alpha$</td>
</tr>
</tbody>
</table>

Fig. 1. Non-Boolean operators for the classes
For example a countable QA_\omega has a countable signature, while a countable PA_\omega has an uncountable signature having the same cardinality as (substitutions in) \omega. The class of completely representable K_\alpha s (K any of the above classes) is denoted by CRK_\alpha. We recall the definition for CAs of finite dimension. The rest of the cases are defined similarly.

**Definition 2.4.** Let n < \omega. Then \mathcal{A} \in CA_n is completely representable, if there exists \mathcal{B} \in GS_n and an isomorphism f: \mathcal{A} \rightarrow \mathcal{B} such that for all X \subseteq \mathcal{A}, f(\prod X) = \bigcap_{x \in X} f(x) whenever \prod X exists.

If \mathcal{A} is an atomic CA_n, then an isomorphism f: \mathcal{A} \rightarrow \mathcal{B}, where \mathcal{B} \in GS_n has top element V, is atomic, if \bigcup_{a \in At\mathcal{A}} f(a) = V. It can be easily shown that f is a complete representation of \mathcal{A} if and only if \mathcal{A} is atomic and f is an atomic representation. Considering polyadic algebras, we will encounter PEA_\alpha and PA_\alpha, \alpha an infinite ordinal (having all substitutions and infinitary cylindrifications) only once in Theorem 3.14. We deal mostly with QAs and QEAs. For a BAO, \mathcal{A} say, for any ordinal \alpha, PEA_\alpha A denotes the cylindric reduct of \mathcal{A} if it has one, PE_{sc}\alpha A denotes the Sc reduct of \mathcal{A} if it has one, and PE_{df}\alpha A denotes the reduct of \mathcal{A} obtained by discarding all the operations except for cylindrifications. If \mathcal{A} is any of the above classes, it is always the case that PE_{df}\alpha A \in DF_\alpha. If \mathcal{A} \in CA_\alpha, then PE_{sc}\alpha A \in Sc_\alpha, and if \mathcal{A} \in QEA_\alpha then PE_{ca}\alpha A \in CA_\alpha. Roughly speaking for an ordinal \alpha, CA_\alpha s are not expansions of Sc_\alpha s, but they are definitionally equivalent to expansions of Sc_\alpha, because the s_\alpha s are term definable in CA_\alpha s by s_\alpha(x) = c_\alpha(x \cdot d_{ij}) (i, j < \alpha). This operation reflects algebraically the substitution of the variable v_j for v_i in a formula such that the substitution is free; this can be always done by reindexing bounded variables. In such situation, we say that Scs are generalized reducts of CAs. However, CA_\alpha s and QA_\alpha are (real) reducts of QEAs, (in the universal algebraic sense) simply obtained by discarding the operations in their signature not in the signature of their common expansion QEA_\alpha. We give a finite approximate equational axiomatization of the concrete algebras defined above, which are the prime source of inspiration for these axiomatizations introduced to capture representability. However, like for CAs, this works only for certain special cases like the locally finite dimensional algebras, but does not generalize much further, cf. Proposition 2.7.

**Definition 2.5.** Let \alpha be an ordinal. We say that a variety V is a variety between DF_\alpha and QEA_\alpha if the signature of V expands that of DF_\alpha and is contained in the signature of QEA_\alpha. Furthermore, any equation formulated in the signature of DF_\alpha that holds in V also holds in Sc_\alpha and all equations that hold in V holds in QEA_\alpha.

Proper examples include Sc, CA_\alpha and QA_\alpha (meaning strictly between). Analogously we can define varieties between Sc_\alpha and CA_\alpha or QA_\alpha and QEA_\alpha, and more generally between a class K of BAOs and a generalized reduct of it. Notions like neat reducts generalize verbatim to such algebras, namely, to DFs and QEAs, and in any variety in between. This stems from the obser-
oration that for any pair of ordinals \( \alpha < \beta \), \( \mathfrak{A} \in \text{QEA}_\beta \) and any non-Boolean extra operation in the signature of \( \text{QEA}_\beta \), \( f \) say, if \( x \in \mathfrak{A} \) and \( \Delta x \subseteq \alpha \), then \( \Delta (f(x)) \subseteq \alpha \). Here \( \Delta x = \{ i \in \beta : c_i x \neq f \} \) is referred as the dimension set of \( x \); it reflects algebraically the essentially free variables occurring in a formula \( \phi \). A variable is essentially free in a formula \( \Psi \iff \) it is free in every formula equivalent to \( \Psi \).\(^1\) Therefore given a variety \( \mathcal{V} \) between \( \text{Sc}_\beta \) and \( \text{QEA}_\beta \), if \( \mathfrak{B} \in \mathcal{V} \) then the algebra \( \mathfrak{N}_n \mathfrak{B} \) having universe \( \{ x \in \mathfrak{B} : \Delta x \subseteq \alpha \} \) is closed under all operations in the signature of \( \mathcal{V} \).

**Definition 2.6.** Let \( 2 < n < \omega \). For a variety \( \mathcal{V} \) between \( \text{Df}_n \) and \( \text{QEA}_n \), a \( \mathcal{V} \) set algebra is a subalgebra of an algebra, having the same signature as \( \mathcal{V} \), of the form \( \langle \mathfrak{B}(n U), f_i^U \rangle \), say, where \( f_i^U \) is identical to the interpretation of \( f_i \) in the class of quasipolyadic equality set algebras. Let \( \mathfrak{A} \) be an algebra having the same signature of \( \mathcal{V} \); then \( \mathfrak{A} \) is a representable \( \mathcal{V} \) algebra, or simply representable \( \iff \mathfrak{A} \) is isomorphic to a subdirect product of \( \mathcal{V} \) set algebras. We write \( \mathcal{R} \mathcal{V} \) for the class of representable \( \mathcal{V} \) algebras.

It can be proved that the class \( \mathcal{R} \mathcal{V} \), as defined above, is also closed under \( \mathcal{H} \), so that it is a variety. This can be proved using the same argument to show that \( \text{RCA}_n \) is a variety, cf. Corollary [11, 3.1.77]. Take \( \mathfrak{A} \in \mathcal{R} \mathcal{V} \), an ideal \( J \) of \( \mathfrak{A} \), then show that \( \mathfrak{A}/J \) is in \( \mathcal{R} \mathcal{V} \). Ideals in \( \text{BAO} \)s are defined as follows. We consider only \( \text{BAO} \)s with extra unary non-Boolean operators to simplify notation. If \( \mathfrak{A} \) is a \( \text{BAO} \), then \( J \subseteq \mathfrak{A} \) is an ideal in \( J \) if is a Boolean ideal and for any extra non-Boolean operator \( f \), say, in the signature of \( \text{BAO} \), and \( x \in \mathfrak{A} \), \( f(x) \in \mathfrak{A} \); the quotient algebra \( \mathfrak{A}/J \) is defined the usual way since ideals defined in this way correspond to congruence relations defined on \( \mathfrak{A} \).

**Proposition 2.7.** Let \( 2 < n < \omega \). Let \( \mathcal{V} \) be a variety between \( \text{Df}_n \) and \( \text{QEA}_n \). Then \( \mathcal{R} \mathcal{V} \) is not a finitely axiomatizable variety.

**Proof.** In [20] a sequence \( \langle \mathfrak{A}_i : i \in \omega \rangle \) of algebras is constructed such that \( \mathfrak{A}_i \in \text{QEA}_n \) and \( \mathfrak{N} \text{Df}_n \mathfrak{A}_i \notin \mathcal{R} \text{Df}_n \), but \( \Pi_{i \in \omega} \mathfrak{A}_i/F \in \text{RQEA}_n \) for any non principal ultrafilter on \( \omega \). An application of Los’ Theorem, taking the ultraproduct of \( \mathfrak{A}_i \)s, finishes the proof. In more detail, let \( \mathfrak{N} \mathcal{V} \) denote restricting the signature to that of \( \mathcal{V} \). Then \( \mathfrak{N} \mathfrak{A}_i \notin \mathcal{R} \mathcal{V} \) and \( \mathfrak{N} \Pi_{i \in I} (\mathfrak{A}_i/F) \in \mathcal{R} \mathcal{V} \).

The last result generalizes to infinite dimensions replacing finite axiomatization by axiomatized by a finite schema, cf. [11, 18]. We consider relation algebras as algebras of the form \( \mathcal{R} = \langle R, +, -, 1', \cdot, \circ, \bowtie, \cdot, \rangle \), where \( \langle R, +, -, \cdot \rangle \) is a Boolean algebra \( 1' \in R \), \( \cdot \) is a unary operation and \( \circ \) is a binary operation. A relation algebra is representable \( \iff \) it is isomorphic to a subalgebra of the form \( \langle \phi(X), \cup, \cap, \cdot, \bowtie, \circ, \text{Id} \rangle \), where \( X \) is an equivalence relation, \( 1' \) is interpreted as the identity relation, \( \cdot \) is the operation of forming converses, and; is interpreted as composition of relations. Following standard

\(^1\)It can well happen that a variable is free in formula that is equivalent to another formula in which this same variable is not free.
notation, \((R)\ra\) denotes the class of (representable) relation algebras. The class \(\ra\) is a discriminator variety that is finitely axiomatizable, cf. [15, Definition 3.8, Theorems 3.19]. We let \(\text{CRRA}\) and \(\text{LRRA}\), denote the classes of completely representable \(\ra\)s, and its elementary closure, namely, the class of \(\ra\)s satisfying the Lyndon conditions as defined in [15, §11.3.2], respectively. Complete representability of \(\ra\)s is defined like the \(\ca\) case. All of the above classes of algebras are instances of \(\text{BAO}\)s. The action of the non-Boolean operators in a completely additive (where operators distribute over arbitrary joins componentwise) atomic \(\text{BAO}\), is determined by their behavior over the atoms, and this in turn is encoded by the \(\text{atom structure}\) of the algebra.

**Definition 2.8 (Atom Structure).** Let \(\mathfrak{A} = \langle A, +, -, 0, 1, \Omega_i : i \in I \rangle\) be an atomic \(\text{BAO}\) with non-Boolean operators \(\Omega_i : i \in I\). Let the rank of \(\Omega_i\) be \(\rho_i\). The \textit{atom structure} \(\text{At}\mathfrak{A}\) of \(\mathfrak{A}\) is a relational structure

\[
\langle \text{At}\mathfrak{A}, R_{\Omega_i} : i \in I \rangle
\]

where \(\text{At}\mathfrak{A}\) is the set of atoms of \(\mathfrak{A}\) and \(R_{\Omega_i}\) is a \((\rho(i) + 1)\)-ary relation over \(\text{At}\mathfrak{A}\) defined by

\[
R_{\Omega_i}(a_0, \cdots, a_{\rho(i)}) \iff \Omega_i(a_1, \cdots, a_{\rho(i)}) \geq a_0.
\]

**Definition 2.9 (Complex algebra).** Conversely, if we are given an arbitrary first order structure \(S = \langle S, r_i : i \in I \rangle\) where \(r_i\) is a \((\rho(i) + 1)\)-ary relation over \(S\), called an \textit{atom structure}, we can define its \textit{complex algebra}

\[
\text{Cm}(S) = \langle \wp(S), \cup, \setminus, \phi, S, \Omega_i \rangle_{i \in I},
\]

where \(\wp(S)\) is the power set of \(S\), and \(\Omega_i\) is the \(\rho(i)\)-ary operator defined by

\[
\Omega_i(X_1, \cdots, X_{\rho(i)}) = \{ s \in S : \exists s_1 \in X_1 \cdots \exists s_{\rho(i)} \in X_{\rho(i)}, r_i(s, s_1, \cdots, s_{\rho(i)}) \},
\]

for each \(X_1, \cdots, X_{\rho(i)} \in \wp(S)\).

It is easy to check that, up to isomorphism, \(\text{At}(\text{Cm}(S)) \cong S\) alway. If \(\mathfrak{A}\) is finite then of course \(\mathfrak{A} \cong \text{Cm}(\text{At}\mathfrak{A})\). An atom structure will be denoted by \(\text{At}\). An atom structure \(\text{At}\) has the signature of a class \(K\) of \(\text{BAO}\)s, if \(\text{CmAt} \in K\).
3. Non-atom canonicity of infinitely many varieties between $\text{RCA}_n$ and $\text{CA}_n$

3.1. Clique guarded semantics

Fix $2 < n < \omega$, and $K$ a variety between $\text{Sc}_n$ and $\text{QEA}_n$. We study three approaches to approximating the class $\text{RK}$ of representable $K$s, by (a) basis, (b) existence of dilations and finally (c) (locally well-behaved) relativized representations, in analogy to the relation algebra case dealt with in [15, Chapter 13]. Examples include $m$-flat and $m$-square representations, where $2 < n < m < \omega$. It will always be the case, unless otherwise explicitly indicated, that $1 < n < m < \omega$; $n$ denotes the dimension.

We identify notationally a set algebra with its universe. Let $M$ be a relativized representation of $A \in \text{CA}_n$, that is, there exists an injective homomorphism $f: A \to \varphi(V)$ where $V \subseteq {}^nM$ and $\bigcup_{s \in V} \text{rng}(s) = M$. For $s \in V$ and $a \in A$, we may write $a(s)$ for $s \in f(a)$. This notation does not refer to $f$, but whenever used then either $f$ will be clear from context, or immaterial in the context. We may also write $1^M$ for $V$. Let $\mathcal{L}(A)^m$ be the first order signature using $m$ variables and one $n$-ary relation symbol for each element of $A$. Allowing infinitary conjunctions, we denote the resulting signature taken in $L_{\infty,\omega}$ by $\mathcal{L}(A)^m_{\infty,\omega}$.

An $n$-clique, or simply a clique, is a set $C \subseteq M$ such $(a_0, \ldots, a_{n-1}) \in V = 1^M$ for all distinct $a_0, \ldots, a_{n-1} \in C$. Let

$$C^m(M) = \{ s \in {}^nM : \text{rng}(s) \text{ is an } n \text{ clique} \}.$$  

Then $C^m(M)$ is called the $n$-Gaifman hypergraph, or simply Gaifman hypergraph of $M$, with the $n$-hyperedge relation $1^M$. The $n$-clique-guarded semantics, or simply clique-guarded semantics, $\models_c$, are defined inductively. Let $f$ be as above. For an atomic $n$-ary formula $a \in A$, $i \in {}^nM$, and $s \in {}^nM$, $M, s \models_c a(x_{i_0}, \ldots, x_{i_{n-1}}) \iff (s_{i_0}, \ldots, s_{i_{n-1}}) \in f(a)$. For equality, given $i < j < m$, $M, s \models_c x_i = x_j \iff s_i = s_j$. Boolean connectives, and infinitary disjunctions, are defined as expected. Semantics for existential quantifiers (cylindrifiers) are defined inductively for $\phi \in \mathcal{L}(A)^m_{\infty,\omega}$ as follows: For $i < m$ and $s \in {}^mM$, $M, s \models_c \exists x_i \phi \iff$ there is a $t \in C^m(M)$, $t \equiv_i s$ such that $M, t \models_c \phi$.

**Definition 3.1.** Let $A \in \text{CA}_n$, $M$ a relativized representation of $A$ and $\mathcal{L}(A)^m$ be as above.

1. Then $M$ is said to be $m$-square, if witnesses for cylindrifiers can be found on $n$-cliques. More precisely, for all $\bar{s} \in C^m(M), a \in A, i < n$, and for any injective map $l: n \to m$, if $M \models_c a(s_{i(0)}, \ldots, s_{i(n-1)})$, then there exists $\bar{t} \in C^m(M)$ with $\bar{t} \equiv_i \bar{s}$, and $M \models a(t_{i(0)}, \ldots, t_{i(n-1)})$. 

**Remark.** When $i = n$, instead of $\bar{s} \models_c a(\bar{t}_{i(0)}, \ldots, \bar{t}_{i(n-1)})$, we mean $\bar{s} \models_c a(t_{i(0)}, \ldots, t_{i(n-1)})$.
ATOM CANONICITY AND FIRST ORDER DEFINABILITY

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(2) $M$ is said to be (infinitary) $m$-flat if it is $m$-square and for all $\phi \in (\Sigma(\mathcal{A}^\omega)\Sigma(\mathcal{A}))^m$, for all $s \in C^m(M)$, for all distinct $i, j < m$, we have $M \models_c [\exists x_i \exists x_j \phi \iff \exists x_j \exists x_i \phi](s)$.

**Definition 3.2.** An $n$-dimensional atomic network on an atomic algebra $\mathcal{A} \in \text{QEA}_n$ is a map $N : \nu \Delta \rightarrow \text{At}\mathcal{A}$, where $\Delta$ is a non-empty finite set of nodes, denoted by $\text{nodes}(N)$, satisfying the following consistency conditions for all $i < j < n$:

(i) If $\bar{x} \in \nu \text{nodes}(N)$ then $N(\bar{x}) \leq d_{ij} \iff \bar{x}_i = \bar{x}_j$.

(ii) If $\bar{x}, \bar{y} \in \nu \text{nodes}(N), i < n$ and $\bar{x} \equiv_i \bar{y}$, then $N(\bar{x}) \leq c_i N(\bar{y})$.

(iii) (Symmetry): if $\bar{x} \in \nu \text{nodes}(N)$, then $s_{[i,j]} N(\bar{x}) = N(\bar{x} \circ [i,j])$.

If $\mathcal{A} \in \text{CA}_n$, then an $\mathcal{A}$ network is a map defined like above satisfying only (i) and (ii). If $\mathcal{A} \in \text{QA}_n$, then an $\mathcal{A}$ network satisfies (ii) and (iii) together with the condition that if $\bar{x} \in \nu \text{nodes}(N)$, then $s_{[i,j]} N(\bar{x}) = N(\bar{x} \circ [i,j])$ (instead of (i)). Finally, if $\mathcal{A} \in \text{Sc}_n$ than an $\mathcal{A}$ network satisfies the last condition together with (ii).

The proof of the following lemma can be distilled from its RA analogue as formulated in [15, Theorem 13.20], by reformulating deep concepts originally introduced by Hirsch and Hodkinson for RAs in the CA context, involving the notions of hypernetworks and hyperbasis. This can (and will) be done. In the coming proof, we highlight the main ideas needed to perform such a transfer from RAs to CAs [15, Definitions 12.1, 12.9, 12.10, 12.25, Propositions 12.25, 12.27]. In all cases, the $m$-dimensional dilation stipulated in the statement of the theorem, will have top element $C^m(M)$, where $M$ is the $m$-relativized representation of the given algebra, and the operations of the dilation are induced by the $n$-clique-guarded semantics. For a class $K$ of BAOs, $K \cap \text{At}$ denotes the class of atomic algebras in $K$.

A set $V (\subseteq n^U)$ is diagonalizable if $s \in V \implies s \circ [i|j] \in V$. We write $S_c$ for the operation of forming complete subalgebras. That is to say, for a class $K$ of BAOs, $\mathcal{B} \in S_c K$ there is an $\mathcal{A} \in K$ such that $\mathcal{A} \subseteq \mathcal{B}$ and for all $X \subseteq \mathcal{A}$, if $\sum^\mathcal{A} X = 1$, then $\sum^\mathcal{B} X = 1$. For two BAOs, $\mathcal{A}$ and $\mathcal{B}$ having the same signature, we write $\mathcal{A} \subseteq_c \mathcal{B}$, if $\mathcal{A}$ is a complete subalgebra of $\mathcal{B}$.

**Lemma 3.3 ([15, Theorems 13.45, 13.36]).** Assume that $2 < n < m < \omega$ and let $\mathcal{A}$ be a BAO having the same signature as $\text{CA}_n$ and satisfying all the $\text{CA}_n$ axioms except possibly for comutativity of cylindrifications.

1. Then $\mathcal{A} \in \text{SNr}_n \text{CA}_m$ $\iff$ $\mathcal{A}$ has an infinitary $m$-flat representation $\iff$ $\mathcal{A}$ has an $m$-flat representation. Furthermore, if $\mathcal{A}$ is atomic, then $\mathcal{A}$ has a complete infinitary $m$-flat representation $\iff \mathcal{A} \in S_c \text{Nr}_n(\text{CA}_m \cap \text{At})$.

2. We can replace infinitary $m$-flat and $\text{CA}_m$ by $m$-square and $\text{D}_m$, respectively, where $\text{D}_m$ are set algebras having a diagonalizable top element $V$ with operations defined like $\text{Cs}_m$ restricted to $V$. 
In [5] a single blow up and blur construction was used to prove non-atom-canonicity of RRA and RCA\(_n\) for \(2 < n < \omega\). To obtain finer results, we use two blow up and blur constructions applied to rainbow algebras. To put things into a unified perspective, we formulate a definition:

**Definition 3.4.** Let \(\mathbf{M}\) be a variety of completely additive BAOs.

1. Let \(\mathfrak{A} \in \mathbf{M}\) be a finite algebra. We say that \(\mathfrak{D} \in \mathbf{M}\) is obtained by blowing up and blurring \(\mathfrak{A}\) if \(\mathfrak{D}\) is atomic, \(\mathfrak{A}\) does not embed in \(\mathfrak{D}\), but \(\mathfrak{A}\) embeds into \(C_{m} \text{At} \mathfrak{D}\).

2. Assume that \(K \subseteq L \subseteq \mathbf{M}\), such that \(S L = L\).

   a. We say that \(K\) is *not* atom-canonical with respect to \(L\) if there exists an atomic \(\mathfrak{D} \in K\) such that \(C_{m} \text{At} \mathfrak{D} \in L\). In particular, \(K\) is not atom-canonical \(\iff\) \(K\) not atom-canonical with respect to itself.

   b. We say that a finite algebra \(\mathfrak{A} \in \mathbf{M}\) detects that \(K\) is not atom-canonical with respect to \(L\), if \(\mathfrak{A} \in L\), and there is a(n atomic) \(\mathfrak{D} \in K\) obtained by blowing up and blurring \(\mathfrak{A}\).

The next proposition and its proof present the construction in [5] in the framework of definition 3.4.

**Proposition 3.5.** Let \(2 < n < \omega\). Then for any finite \(j > 0\), \(\text{RRA} \cap \text{RaCA}_{2+j}^+\) is not atom-canonical with respect to \(\text{RRA}\), and \(\text{RCA}_n \cap \text{Nr}_n \text{CA}_{n+j}^+\) is not atom-canonical with respect to \(\text{RCA}_n\).

From now on, unless otherwise indicated, \(n\) is fixed to be a finite ordinal > 2.

**Definition 3.6.** (1) Let \(K_n\) be any variety between \(\text{Sc}_n\) and \(\text{QEA}_n\). Assume that \(\mathfrak{A} \in K_n\) is atomic and that \(m, k \leq \omega\). The *atomic game* \(G_k^m(\text{At}\mathfrak{A})\), or simply \(G_k^m\), is the game played on atomic networks of \(\mathfrak{A}\) using \(m\) nodes and having \(k\) rounds [17, Definition 3.3.2], where \(\forall\) is offered only one move, namely, a *cylindrifier move*: Suppose that we are at round \(t > 0\). Then \(\forall\) picks a previously played network \(N_i:\ (\text{nodes}(N_i) \subseteq m), i < n, a \in \text{At}\mathfrak{A}, \bar{x} \in \text{nodes}(N_i)\), such that \(N_i(\bar{x}) \leq c_i a\). For her response, \(\exists\) has to deliver a network \(M\) such that \(\text{nodes}(M) \subseteq m, M \equiv_i N, \text{and there is } \bar{y} \in \text{nodes}(M)\) that satisfies \(\bar{y} \equiv_i \bar{x}\) and \(M(\bar{y}) = a\).

We write \(G_k(\text{At}\mathfrak{A})\), or simply \(G_k\), for \(G_k^m(\text{At}\mathfrak{A})\) if \(m \geq \omega\).

2. The \(\omega\)-rounded game \(G^m(\text{At}\mathfrak{A})\) or simply \(G^m\) is like the game \(G^m(\text{At}\mathfrak{A})\) except that \(\forall\) has the option to reuse the \(m\) nodes in play.

Observe that for \(k, m \leq \omega\), the games \(G_k^m(\text{At}\mathfrak{A})\) and \(G^m(\text{At}\mathfrak{A})\) depend on the signature of \(\mathfrak{A}\).

For a class \(K\) of BAOs, we denote by \(K_{\text{ad}}\) the class of completely additive algebras in \(K\). The following Lemma is proved in [35, Lemma 5.7] and [36, Lemma 4.3].

**Lemma 3.7.** Let \(2 < n < \omega\), and assume that \(m > n\). Let \(K\) be any variety between \(\text{Sc}_n\) and \(\text{QEA}_n\). If \(\mathfrak{A} \in S_n \text{Nr}_n K_{\text{ad}}^m\) is atomic, then \(\exists\) has a winning
strategy in $G^m(\text{At}\mathfrak{A})$. If $\mathfrak{A} \in \mathbb{K}$, and $\mathfrak{A}$ has a complete $m$-square representation then $\exists$ has a winning strategy in $G^m_\omega(\text{At}\mathfrak{A})$.

3.2. Blowing up and blurring finite rainbow relation and cylindric algebras

Rainbow construction: The rainbow construction in algebraic logic, invented by Hirsch and Hodkinson, reduces finding subtle differences between seemingly related notions or concepts using a very simple Ehrenfeucht–Fraïssé forth pebble game between two players $\forall$ and $\exists$ on two very simple structures. From those structures a relation or cylindric algebra can be constructed and a winning strategy for either player lifts to a winning strategy on the atom structure of the algebra for the same player, though the number of ‘pebble pairs’ and rounds may increase in the algebra. In the case of relation algebras, the atoms are coloured, so that the games are played on colours. For cylindric algebras, matters are a little bit more complicated because games are played on so-called coloured graphs, which are models of the rainbow signature coded in an $L_{\omega_1,\omega}$ theory based on a graph $\Gamma$ (which we denote by $R$) [17, Definition 3.6.9]. The atom structure consists of finite coloured graphs rather than colours. Nevertheless, the essence of the two construction is very similar, because in the cylindric algebra constructed from the relational structures $G$ (the greens) and $R$ (the reds), the relation algebra atom structure based also on $G$ and $R$ is coded in the cylindric atom structure, but the latter has additional ‘shades of yellow’ that are used to label $n-1$ hyperedges coding the additional cylindric information.

The strategy for $\exists$ in a rainbow game for relation algebras is try white, if it doesn’t work try black, and finally if it doesn’t work try red. In the latter case she is kind of cornered, so it is the hardest part in the game. She never uses green. In the cylindric algebra case, the most difficult part for $\exists$ is to label the edge between apexes of two cones (a cone is a special coloured graph) having a common base, when she is also forced a red. So in both cases the choice of a red, when she is forced one, is the most difficult part for $\forall$ and if she succeeds to choose a suitable red label in every round then she wins. Indeed, it is always the case that $\forall$ wins on a red clique, using his greens to force $\exists$ play an inconsistent triple of reds.

The most general exposition of CA rainbow constructions is given in [17, Section 6.2, Definition 3.6.9] in the context of constructing atom structures from classes of models. Our models are just coloured graphs [14]. Let $G$ be a relational structure. Let $2 < n < \omega$. Then the colours used are:

- greens: $g_i$ ($1 \leq i \leq n - 2$), $g_0$, $i \in G$,
- whites: $w_i$: $i \leq n - 2$,
- reds: $r_{ij}$ ($i < j \leq n$),
shades of yellow: $y_S$: $S$ a finite subset of $\omega$ or $S = \omega$.

A coloured graph is a graph such that each of its edges is labelled by the colours in the above first three items, greens, whites or reds, and some $n - 1$ hyperedges are also labelled by the shades of yellow. Certain coloured graphs will deserve special attention.

Definition 3.8. Let $i \in G$, and let $M$ be a coloured graph consisting of $n$ nodes $x_0, \ldots, x_{n-2}, z$. We call $M$ an $i$-cone if $M(x_0, z) = g_i^0$ and for every $1 \leq j \leq n - 2$, $M(x_j, z) = g_j$, and no other edge of $M$ is coloured green. $(x_0, \ldots, x_{n-2})$ is called the base of the cone, $z$ the apex of the cone and $i$ the tint of the cone.

The rainbow algebra depending on $G$ and $R$ from the class $K$ consisting of all coloured graphs $M$ such that:

1. $M$ is a complete graph and $M$ contains no triangles (called forbidden triples) of the following types:

   (1) $(g, g', g^*)$, $(g_i, g_i, w_i)$ any $1 \leq i \leq n - 2$,

   (2) $(g'_0, g_k^0, w_0)$ any $j, k \in G$,

   (3) $(r_{ij}, r_{j'k'}, r_{j^*k^*})$ unless $|(j, k), (j', k'), (j^*, k^*)| = 3$

and no other triple of atoms is forbidden.

2. If $a_0, \ldots, a_{n-2} \in M$ are distinct, and no edge $(a_i, a_j)$ ($i < j < n$) is coloured green, then the sequence $(a_0, \ldots, a_{n-2})$ is coloured a unique shade of yellow. No other $(n - 1)$ tuples are coloured shades of yellow. Finally, if $D = \{d_0, \ldots, d_{n-2}, \delta\} \subseteq M$ and $M \upharpoonright D$ is an $i$ cone with apex $\delta$, inducing the order $d_0, \ldots, d_{n-2}$ on its base, and the tuple $(d_0, \ldots, d_{n-2})$ is coloured by a unique shade $y_S$ then $i \in S$.

Let $G$ and $R$ be relational structures as above. Take the set $J$ consisting of all surjective maps $a: n \to \Delta$, where $\Delta \in K$ and define an equivalence relation $\sim$ on this set relating two such maps iff they essentially define the same graph [14]; the nodes are possibly different but the graph structure is the same. Let $At$ be the atom structure with underlying set $J \sim$. We denote the equivalence class of $a$ by $[a]$. Then define, for $i < j < n$, the accessibility relations corresponding to $ij$th-diagonal element, $ij$th-cylindrifier, and substitution operator corresponding to the transposition $[i, j]$, as follows:

1. $[a] \in E_{ij}$ iff $a(i) = a(j)$,

2. $[a]T_{i}[b]$ iff $a \upharpoonright n \smallsetminus \{i\} = b \upharpoonright n \smallsetminus \{i\}$,

3. $[a]S_{[i,j]}[b]$ iff $a \circ [i,j] = b$.

This, as easily checked, defines a QEA$_n$ atom structure. The complex QEA$_n$ over this atom structure will be denoted by $\mathfrak{A}_{G,R}$. The dimension of $\mathfrak{A}_{G,R}$, always finite and $> 2$, will be clear from context. For rainbow atom structures, there is a one to one correspondence between atomic networks and coloured graphs [14, Lemma 30], so for $2 < n < m \leq \omega$, we use the graph
versions of the games $G^m_k$, $k \leq \omega$, and $G^m$ played on rainbow atom structures of dimension $m$ [14, pp.841–842]. We start by translating the atomic $k$ rounded game $G^m_k$ where the number of nodes are limited to $n$ to games on coloured graphs [14, lemma 30].

Let $\mathfrak{C}$ be a rainbow algebra. Let $N$ be an atomic $\mathfrak{C}$ network. Let $x, y$ be two distinct nodes occurring in the $n$ tuple $\bar{z}$. $N(\bar{z})$ is an atom of $\mathfrak{C}$ which defines an edge colour of $x, y$. Using the fact that the dimension is at least 3, the edge colour depends only on $x$ and $y$ not on the other elements of $\bar{z}$ or the positions of $x$ and $y$ in $\bar{z}$. Similarly $N$ defines shades of white for certain $(n - 1)$ tuples. In this way $N$ translates into a coloured graph. This translation has an inverse. More precisely, letting $\text{CRG}$ be the class of coloured graphs in a rainbow signature, we have:

Let $M \in \text{CRG}$ be arbitrary. Define $N_M$ whose nodes are those of $M$ as follows. For each $a_0, \ldots, a_{n-1} \in M$, define $N_M(a_0, \ldots, a_{n-1}) = [\alpha]$ where $\alpha: n \to M \mid \{a_0, \ldots, a_{n-1}\}$ is given by $\alpha(i) = a_i$ for all $i < n$. Then, as easily checked, $N_M$ is an atomic $\mathfrak{C}$ network. Conversely, let $N$ be any non empty atomic $\mathfrak{C}$ network. Define a complete coloured graph $M_N$ whose nodes are the nodes of $N$ as follows:

- For all distinct $x, y \in M_N$ and edge colours $\eta$, $M_N(x, y) = \eta$ if and only if for some $\bar{z} \in ^n N$, $i, j < n$, and atom $[\alpha]$, we have $N(\bar{z}) = [\alpha]$, $z_i = x$, $z_j = y$ and the edge $(\alpha(i), \alpha(j))$ is coloured $\eta$ in the graph $\alpha$.

- For all $x_0, \ldots, x_{n-2} \in ^{n-1} M_N$ and all yellows $y_S$, $M_N(x_0, \ldots, x_{n-2}) = y_S \iff$ for some $\bar{z}$ in $^n N$, $i_0, \ldots, i_{n-2} < n$ and some atom $[\alpha]$, we have $N(\bar{z}) = [\alpha]$, $z_{i_j} = x_j$ for each $j < n - 1$ and the $n - 1$ tuple $< \alpha(i_0), \ldots, \alpha(i_{n-2}) >$ is coloured $y_S$. Then $M_N$ is well defined and is in CRG.

The following is then, though tedious and long, easy to check: For any $M \in \text{CGR}$, we have $M_{N_M} = M$, and for any $\mathfrak{C}$ network $N$, $N_{M_N} = N$. This translation makes the following equivalent formulation of the games $G^m_{nk}(\text{At}_{\mathfrak{C}})$ originally defined on networks. The new graph version of the game [14, p.27–29] builds a nested sequence $M_0 \subseteq M_1 \subseteq M_i, \ldots, i < k$ ($k$ the number of rounds $\leq \omega$) of coloured graphs such that nodes($M_i$) $\subseteq m$.

$\forall$ picks a graph $M_0 \in \text{CRG}$ with $M_0 \subseteq m$ and $\exists$ makes no response to this move. In a subsequent round, let the last graph built be $M_i$ ($i < k$). $\forall$ picks

- a graph $\Phi \in \mathcal{G}$ with $|\Phi| = n$,

- a single node $t \in \Phi$,

- a coloured graph embedding $\theta$: $\Phi \setminus \{t\} \to M_i$. Let $F = \phi \setminus \{t\}$. Then $\exists$ must respond by amalgamating $M_i$ and $\Phi$ with the embedding $\theta$. In other words, she has to define a graph $M_{i+1} \in \text{CRG}$ and embeddings $\lambda: M_i \to M_{i+1}$, $\mu: \phi \to M_{i+1}$, such that $\lambda \circ \theta = \mu \mid F$.

Summarizing we have:
Proposition 3.9. Let $2 < n < \omega$. Let $k, m \leq \omega$, and $C$ be a rainbow $\text{CA}_n$. Then $\exists$ has a winning strategy in $G^m_k(\text{At}C) \iff \exists$ has a winning strategy in the above $k$-rounded graph game played on $C$ where the size of graphs during the play is limited to $m$ nodes.

The game $G^m$ lifts to a game on coloured graphs, that is like the graph games $G^m_\omega$ \cite{14}, where the number of nodes of graphs played during the $\omega$ rounded game does not exceed $m$, but $\forall$ has the option to re-use nodes. The typical winning strategy for $\forall$ in the graph version of both atomic games is bombarding $\exists$ with cones having a common base and green tints until she runs out of (suitable) reds, that is to say, reds whose indicies do not match, cf. \cite[4.3]{14}.

Let $K_n$ be a variety between $\text{Sc}_n$ and $\text{QEA}_n$. A $K_n$ atom structure $\text{At}$ is weakly representable if there is an atomic $\mathfrak{A} \in RK_n$ such that $\text{At} = \text{At}\mathfrak{A}$; it is strongly representable if $\mathfrak{C}m\text{At} \in RK_n$. These two notions are distinct, cf. \cite{19} and the following Theorem 3.10; see also the forthcoming Theorem 3.16.

With these preliminaries out of the way we are now ready to start digging deeper:

Theorem 3.10. Let $2 < n < \omega$.

1. The variety $\text{RRA}$ is not atom-canonical with respect to $\text{SRaCA}_k$, for any $k \geq 6$,

2. Let $K$ be any variety between $\text{Sc}$ and $\text{QEA}$. Let $t(n) = n(n + 1)/2 + 1$. Then $\text{RK}_n$ is not-atom canonical with respect to $\text{SNr}_nK_{t(n)}$. In fact, there is a countable atomic simple $\mathfrak{A} \in R\text{QEA}_n$ such that $\mathfrak{Rd}_{sc}\mathfrak{C}m\text{At}\mathfrak{A}$ does not have an $t(n)$-square, a fortiori $t(n)$-flat, representation.

3. $\text{RDF}_n$ is not atom-canonical.

Proof. For item (1) cf. \cite[Lemmata 17.32, 17.34, 17.35, 17.36]{17}.

Item (2): The proof is long and uses many ideas in \cite{19}. We will highlight only the differences in detail from the proof in \cite{19} needed to make our result work. When parts of the proof coincide we will be more sketchy. The proof is divided into four parts:

1. **Blowing up and blurring $B_f$ forming a weakly representable atom structure $\text{At}$**: Take the finite rainbow $\text{QEA}_n$, $B_f$ where the reds is the complete irreflexive graph $n$, and the greens are $\{g_i: 1 \leq i < n - 1\} \cup \{g_0: 1 \leq i \leq n(n - 1)/2 + 2\}$, endowed with the quasi-polyadic operations. We will show $\mathfrak{Rd}_K B_f$ detects that $\text{RK}_n$ is not atom-canonical with respect to $\text{SNr}_nK_{t(n)}$ with $t(n)$ as specified in the statement of the theorem. Denote the finite atom structure of $B_f$ by $\text{At}_f$; so that $\text{At}_f = \text{At}(B_f)$. One then defines a larger the class of coloured graphs like in \cite[Definition 2.5]{19}. Let $2 < n < \omega$. Then the colours used are like above except that each red is ‘split’ into $\omega$ many having ‘copies’ the form $r^l_{ij}$ with $i < j < n$ and $l \in \omega$, with an additional shade of red $\rho$ such that the consistency conditions for
the new reds (in addition to the usual rainbow consistency conditions) are as follows:

- \((r^i_{jk}, r^i_{j^*k^*}, r^i_{j^*k^*})\) unless \(i = i' = i^*\) and \(|\{(j, k), (j', k'), (j^*, k^*)\}| = 3\)
- \((r, \rho, \rho)\) and \((r, r^*, \rho)\), where \(r, r^*\) are any reds.

The consistency conditions can be coded in an \(L_{\omega, \omega}\) theory \(T\) having signature the reds with \(\rho\) together with all other colours like in [17, Definition 3.6.9]. The theory \(T\) is only a first order theory (not an \(L_{\omega_1, \omega}\) theory) because the number of available greens is finite which is not the case with [17] where the number of available greens are countably infinite coded by an infinite disjunction. One construct an \(n\)-homogeneous model \(M\) is as a countable limit of finite models of \(T\) using a game played between \(\exists\) and \(\forall\) like in [19, Theorem 2.16]. In the rainbow game \(\forall\) challenges \(\exists\) with cones having green tints \((g^i_0)\), and \(\exists\) wins if she can respond to such moves. This is the only way that \(\forall\) can force a win. \(\exists\) has to respond by labelling apexes of two successive cones, having the same base played by \(\forall\). By the rules of the game, she has to use a red label. She resorts to \(\rho\) whenever she is forced a red while using the rainbow reds will lead to an inconsistent triangle of reds; [19, Proposition 2.6, Lemma 2.7]. The number of greens make [19, Lemma 3.10] work with the same proof using only finitely many green and not infinitely many. The winning strategy implemented by \(\exists\) using the red label \(\rho\) that comes to her rescue whenever she runs out of ‘rainbow reds’, so she can always and consistently respond with an extended coloured graph.

2. Representing a term algebra (and its completion) as (generalized) set algebras: Having \(M\) at hand, one constructs two atomic \(n\)-dimensional set algebras based on \(M\), sharing the same atom structure and having the same top element. The atoms of each will be the set of coloured graphs, seeing as how, quoting Hodkinson [19] such coloured graphs are ‘literally indivisible’. Now \(L_n\) and \(L_{\infty, \omega}^n\) are taken in the rainbow signature (without \(\rho\)). Continuing like in op.cit, deleting the one available red shade, set \(W = \{\bar{a} \in {}^nM: M \models (\wedge_{i<j<n} \neg \rho(x_i, x_j))(\bar{a})\}\), and for \(\phi \in L_{\infty, \omega}^n\), let \(\phi^W = \{s \in W: M \models_W \phi[s]\}\). Here \(W\) is the set of all \(n\)-ary assignments in \({}^nM\), that have no edge labelled by \(\rho\) and \(\models_W\) is first order semantics with quantifiers relativized to \(W\), cf. [19, §3.2 and Definition 4.1]. Let \(\mathfrak{A}\) be the relativized set algebra with domain \(\{\varphi^W: \varphi\) a first-order \(L_n\)-formula\} and unit \(W\), endowed with the usual concrete cylindric operations read off the connectives. Classical semantics for \(L_n\) rainbow formulas and their semantics by relativizing to \(W\) coincide as proved in [19, Proposition 3.13] but not with respect to \(L_{\infty, \omega}^n\) rainbow formulas. Hence the set algebra \(\mathfrak{A}\) is isomorphic to a cylindric set algebra of dimension \(n\) having top element \({}^nM\), so \(\mathfrak{A}\) is simple, in fact its Df reduct is simple. Let \(\mathfrak{E} = \{\phi^W: \phi \in L_{\infty, \omega}^n\}\) defined similarly to [19, Definition 4.1] with the operations defined like on \(\mathfrak{A}\) the usual way. \(\mathfrak{CmAt}\) is a complete \(\mathfrak{CA}_n\) and, so like in [19, Lemma 5.3] we have an iso-
morphism from \( \mathcal{CmAt} \) to \( \mathcal{E} \) defined via \( X \mapsto \bigcup X \). Since \( \mathsf{AtA} = \mathsf{AtIm}(\mathsf{AtA}) \), which we refer to only by \( \mathsf{AtA} \), and \( \mathsf{ImAtA} \subseteq \mathsf{AtA} \), hence \( \mathsf{ImAtA} = \mathsf{ImAtA} \) is representable. The atoms of \( \mathsf{AtA} \), \( \mathsf{ImAtA} \) and \( \mathsf{ImAtA} = \mathsf{ImAtA} \) are the coloured graphs whose edges are not labelled by \( \rho \). These atoms are uniquely determined by the interpretation in \( \mathsf{M} \) of so-called MCA formulas in the rainbow signature of \( \mathsf{AtA} \) as in [19, Definition 4.3].

3. Embedding \( \mathfrak{A}_{n+1,n} \) into \( \mathcal{Cm}(\mathsf{At}(\mathcal{Bb}(\mathfrak{A}_{n+1,n}, r, \omega))) \): Let \( \mathsf{CRG}_f \) be the class of coloured graphs on \( \mathsf{AtA}_f \) and \( \mathsf{CRG} \) be the class of coloured graph on \( \mathsf{AtA} \). We can (and will) assume that \( \mathsf{CRG}_f \subseteq \mathsf{CRG} \). Write \( M_a \) for the atom that is the (equivalence class of the) surjection \( a: n \to M, M \in \mathsf{CRG} \). Here we identify \( a \) with \( |a| \); no harm will ensue. We define the (equivalence) relation \( \sim \) on \( \mathsf{AtA} \) by \( M_b \sim N_a, (M, N \in \mathsf{CRG}) \):

- \( a(i) = a(j) \iff b(i) = b(j) \),
- \( M_a(a(i), a(j)) = r^l \iff N_b(b(i), b(j)) = r^k \), for some \( l, k \in \omega \),
- \( M_a(a(i), a(j)) = N_b(b(i), b(j)) \), if they are not red,
- \( M_a(a(k_0), \ldots, a(k_{n-2})) = N_b(b(k_0), \ldots, b(k_{n-2})) \), whenever defined.

We say that \( M_a \) is a copy of \( N_b \) if \( M_a \sim N_b \) (by symmetry \( N_b \) is a copy of \( M_a \)). Indeed, the relation \( \sim \) of an equivalence relation on \( \mathsf{AtA} \). An atom \( M_a \) is called a red atom, if \( M_a \) has at least one red edge. Any red atom has \( \omega \) many copies, that are cylindrically equivalent, in the sense that, if \( N_a \sim M_b \) with one (equivalently both) red, with \( a: n \to N \) and \( b: n \to M \), then we can assume that \( \text{nodes}(N) = \text{nodes}(M) \) and that for all \( i < n \), \( a \upharpoonright n \sim \{i\} \) and \( b \upharpoonright n \sim \{i\} \). In \( \mathcal{CmAt} \), we write \( M_a \) for \( \{M_a\} \) and we denote suprema taken in \( \mathcal{CmAt} \), possibly finite, by \( \sum \). Define the map \( \Theta \) from \( \mathcal{A}_{n+1,n} = \mathcal{CmAt}_f \) to \( \mathcal{CmAt} \), by specifying first its values on \( \mathsf{AtA}_f \), via \( M_a \mapsto \sum_j M_a^{(j)} \) where \( M_a^{(j)} \) is a copy of \( M_a \). So each atom maps to the suprema of its copies. This map is well-defined because \( \mathcal{CmAt} \) is complete. We check that \( \Theta \) is an injective homomorphism. Injectivity is easy. We check preservation of all the \( \mathsf{CA}_n \) extra Boolean operations.

- Diagonal elements. Let \( l < k < n \). Then:

\[
M_x \leq \Theta(d_{lk}^{\mathcal{CmAt}_f}) \iff M_x \leq \bigcup_j \bigcup_{a_l = a_k} M_a^{(j)}
\]

\[
\iff M_x \leq \bigcup_{a_l = a_k} \sum_j M_a^{(j)}
\]

\[
\iff M_x = M_a^{(j)} \text{ for some } a: n \to M
\]

such that \( a(l) = a(k) \)

\[
\iff M_x \in d_{lk}^{\mathcal{CmAt}}.
\]
• Cylindrifiers. Let \( i < n \). By additivity of cylindrifiers, we restrict our attention to atoms \( M_a \in \mathsf{At}_f \) with \( a : n \to M \), and \( M \in \mathsf{CRG}_f \subseteq \mathsf{CRG} \). Then:

\[
\Theta(c_i^{\mathsf{emAt}_f} M_a) = f \left( \bigcup_{[c] = i[a]} M_c \right) = \bigcup_{[c] = i[a]} \Theta(M_c)
\]

\[
= \bigcup_{[c] = i[a]} \sum_j M_c^{(j)} = \sum_j \bigcup_{[c] = i[a]} M_c^{(j)} = \sum_j c_i^{\mathsf{emAt}} M_a^{(j)}
\]

\[
= c_i^{\mathsf{emAt}} \left( \sum_j M_a^{(j)} \right) = c_i^{\mathsf{emAt}} \Theta(M_a).
\]

• Substitutions: Let \( i, k < n \). By additivity of the \( s_{[i,k]} \)'s, we again restrict ourselves to atoms of the form \( M_a \) as specified in the previous items. Now computing we get:

\[
\Theta(s_i^{\mathsf{emAt}_f} M_a) = \Theta(M_{a_0[i,k]}) = \sum_j s_i^{\mathsf{emAt}} M_a^{(j)} = s_i^{\mathsf{emAt}} \left( \sum_j M_a^{(j)} \right) = s_i^{\mathsf{emAt}} \Theta(M_a).
\]

4. \( \forall \) has a winning strategy in \( G^{t(n)} \mathsf{At}(Rd\mathfrak{B}_f) \); and the required result: It is straightforward to show that \( \forall \) has winning strategy first in the Ehrenfeucht–Fraïssé forth private game played between \( \exists \) and \( \forall \) on the complete irreflexive graphs \( n(n-1)/2 + 2 \) and \( n \in n(n-1)/2 + 2 \) rounds \( EF_{n(n-1)+2}^{n(n-1)+2} \) \( n+1, n \), cf. \[17, Definition 16.2\] since \( n(n-1)/2 + 2 \) is ‘longer’ than \( n \). Using (any) \( p > n \) many pairs of pebbles available on the board \( \forall \) can win this game in \( n+1 \) many rounds. For brevity, write \( \mathcal{D} \in \mathcal{Sc}_n \) instead of \( \mathfrak{Md}_{\mathbb{Z}^2} \mathfrak{B} \). \( \forall \) lifts his winning strategy from the last private Ehrenfeucht–Fraïssé forth game to the graph game on \( \mathsf{At}_f = \mathsf{At}(\mathcal{D}) \) see \[14, p. 841\] forcing a win using \( t(n) \) nodes. One uses the \( n(n-1)/2 + 2 \) green relations in the usual way to force a red clique \( C \), say with \( n(n-1)/2 + 2 \). Pick any point \( x \in C \). Then there are \( > n(n-1)/2 \) points \( y \) in \( C \setminus \{x\} \). There are only \( n(n-1)/2 \) red relations. So there must be distinct \( y, z \in C \setminus \{x\} \) such that \( (x,y) \) and \( (x,z) \) both have the same red label (it will be some \( r^m_{ij} \) for \( i < j < n \)). But \((y,z)\) is also red, and this contradicts (1.3) above. In more detail, \( \forall \) bombards \( \exists \) with cones having common base and distinct green tints until \( \exists \) is forced to play an inconsistent red triangle (where indicies of reds do not match). He needs \( n-1 \) nodes as the base of cones, plus \( |P| + 2 \) more nodes, where \( P = \{(i,j) : i < j < n \} \) forming a red clique, triangle with two edges satisfying the same \( r^m_p \) for \( p \in P \). Calculating, we get \( t(n) = n - 1 + n(n-1)/2 + 2 = n(n+1)/2 + 1 \). By Lemma 3.7, \( \mathcal{D} \notin \mathcal{Sc}_n \mathcal{Nr}_n \mathcal{Sc}_t^{ad}(n) \) when \( 2 < n \). Since \( \mathcal{D} \) is finite, then \( \mathcal{D} \notin \mathcal{SNr}_n \mathcal{Sc}_t^{ad}(n) \), because \( \mathcal{D} \) coincides with its canonical extension and for
any $\mathcal{D} \in \mathcal{S}_{c,n}, \mathcal{D} \in \mathcal{SNr}_n \mathcal{S}_{c_l(n)} \implies \mathcal{D}^+ \in \mathcal{S}_c \mathcal{N}_r_n \mathcal{S}_{c_l(n)}$. To see why, we could omit the superscript $a_d$, abbreviating additivity, assume that $\mathcal{D} \subseteq \mathcal{N}_r_n \mathcal{E}^a_d$, $\mathcal{E} \in \mathcal{S}_{c,n+3}$. Let $\mathcal{E}' = \mathcal{S}g^e \mathcal{D}$, then $\mathcal{E}'$ is finite, hence completely additive and $\mathcal{D} \subseteq \mathcal{N}_r_n \mathcal{E}'$. But $\mathcal{B}_f$ embeds into $\mathcal{FaAt}_{\mathcal{A}}$, hence $\mathcal{FaAt}_{\mathcal{A}}$ embeds into the variety $\mathcal{SNr}_n \mathcal{S}_{c_l(n)}$, as well. Since $\mathcal{Fa}_{sc} \mathcal{A}$ is completely additive because it is a reduct of a $QEA_n$, then $\mathcal{FaAt}_{\mathcal{Fa}_{sc} \mathcal{A}}$ is the Dedekind-MacNeille completion of $\mathcal{Fa}_{sc} \mathcal{A}$. By the second part of Lemma 3.7, the required follows. But $\mathcal{D}$ embeds into $\mathcal{Fa}_{sc} \mathcal{FaAt}_{\mathcal{A}}$, hence $\mathcal{FaAt}_{\mathcal{Fa}_{sc} \mathcal{A}}$ embeds into the variety $\mathcal{SNr}_n \mathcal{S}_{c_l(n)}$, as well. By the second part of Lemma 3.7, the required follows.

Now we prove the last item, namely, that $\mathcal{RDf}_n$ is not atom-canonical. Using essentially the argument in [11, Lemma 5.1.50, Theorem 5.1.51] by considering closure under infinite intersections instead of intersections, it is enough to show that $\mathcal{FaAt}_{\mathcal{A}}$ is generated by elements whose dimension sets have cardinality $< n$ using infinite unions. We show that for any rainbow atom $[a]$, $a : n \to \Gamma$, $\Gamma$ a coloured graph, that $[a] = \prod_{i<n} c_i[a]$. Clearly $\leq$ holds. Assume that $b : n \to \Delta$, $\Delta$ a coloured graph, and $[a] \neq [b]$. We show that $[b] \notin \prod_{i<n} c_i[a]$ by which we will be done. Because $a$ is not equivalent to $b$, we have one of two possibilities; either $(\exists i,j < n) \left( \Delta(b(i), b(j)) \neq \Gamma(a(i), a(j)) \right)$ or $(\exists i_1, \ldots, i_{n-1} < n) \left( \Delta(b_{i_1}, \ldots, b_{i_{n-1}}) \neq \Gamma(a_{i_1}, \ldots, a_{i_{n-1}}) \right)$. Assume the first possibility (the second is similar): Choose $k \notin \{i, j\}$. This is possible because $n > 2$. Assume for contradiction that $[b] \in c_k[a]$. Then $(\forall i, j \in n \setminus \{k\}) \left( \Delta(b(i), b(j)) = \Gamma(a(i)a(j)) \right)$. By assumption and the choice of $k$, $(\exists i, j \in n \setminus \{k\}) \left( \Delta(b(i), b(j)) \neq \Gamma(a(i)a(j)) \right)$, contradiction.

In [18] it is proved that for any $2 < n < \omega$, for any variety $K_n$ (consisting of $n$ dimensional algebras) between $\mathcal{S}_{c,n}$ and $QEA_n$, and for any $k \geq 1$, $\mathcal{SNr}_n K_{n+k+1} \subseteq \mathcal{SNr}_n K_{n+k}$, and in fact the gap cannot be finitely axiomatized. Hence we readily conclude:

**Proposition 3.11.** For $2 < n < \omega$, and any variety $V$ between $\mathcal{S}_{c,n}$ and $QEA_n$, there are infinitely many varieties containing (and including) $RV$ and contained in $V$ that are not atom canonical.

Sahlqvist equations are the algebraic translations of Sahlqvist formulas [15, Definition 2.92]. Any Sahlqvist equation (formula) is a canonical one, i.e. preserved in canonical extensions, but the converse is not true. Sahlqvist formulas are a certain kind of modal formula with remarkable properties, cf. [8]. Using [41], by observing that $\mathcal{RDf}_n$ is the variety of modal algebras corresponding to $\mathcal{S}S^m$, we immediately get:

**Corollary 3.12.** Let $2 < n < \omega$, and $m \geq t(n)$ with $t(n)$ as specified in Theorem 3.10. Then the $n$-dimensional multimodal logics $\mathcal{S}S^m$ and the $m$-clique guarded fragment of $L_n$ determined by $m$-square models or $m$-flat models are not Sahlqvist.
Let $2 < n < \omega$. Two stronger distinct results on the complexity of axiomatizations for $L_n$ and $S5^n$ are the following. Though canonical (equivalently $RCA_n$ is closed under canonical extensions), $L_n$ cannot be axiomatized by canonical formulas, cf. [7], so as indicated in [7]: The variety $RCA_n$ is barely canonical; any equational axiomatization of $RCA_n$ necessarily contains infinitely many non-canonical equations. The same holds for $RDF_n$, cf. [7]. Observe that the other way round; $V$ is defined by canonical equations $\Rightarrow V$ is canonical, is true. The second result is that neither $S5^n$ nor $L_n$ can be axiomatized by modal formulas whose class of frames is first order definable (this is stronger than excluding Sahlqvist axiomatizability). Proved algebraically, this follows from the fact that the class of strongly representable atom structures of $CA_n$ and $Df_n$, cf. [7].

In both cases of $RA$ addressed in [15, Lemmata 17.32, 17.34, 17.35, 17.36] and $CA$s addressed in Theorem 3.10 pproving non atom canoicity for infinitely many varieties of $RA$s and $CA_n$s, respectively, the relational structures $G$ and $R$ used satisfy $|G| > |R|$ both $G$ (the greens) and $R$ (the reds) are finite. For $RA$, $R = 3$ and for $CA_n$s, $R = n$ (the dimension), where the finite ordinals 3 and $n$ are viewed as complete irreflexive graphs. Using the rainbow algebras based on such graphs, we have proved that $B_f(R_{4,3})$ which is the rainbow relation algebra based on the complete irreflexive graphs with nodes $m$ and $n$ defined the obvious way, cf.[15, Definition 17.31]) detects that $RCA_n$ ($RRA$) is not atom-canonical with repect to $SN_{n,CA_{t(n)}}(SRaCA_6)$ with $t(n)$ as defined in the statement of Theorem 3.10 Worthy of note, is that it is commonly accepted that relation algebras have dimension three being a natural habitat for three variable first order logic. Nevertheless, sometimes it is argued that the dimension should be three and a half in the somewhat loose sense that $RA$s lie ‘halfway’ between $CA_3$ and $CA_4$ manifesting behaviour of each; for example associativiy in $RA$s needs 4 variables to be proved. From Hodkinson’s construction in [19], we know that $EmAt \not\subseteq SN_{n,CA_m}$ for some finite $m > n$, but the (semantical) argument used in [19] does not give any information on the value of such $m$. By truncating the greens to be $n(n - 1)/2 + 2$ (instead of the ‘overkill’ of infinitely many in [19]), and using a syntactical blow up and blur construction, we could pin down such a value of $m$, namely, $m = t(n)$ as specified in the statement of Theorem 3.10, by showing that $EmAt \not\subseteq SN_{n,CA_{t(n)}}$. Results involving notions like atom-canonicity, Dedekind-MacNeille completions, complete representations for the infinite dimensional case, are extremely rare in algebraic [17, Problem 3.8.3]; in fact, almost non-existent. In this direction we present a conditional result (the condition is very likely to be true). For each finite $k \geq 3$, let $A(k)$ be an atomic countable simple representable $CA_k$ such that $B(k) = EmAtA(k) \not\subseteq SN_{k,CA_{k+3}}$. We know that such algebras exist by Theorem 3.10.
We make the following assumption: (*) Assume that $\mathcal{B}_m$ embeds into $\mathfrak{R}_m\mathcal{B}_t$, whenever $3 \leq m < t < \omega$. Our next theorem lifts item (2) of Theorem 3.10 to the transfinite conditionally (modulo (*)).

**Theorem 3.13.** Assume that (*) above holds for the algebras constructed in Theorem 3.10 (or any other algebras). Then for $k \geq 3$, $\mathbf{SNr}_\omega \mathbf{CA}_{\omega+k}$ is not atom-canonical. In particular, $\mathbf{RCA}_\omega$ cannot be axiomatized by (a necessarily infinite schema of) Sahlqvist equations.

**Proof.** Throughout the proof $F$ denote a non-principal ultrafilter on $\omega \setminus 3$. For each finite $k \geq 3$, let $\mathfrak{A}(k)$ and $\mathcal{B}(k)$ be the algebras constructed in Theorem 3.10 (of dimension $k$) and assume further that the assumption abbreviated by (*) preceding the theorem holds for the algebras constructed in *op.cit.* Let $\mathfrak{A}_k$ be an (atomic) algebra having the signature of $\mathbf{CA}_\omega$ such that $\mathfrak{R}_k\mathfrak{A}_k = \mathfrak{A}(k)$. Analogously, let $\mathcal{B}_k$ be an algebra having the signature of $\mathbf{CA}_\omega$ such that $\mathfrak{R}_k\mathcal{B}_k = \mathcal{B}(k)$, and we require in addition that $\mathcal{B}_k = \mathcal{Cm}(\mathfrak{A}_k)$. Let $\mathcal{B} = \Pi_{i \in \omega\setminus 3}\mathcal{B}_i/F$. It is easy to show that $\mathfrak{A} = \Pi_{i \in \omega\setminus 3}\mathfrak{A}_i/F \in \mathbf{RCA}_\omega$. Furthermore, a direct computation gives: $\mathcal{Cm}\mathfrak{At}\mathfrak{A} = \mathcal{Cm}(\mathfrak{At}[\Pi_{i \in \omega\setminus 3}\mathfrak{A}_i/F]) = \mathcal{Cm}[\Pi_{i \in \omega\setminus 3}(\mathfrak{At}\mathfrak{A}_i)/F] = \Pi_{i \in \omega\setminus 3}(\mathcal{Cm}(\mathfrak{At}\mathfrak{A}_i)/F) = \Pi_{i \in \omega\setminus 3}\mathfrak{B}_i/F = \mathcal{B}$. By the same token, $\mathcal{B} \in \mathbf{CA}_\omega$.

Assume for contradiction that $\mathfrak{B} \in \mathbf{SNr}_\omega \mathbf{CA}_{\omega+3}$. Then $\mathfrak{B} \subseteq \mathbf{Rr}_\omega \mathfrak{C}$ for some $\mathfrak{C} \in \mathbf{CA}_{\omega+3}$. Let $3 \leq m < \omega$ and let $\lambda \colon m+3 \to \omega + 3$ be the function defined by $\lambda(i) = i$ for $i < m$ and $\lambda(m+i) = \omega + i$ for $i < 3$. Then we get (**): $\mathfrak{R}_m^\lambda \mathfrak{C} \in \mathbf{CA}_{m+3}$ and $\mathfrak{R}_m\mathfrak{B} \subseteq \mathfrak{R}_m\mathfrak{R}_m^\lambda \mathfrak{C}$. By assumption let $I_t \colon \mathfrak{B}_m \to \mathfrak{R}_m\mathcal{B}_t$ be an injective homomorphism for $3 \leq m < t < \omega$. Let $\iota(b) = (I_t b \colon t \geq m)/F$ for $b \in \mathcal{B}_m$. Then $\iota$ is an injective homomorphism that embeds $\mathcal{B}_m$ into $\mathfrak{R}_m\mathcal{B}$. By (***) we know that $\mathfrak{R}_m\mathfrak{B} \in \mathbf{SNr}_m \mathbf{CA}_{m+3}$, hence $\mathfrak{B}_m \in \mathbf{SNr}_m \mathbf{CA}_{m+3}$, too. This is a contradiction, and we are done. $\square$

The next theorem shows us that atom-canonicity and complete representations have the habit to go hand in hand. Indeed, in contrast to the cylindric paradigm, for polyadic (equality) algebras of infinite dimension, we have the followingm proved in [36]:

**Theorem 3.14.** 1. Let $\alpha$ be an infinite ordinal and $n < \omega$. If $\mathfrak{D} \in \mathbf{PEA}_\alpha$ is atomic, then any complete subalgebra of $\mathfrak{R}_n \mathfrak{D}$ is completely representable as a $\mathbf{PEA}_n$.

2. Let $\alpha$ be an infinite ordinal and $n \leq \alpha$. Let $\mathfrak{A} \in \mathbf{PA}_\alpha$ be completely additive and atomic. Then any complete subalgebra of $\mathfrak{R}_n \mathfrak{A}$ is completely representable. In particular, $\mathbf{S}_c \mathbf{PA}_\alpha^{ad} \cap \mathfrak{At} = \mathbf{PA}_\alpha^{ad} \cap \mathfrak{At} = \mathbf{CRPA}_\alpha$ and the class $\mathbf{CRPA}_\alpha$ is elementary.

3. For any pair of infinite ordinals $\alpha < \beta$, and $K \in \{\mathbf{PA}, \mathbf{PEA}\}$ the variety $\mathbf{SNr}_\alpha K_\beta$ is closed under Dedekind-MacNeille completions, a fortiori, it is atom-canonical.

**Proof.** The first and second items are proved in [36, Theorem 1.3]. The last item follows from that $\mathbf{SNr}_\alpha K_\beta = K_\alpha$ which is Sahlqvist axiomatizable;
with equational axiomatization given (for polyadic algebras with and without equality) in [11, Definition 5.4.1]. Applying [41], we get the required. □

3.3. An application on omitting types for the clique guarded fragment of $L_n$

The following definition to be used in the sequel is taken from [5]:

**Definition 3.15 ([5, Definition 3.1])**. Let $\mathcal{R}$ be a relation algebra, with non-identity atoms $I$ and $2 < n < \omega$. Assume that $J \subseteq \wp(I)$ and $E \subseteq 3^{\omega}$.

1. We say that $(J, E)$ is an $n$-blur for $\mathcal{R}$, if $J$ is a complex $n$-blur defined as follows:

   (1) Each element of $J$ is non-empty,
   (2) $\bigcup J = I$,
   (3) $(\forall P \in I)(\forall W \in J)(I \subseteq P; W),$
   (4) $(\forall V_1, \ldots, V_n, W_2, \ldots, W_n \in J)(\exists T \in J)(\forall 2 \leq i \leq n)\text{safe}(V_i, W_i, T)$, that is there is for $v \in V_i, w \in W_i$ and $t \in T$, we have $v; w \leq t,$
   (5) $(\forall P_2, \ldots, P_n, Q_2, \ldots, Q_n \in I)(\forall W \in J)W \cap P_2; Q_2 \cap \ldots P_n; Q_n \neq \emptyset$.

   and the tenary relation $E$ is an index blur defined as in item (ii) of [5, Definition 3.1].

2. We say that $(J, E)$ is a strong $n$-blur, if it $(J, E)$ is an $n$-blur, such that the complex $n$-blur satisfies:

   $(\forall V_1, \ldots, V_n, W_2, \ldots, W_n \in J)(\forall T \in J)(\forall 2 \leq i \leq n)\text{safe}(V_i, W_i, T)$.

   Here $\text{safe}(V_i, W_i, T)$ means that for all $v \in V_i$, $w \in W_i$ and $t \in T_i$, $v; w \geq t$. Let $2 < n \leq l < m \leq \omega$. Consider the statements $\Psi(l, m)$ and $\mathsf{VT}(l, m) = \neg \Psi(l, m)$ as defined in the introduction. Recall that $\mathsf{VT}(\omega, \omega)$ is just Vaught’s theorem, namely, countable atomic theories have atomic countable models. From Theorem 3.10 and the construction in [5] using essentially the argument in [33, Proof of Theorem 3.1.1] one obtains:

**Theorem 3.16.** For $2 < n < \omega$ and $n \leq l < \omega$, $\Psi(n, t(n))$, with $t(n) = n(n + 1)/2 + 1$ and $\Psi(l, \omega)$ hold. Furthermore, if for each $n < m < \omega$, there exists a finite relation algebra $\mathcal{R}_m$ having $m - 1$ strong blur and no $m$-dimensional relational basis, then for $2 < n \leq l < m \leq \omega$ and $l = m = \omega$, $\mathsf{VT}(l, m) \iff l = m = \omega$.

**Proof.** We start by the last part. Let $\mathcal{R}_m$ be as in the hypothesis with strong $m - 1$-blur $(J, E)$ and $m$-dimensional relational basis. We ‘blow up and blur’ $\mathcal{R}_m$ in place of the Maddux algebra $\mathcal{E}_k(2, 3)$ blown up and blurred in [5, Lemma 5.1], where $k < \omega$ is the number of non-identity atoms and $k$ depends recursively on $l$, giving the desired ‘strong’ $l$-bluness, cf. [5, Lemmata 4.2, 4.3]. Here we assume familiarity with the construction of atomic
relational algebras from forbidden (or their complement the consistent) triples of atoms, cf. [15, Definition 3.25]. In the case of \( \mathcal{E}_k(2,3) \) used in [5], the set of blurs is the set of all subsets of non-identity atoms having the same size \( l < \omega \), where \( k = f(l) \geq l \) for some recursive function \( f \) from \( \omega \to \omega \), so that \( k \) depends recursively on \( l \). One (but not the only) way to define the index blur \( E \subseteq 3^\omega \) is as follows [33, Theorem 3.1.1]: \( E(i,j,k) \iff (\exists p,q,r)(\{p,q,r\} = \{i,j,k\} \text{ and } r - q = q - p) \). This is a concrete instance of an index blur as defined in [5, Definition 3.1(iii)] (recalled in Definition 3.15 above), but defined uniformly, it does not depend on the blurs. The underlying set of \( \mathbb{A}_t \), the atom structure of blowing up and blurring \( \mathfrak{R}_m \) is the following set consisting of triplets: \( \mathbb{A}_t = \{(i,P,W) : i \in \omega, P \in \mathbb{A}_t \mathfrak{R}_m \sim \{\text{id}\}, W \in J \cup \{\text{ld}\} \). When \( \mathfrak{R}_m = \mathcal{E}_k(2,3) \) (some finite \( k > 0 \)), composition is defined by singling out the following (together with their Peircean transforms), as the consistent triples: \((a,b,c)\) is consistent \( \iff \) one of \( a, b, c \) is \( \text{id} \) and the other two are equal, or if \( a = (i,P,S), b = (j,Q,Z), c = (k,R,W) \)

\[
S \cap Z \cap W \neq \emptyset \implies E(i,j,k) \& \{|P,Q,R|\} \neq 1.
\]

(We are avoiding monochromatic triangles). That is if for \( W \in J, \ E^W = \{(i,P,W) : i \in \omega, P \in W\} \), then

\[
(i,P,S); (j,Q,Z) = \bigcup \{E^W : S \cap Z \cap W = \emptyset\}
\]

\[
 \bigcup \{(k,R,W) : E(i,j,k), \{|P,Q,R|\} \neq 1\}.
\]

More generally, for the \( \mathfrak{R}_m \) as postulated in the hypothesis, composition in \( \mathbb{A}_t \) is defined as follow. First the index blur \( E \) can be taken to be like above. Now the triple \( ((i,P,S),(j,Q,Z),(k,R,W)) \) in which no two entries are equal, is consistent if either \( S,Z,W \) are safe, briefly safe\((S,Z,W)\), witness item (4) in definition 3.15 (which vacuously hold if \( S \cap Z \cap W = \emptyset \)), or \( E(i,j,k) \) and \( P; Q \leq R \) in \( \mathfrak{R}_m \). This generalizes the above definition of composition, because in \( \mathcal{E}_k(2,3) \), the triple of non-identity atoms \( (P,Q,R) \) is consistent \( \iff \) they do not have the same colour \( \iff \) \(|\{P,Q,R\}| \neq 1\). The relation algebra \( \mathfrak{R}_m, J, E \), obtained by blowing up and blurring \( \mathfrak{R}_m \) with respect to \( (J,E) \), is \( \mathfrak{I}m\mathfrak{A}t \) (the term algebra). For brevity call it \( \mathcal{R} \). The universe of \( \mathcal{R} \) is the set \( \{X \subseteq H \cup \{\text{id}\}: X \cap E^W \in \text{Cof}(E^W), \text{ for all } W \in J\} \), where Cof\((E^W)\) denotes the set of co-finite subsets of \( E^W \), that is subsets of \( E^W \) whose complement is infinite, with \( E^W \) as defined above. The relation algebra operations are lifted from \( \mathbb{A}_t \) the usual way, cf. Definition 2.9. It is proved in [5] that \( \mathcal{R} \) is representable as shown next. For \( a \in \mathbb{A}_t \), and \( W \in J \), set \( U^a = \{X \in R: a \in X\} \) and \( U^W = \{X \in R: |X \cap E^W| \geq \omega\} \). Then the principal ultrafilters of \( \mathcal{R} \) are exactly \( U^a, a \in H \) and \( U^W \) are non-principal ultrafilters for \( W \in J \) when \( E^W \) is infinite. Let \( J' = \{W \in J: |E^W| \geq \omega\} \), and let \( \text{Uf} = \{U^a : a \in F\} \cup \{U^W : W \in J'\} \). \( \text{Uf} \) is the set of ultrafilters of \( \mathcal{R} \) which is used as colours to represent \( \mathcal{R} \), cf. [5, pp. 75-77]. The representation is built from coloured graphs whose edges are labelled by elements in \( \text{Uf} \) in a fairly standard step-by-step construction.
Now we show why the Dedekind-MacNeille completion $\text{CmA}t$ does not have an $m$ square representation. For $P \in I$, let $H^P = \{ (i, P, W) : i \in \omega, W \in J, P \in W \}$. Let $P_1 = \{ H^P : P \in I \}$ and $P_2 = \{ E^W : W \in J \}$. These are two partitions of $\text{At}$. The partition $P_2$ was used to represent $\mathcal{R}$, in the sense that the tenary relation corresponding to composition was defined on $\text{At}$, in a such a way so that the singletons generate the partition $(E^W : W \in J)$ up to “finite deviations.” The partition $P_1$ will now be used to show that the Dedekind-MacNeille completion of $\mathcal{R}$, namely, $\text{CmA}t(\mathbb{B}b(\mathcal{R}, J, E)) = \text{Cm}(\text{At})$ lacks an $m$ square representation. This follows by observing that composition restricted to $P_1$ satisfies: $H^P; H^Q = \bigcup \{ H^Z : P \leq Q \text{ in } \mathcal{R} \}$, which means that $\mathcal{R}$ embeds into the complex algebra $\text{CmA}t$ prohibiting the existence of an $m$-square representation because $\mathcal{R}_m$ does not have an $m$-square representation by hypothesis. Now take $\mathfrak{A} = \mathbb{B}b_m(\mathcal{R}_m, J, E)$ as defined in [5] to be the $\text{CA}_n$ obtained after blowing up and blurring $\mathcal{R}_m$ to a weakly representable relation algebra atom structure, namely, $\text{At} = \text{At}\mathcal{R}$. Here by [5, Theorem 3.2 9(ii)], $\text{Mat}_m(\text{At}\mathcal{R})$ is a $\text{CA}_n$ atom structure and $\mathfrak{A}$ is an atomic subalgebra of $\text{CmA}t_n(\text{At}\mathcal{R})$ containing $\text{CmA}t_n(\text{At}\mathcal{R})$, cf. [5]. Then $\mathfrak{A} \in \text{RCA}_n \cap \text{Nr}_n \text{CA}_n$, but $\mathfrak{A}$ has no complete $m$-square representation. In fact, by [5, item (3) pp.80], $\mathfrak{A} \cong \text{Nr}_m \mathbb{B}b(\mathcal{R}_m, J, E)$ is defined by [5, p.80]. Surjectiveness uses the condition $(J5)_t$ formulated in the second item of definition 3.15 of strong $l$-blurrness. A complete $m$-square representation of an atomic $\mathfrak{B} \in \text{CA}_n$ induces an $m$-square representation of $\text{CmA}t\mathfrak{B}$. To see why, assume that $\mathfrak{B}$ has an $m$-square complete representation via $f : \mathfrak{B} \to \mathfrak{D}$, where $\mathfrak{D} = \varphi(V)$ and the base of the representation $M = \bigcup_{s \in V} \text{rng}(s)$ is $m$-square. Let $\mathcal{C} = \text{CmA}t\mathfrak{B}$. For $c \in C$, let $c \upharpoonright = \{ a \in \text{At}\mathcal{C} : a \leq c \} = \{ a \in \text{At}\mathfrak{B} : a \leq c \}$. Define, representing $\mathcal{C}$, $g : \mathcal{C} \to \mathfrak{D}$ by $g(c) = \sum_{x \in c \upharpoonright} f(x)$, then $g$ is the required homomorphism into $\varphi(V)$ having base $M$. But $\text{CmA}t\mathfrak{A}$ does not have an $m$-square representation, because $\mathfrak{A}$ does not have an $m$-dimensional relational basis, and $\mathcal{R}_m \subseteq \text{RaCmA}t\mathfrak{A}$. So an $m$-square representation of $\text{CmA}t\mathfrak{A}$ induces one of $\mathcal{R}_m$ which by [15, Theorem 13.46], the equivalence (1) $\iff$ (5) implies that $\mathcal{R}_m$ has no $m$-dimensional relational basis, a contradiction. We prove $\Psi(m - 1, m)$, hence the required. By [11, §4.3], we can (and will) assume that $\mathfrak{A} = \mathfrak{M}T$ for a countable, simple and atomic theory $L_n$ theory $T$. Let $\Gamma$ be the $n$-type consisting of co-atoms of $T$. Then $\Gamma$ is realizable in every $m$-square model, for if $M$ is an $m$-square model omitting $\Gamma$, then $M$ would be the base of a complete $m$-square representation of $\mathfrak{A}$, and so by Lemma 3.3 $\mathfrak{A} \in \mathfrak{S}_c \text{Nr}_n \text{D}_m$ which is impossible. Suppose for contradiction that $\psi$ is an $m - 1$ witness, so that $T \models \phi \supseteq \alpha$, for all $\alpha \in \Gamma$, where recall that $\Gamma$ is the set of coatoms. Then since $\mathfrak{A}$ is simple, we can assume without loss that $\mathfrak{A}$ is a set algebra with base $M$ say. Let $M = (M, R_i)_{i \in \omega}$ be the corresponding model (in a relational signature) to this set algebra in the sense of [11, §4.3].
Let $\phi^M$ denote the set of all assignments satisfying $\phi$ in $M$. We have $M \models T$ and $\phi^M \in \mathfrak{A}$, because $\mathfrak{A} \in \text{Nr}_n\text{CA}_{m-1}$. But $T \models \exists x \phi$, hence $\phi^M \neq 0$, from which it follows that $\phi^M$ must intersect an atom $\alpha \in \mathfrak{A}$ (recall that the latter is atomic). Let $\psi$ be the formula, such that $\psi^M = \alpha$. Then it cannot be the case that $T \models \phi \rightarrow \neg \psi$, contradiction and we are done. Finally, $\Psi(n, n+3)$ and $\Psi(l, \omega)$ ($n \leq l < \omega$) follow from Theorems 3.10 and the above argument taking $R_m$ to be $E_k(2,3)$, respectively. □

Let $2 < n \leq l < m \leq \omega$. In $\text{VT}(l, m)$, while the parameter $l$ measures how close we are to $L_{\omega, \omega}$, $m$ measures the ‘degree’ of squareness of permitted models. Using elementary calculus terminology one can view $\lim_{l \to \infty} \text{VT}(l, \omega) = \text{VT}(\omega, \omega)$ algebraically using ultraproducts as follows. Fix $2 < n < \omega$. For each $2 < n \leq l < \omega$, let $\mathfrak{A}_l$ be the finite Maddux algebra $\mathfrak{E}_{f(l)}(2,3)$, as defined on [5, p.83, §5, in the proof of Theorem 5.1] with $l$-blur $(J_l, E_l)$ as defined in [5, Definition 3.1] and $f(l) \geq l$ as specified in [5, Lemma 5.1] (denoted by $k$ therein). Let $\mathcal{R}_l = \mathfrak{Bb}(\mathfrak{A}_l, J_l, E_l) \in \text{RRA}$ where $\mathcal{R}_l$ is the relation algebra having atom structure denoted $At$ in [5, p. 73] when the blown up and blurred algebra denoted $\mathfrak{A}_l$ happens to be the finite Maddux algebra $\mathfrak{E}_{f(l)}(2,3)$ and let $\mathfrak{A}_l = \mathfrak{N}_n \mathfrak{Bb}_l(\mathfrak{A}_l, J_l, E_l) \in \text{RCA}_n$ as defined in [5, Top of p.80] (with $\mathfrak{R}_l = \mathfrak{E}_{f(l)}(2,3)$). Then $(\text{At}\mathcal{R}_l: l \in \omega \sim n)$, and $(\text{At}\mathfrak{A}_l: l \in \omega \sim n)$ are sequences of weakly representable atom structures that are not strongly representable with a completely representable ultraproduct. Let $\text{LCA}_n$ denote the class of $\text{CA}_n$'s satisfying the Lyndon conditions [17], which is the elementary closure of the class of completely representable $\text{CA}_n$'s. We immediately get:

**Corollary 3.17** (Monk, Maddux, Biró, Hirsch and Hodkinson). Let $2 < n < \omega$. Then the set of equations using only one variable that holds in each of the varieties $\text{RCA}_n$ and $\text{RRA}$, together with any finite first order definable expansion of each, cannot be derived from any finite set of equations valid in the variety [6,22]. Furthermore, $\text{LCA}_n$ is not finitely axiomatizable.

4. Positive OTTs for $L_n$ with standard ‘unguarded’ semantics

Unless otherwise specified, $n$ will denote a finite ordinal $> 2$. Now we turn to proving omitting types theorems for certain (not all) $L_n$ theories. But first a definition:

**Definition 4.1.** Let $\mathfrak{A} \in \text{RCA}_n$ and let $\lambda$ be a cardinal.

1. If $\mathbf{X} = (X_i: i < \lambda)$ is family of subsets of $\mathfrak{A}$, we say that $\mathbf{X}$ is omitted in $\mathfrak{C} \in \text{Crs}_n$, if there exists an isomorphism $f: \mathfrak{A} \rightarrow \mathfrak{C}$ such that $\bigcap f(X_i) = \emptyset$ for all $i < \lambda$. When we want to stress the role of $f$, we say that $\mathbf{X}$ is omitted in $\mathfrak{C}$ via $f$. 
2. If $X \subseteq \mathcal{A}$ and $\prod X = 0$, then we refer to $X$ as a non-principal type of $\mathcal{A}$.

Observe that $\mathcal{A} \in \text{RCA}_n$ is completely representable $\iff$ $\mathcal{A}$ is atomic, and the single non-principal type of co-atoms can be omitted in a $\mathcal{G}_n$.

We prove positive omitting types theorems for $L_n$ by imposing extra conditions on theories considered like quantifier elimination. We need to recall certain cardinals that play a key role in (positive) omitting types theorems for $L_{\omega, \omega}$. Let $\text{covK}$ be the cardinal used in [33, Theorem 3.3.4]. The cardinal $p \satisfaction \omega < p \leq 2^\omega$ and has the following property: If $\lambda < p$, and $(A_i : i < \lambda)$ is a family of meager subsets of a Polish space $X$ (of which Stone spaces of countable Boolean algebras are examples) then $\bigcup_{i \in \lambda} A_i$ is meager.

For the definition and required properties of $p$, witness [10, p. 3, pp. 44-45, corollary 22c]. Both cardinals $\text{covK}$ and $p$ have an extensive literature. It is consistent that $\omega < p < \text{covK} \leq 2^\omega$ [10], so that the two cardinals are generally different, but it is also consistent that they are equal; equality holds for example in the Cohen real model of Solovay and Cohen. Martin’s axiom implies that both cardinals are the continuum. To prove the main result on positive omitting types theorems, we need the following lemma due to Shelah:

**Lemma 4.2.** Assume that $\lambda$ is an infinite regular cardinal. Suppose that $T$ is a first order theory, $|T| \leq \lambda$ and $\phi$ is a formula consistent with $T$, then there exist models $\mathfrak{M}_i : i < \lambda$, each of cardinality $\lambda$, such that $\phi$ is satisfiable in each, and if $i(1) \neq i(2) < \lambda^2$, $\bar{a}_{i(l)} \in M_{i(l)}$, $l = 1, 2$, $\text{tp}(\bar{a}_{i(1)}) = \text{tp}(\bar{a}_{i(2)})$, then there are $p_i \subseteq \text{tp}(\bar{a}_{i(l)})$, $|p_i| < \lambda$ and $p_i \vdash \text{tp}(\bar{a}_{i(l)})$. Here $\text{tp}(\bar{a})$ denotes the complete type realized (over the empty set) by the tuple $\bar{a}$.

**Proof.** [39, Theorem 5.16, Chapter IV].

In the Theorem $n < \omega$:

**Theorem 4.3.** Let $\mathcal{A} \in S_c \text{Nr}_n \text{CA}_\omega$ be countable. Let $\lambda < 2^{\aleph_0}$ and let $X = (X_i : i < \lambda)$ be a family of non-principal types of $\mathcal{A}$. Then the following hold:

1. If $\mathcal{A} \in \text{Nr}_n \text{CA}_\omega$ and the $X_i$s are non-principal ultrafilters, then $X$ can be omitted in a $\mathcal{G}_n$.

2. Every subfamily of $X$ of cardinality $< p$ can be omitted in a $\mathcal{G}_n$; in particular, every countable subfamily of $X$ can be omitted in a $\mathcal{G}_n$.

3. If $\mathcal{A}$ is simple, then every subfamily of $X$ of cardinality $< \text{covK}$ can be omitted in a $\mathcal{C}_n$.

**Proof.** To substantially simplify the proof while retaining the gist of ideas used in the more general case for the first item we assume that $\mathcal{A}$ is countable and simple, that is to say, has no proper ideal. This means that $\mathcal{A}$ ‘algebraically represents’ a complete countable theory. We have $\prod^B X_i = 0$ for all $i < \kappa$ because, $\mathcal{A}$ is a complete subalgebra of $\mathcal{B}$. To
see why, assume that \( S \subseteq \mathfrak{A} \) and \( \sum \mathfrak{A} S = y \), and for contradiction that
there exists \( d \in \mathfrak{B} \) such that \( s \leq d < y \) for all \( s \in S \). Then, assuming that \( A \) generates \( \mathfrak{B} \), we can infer that \( d \) uses finitely many dimensions in \( \omega \sim n \), \( m_1, \ldots, m_n \), say. Now let \( t = y \cdot -c_{m_1} \ldots c_{m_n}(-d) \). We claim that
\( t \in \mathfrak{A} = \mathfrak{Nr}_n \mathfrak{B} \) and \( s \leq t < y \) for all \( s \in S \). This contradicts \( y = \sum \mathfrak{A} S \). The first required follows from the fact that \( \Delta y \subseteq n \) and that all indices in \( \omega \sim n \)
that occur in \( d \) are cylindrical. In more detail, put \( J = \{ m_1, \ldots, m_n \} \) and let \( i \in \omega \sim n \), then \( c_i t = c_i(-c_j(-d)) = c_i - c_j(-d) = c_i - c_i c_j(-d) = -c_i c_j(-d) = -c_j(-d) = t \). We have shown that \( c_i t = t \) for all \( i \in \omega \sim n \), thus \( t \in \mathfrak{Nr}_n \mathfrak{B} = \mathfrak{A} \). If \( s \in S \), we show that \( s \leq t \). We know that \( s \leq y \). Also \( s \leq d \), so \( s \cdot -d = 0 \). Hence \( 0 = c_{m_1} \ldots c_{m_n}(s \cdot -d) = s \cdot c_{m_1} \ldots c_{m_n}(-d) \), so \( s \leq c_{m_1} \ldots c_{m_n}(-d) \), hence \( s \leq t \) as required. We finally check that \( t < y \).
If not, then \( t = y \) so \( y \leq c_{m_1} \ldots c_{m_n}(-d) \) and so \( y \cdot c_{m_1} \ldots c_{m_n}(-d) = 0 \). But
\( -d \leq c_{m_1} \ldots c_{m_n}(-d) \), hence \( y \cdot -d \leq y \cdot c_{m_1} \ldots c_{m_n}(-d) = 0 \). Hence \( y \cdot -d = 0 \) and this contradicts that \( d < y \). We have proved that \( \sum \mathfrak{B} X = 1 \) showing that \( \mathfrak{A} \) is indeed a complete subalgebra of \( \mathfrak{B} \). Since \( \mathfrak{B} \) is a locally finite dimensional algebra, we can assume that \( \mathfrak{B} = \mathcal{Fm}_T \) for some countable consistent theory \( T \). For each \( i < \kappa \), let \( \Gamma_i = \{ \phi/T : \phi \in X_i \} \). Let \( \mathcal{F} = (\Gamma_j : j < \kappa) \) be the corresponding set of types in \( T \). Then each \( \Gamma_j (j < \kappa) \) is a non-principal and complete \( n \)-type in \( T \), because each \( X_j \) is a maximal filter in \( \mathfrak{A} = \mathfrak{Nr}_n \mathfrak{B} \).

(\*\*) Let \( (\mathcal{M}_i : i < 2^{\omega}) \) be a set of countable models for \( T \) that overlap only
on principal maximal types; these exist by lemma 4.2. Assume for contradiction that for all \( i < 2^{\omega} \), there exists \( \Gamma \in \mathcal{F} \), such that \( \Gamma \) is realized in \( \mathcal{M}_i \). Let \( \psi : 2^{\omega} \to \varphi(\mathcal{F}) \), be defined by \( \psi(i) = \{ F \in \mathcal{F} : F \) is realized in \( \mathcal{M}_i \}. \) Then for all \( i < 2^{\omega} \), \( \psi(i) \neq \emptyset \). Furthermore, for \( i \neq j \), \( \psi(i) \cap \psi(j) = \emptyset \), for
if \( F \in \psi(i) \cap \psi(j) \), then it will be realized in \( \mathcal{M}_i \) and \( \mathcal{M}_j \), and so it will be principal. This implies that \( |\mathcal{F}| = 2^{\omega} \) which is impossible. Hence we obtain a model \( \models T \) omitting \( X \) in which \( \varphi \) is satisfiable. The map \( f \) defined from \( \mathfrak{A} = \mathcal{Fm}_T \) to \( Cs_n^M \) (the set algebra based on \( M \), cf. [11, 4.3.4]) via \( \phi_T \mapsto \phi^M \),
where the latter is the set of \( n \)-ary assignments in \( \mathfrak{M} \) satisfying \( \phi \), omits \( X \). Injectivity follows from the facts that \( f \) is non-zero and \( \mathfrak{A} \) is simple.

For (2) and (3), we can assume that \( \mathfrak{A} \subseteq \mathfrak{Nr}_n \mathfrak{B} \), \( \mathfrak{B} \subseteq \mathfrak{L}_\omega \mathfrak{B} \). We work in \( \mathfrak{B} \).
Using the notation on [33, p. 216 of proof of Theorem 3.3.4] replacing \( \mathcal{Fm}_T \nicefrac{\omega} \) by \( \mathfrak{B} \), we have \( \mathcal{H} = \bigcup_{\lambda \in \lambda} \bigcup_{\tau \in V} \mathcal{H}_{i, \tau} \) where \( \lambda < \mathfrak{p} \), and \( V \) is the weak space \( \omega_\mathfrak{H}^{(\text{Id})} \), can be written as a countable union of nowhere dense sets, and so can the countable union \( \mathcal{G} = \bigcup_{j \in \omega} \bigcup_{x \in \mathfrak{B}} \mathcal{G}_{j, x} \). So for any \( a \neq 0 \), there is an ultrafilter \( F \in N_\mathcal{A} \cap (S \setminus \mathcal{H} \cup \mathcal{G}) \) by the Baire category theorem. This induces a homomorphism \( f_a : \mathfrak{A} \to \mathcal{C}_a \), \( \mathcal{C}_a \in Cs_n \) that omits the given types, such that
\( f_a(a) \neq 0 \). (First one defines \( f \) with domain \( \mathfrak{B} \) as on p.216, then restricts \( f \) to \( \mathfrak{A} \) obtaining \( f_a \) the obvious way.) The map \( g : \mathfrak{A} \to \mathfrak{P}_a \{ a \in \mathfrak{A} \setminus \{ 0 \} \} \mathfrak{C}_a \) defined via \( x \mapsto (g_a(x) : a \in \mathfrak{A} \setminus \{ 0 \})(x \in \mathfrak{A}) \) is as required. In case \( \mathfrak{A} \) is simple, then by properties of \( \text{covK}, S \setminus (\mathcal{H} \cup \mathcal{G}) \) is non-empty, so if \( F \in S \setminus (\mathcal{H} \cup \mathcal{G}) \), then
$F$ induces a non-zero homomorphism $f$ with domain $\mathfrak{A}$ into a $C_{sn}$ omitting the given types. By simplicity of $\mathfrak{A}$, $f$ is injective. □

If $\mathfrak{A}$ as above happens to be atomic, then $X$, and in fact any family of non-principal types, will be omitted in a complete representation of $\mathfrak{A}$ which exists by [33, Theorem 5.3.6]. This is an $L_n$ version of the fact that in $L_{\omega,\omega}$, atomic models of atomic theories omit any given family of non-principal types (regardless of their cardinality) which in turn is a model-theoretic expression that atomic and complete representations in the $CA_n$ context are the same. By observing that if $T$ is an $L_n$ theory that admits elimination of quantifiers ($n < \omega$), then $\mathfrak{F}m_T \in Nr_nCA_\omega$, we get using Theorem 4.3 the following corollary:

**Corollary 4.4.** Let $n$ be any finite ordinal. Let $T$ be a countable and consistent $L_n$ theory and $\lambda$ be a cardinal $< p$. Let $F = (\Gamma_i : i < \lambda)$ be a family of non-principal types of $T$. Suppose that $T$ admits elimination of quantifiers. Then the following hold:

1. If $\phi$ is a formula consistent with $T$, then there is a model $M$ of $T$ that omits $F$, and $\phi$ is satisfiable in $M$. If $T$ is complete, then we can replace $p$ by $\text{cov}K$.

2. If the non-principal types constituting $F$ are maximal, then we can replace $p$ by $2^\omega$.

Using the full power of Lemma 4.2 together with the argument in item (1) of Theorem 4.3, one can replace in the last item of the last corollary $\omega$ by any regular uncountable cardinal $\mu$ as explicitly formulated next, cf. [33, Theorem 3.2.9].

**Theorem 4.5.** Let $\kappa$ be a regular infinite cardinal and $n < \omega$. Assume that $\mathfrak{A} \in Nr_nCA_\omega$ with $|A| \leq \kappa$, that $\lambda$ is a cardinal $< 2^\kappa$, and that $X = (X_i : i < \lambda)$ is a family of non-principal types of $\mathfrak{A}$. If the $X_i$s are non-principal ultrafilters of $\mathfrak{A}$, then $X$ can be omitted in a $Gs_n$.

In [36, Theorem 4.5], it is shown (algebraically) that the maximality condition cannot be removed when we consider uncountable theories by proving the following.

**Theorem 4.6.** Let $\kappa$ be an infinite cardinal. Then there exists an atomless $\mathfrak{C} \in CA_\omega$ such that for all $2 < n < \omega$, $|\mathfrak{N}_n\mathfrak{C}| = 2^\kappa$, $\mathfrak{N}_n\mathfrak{C} \in LCA_n(= El\text{CRCA}_n)$, but $\mathfrak{N}_n\mathfrak{C}$ is not completely representable. Thus the non-principal type of co-atoms of $\mathfrak{N}_n\mathfrak{C}$ cannot be omitted. In particular, the condition of maximality in Theorem 4.5 cannot be removed.

This was used in [36, Corollary 4.7] to reprove the main results in [14] in a completely different way using Monk like algebras rather than rainbow ones:

**Corollary 4.7.** For $2 < n < \omega$, the classes $CRCA_n$ and CRRA are not elementary.
For a class \( K \) of BAOs, let \( K \cap \text{Count} \) denote the class of atomic algebras in \( K \) having countably many atoms.

**Proposition 4.8.** Let \( 2 < n < \omega \).

1. For any ordinal \( 0 \leq j \), \( \text{RCA}_n \cap \text{Nr}_n \text{CA}_{n+j} \cap \text{Count} \) is not atom-canonical with respect to \( \text{RCA}_n \iff j < \omega \),

2. For any ordinal \( j \), \( \text{Nr}_n \text{CA}_{n+j} \cap \text{RCA}_n \cap \text{At} \not\subseteq \text{CRCA}_n \),

3. There exists an atomic \( \text{RCA}_n \) such that its Dedekind-MacNeille (minimal) completion does not embed into its canonical extension.\(^2\)

**Proof.** (1): One implication follows from [5] where for each \( 2 < n < l < \omega \) an algebra \( \mathfrak{A}_l \in \text{RCA}_n \cap \text{Nr}_n \text{CA}_l \) is constructed such that \( \text{CmAt} \mathfrak{A}_l \notin \text{RCA}_n \), so \( \mathfrak{A}_l \) cannot be completely representable. Conversely, for any infinite ordinal \( j \), \( \text{Nr}_n \text{CA}_{n+j} = \text{Nr}_n \text{CA}_\omega \) and if \( \mathfrak{A} \in \text{Nr}_n \text{CA}_\omega \cap \text{Count} \), then by [33, Theorem 5.3.6], \( \mathfrak{A} \in \text{CRCA}_n \), so \( \text{CmAt} \mathfrak{A} \in \text{RCA}_n \).

(2): The case \( j < \omega \), follows from the fact that the algebra \( \mathfrak{A}_{n+j} \) used in the previous item is in \( \text{Nr}_n \text{CA}_{n+j} \cap \text{RCA}_n \) but has no complete representation. For infinite \( j \) one uses the construction in Theorem 4.6.

(3): Let \( \mathfrak{A} = \text{CmAt} \mathfrak{A} \) be the \( \text{CA}_n \) as defined in the proof of Theorem 3.10. Since \( \text{CmAt} \mathfrak{A} \notin \text{RCA}_n \), it does not embed into \( \mathfrak{A}^+ \), because \( \mathfrak{A}^+ \in \text{RCA}_n \) since \( \mathfrak{A} \in \text{RCA}_n \) and \( \text{RCA}_n \) is a canonical variety. \( \square \)

In our next table, results on atom-canonicity for various varieties of RAs and CA\(_n\)s are summarized. For CA\(_s\) the dimension \( n \) is finite > 2. In the table \( m \geq 6 \) and \( t(n) \) is as specified in Theorem 3.10.

<table>
<thead>
<tr>
<th>Algebras</th>
<th>Atom-canonical</th>
<th>Citation</th>
</tr>
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<tr>
<td>( \text{RCA}_n, \text{RRA} )</td>
<td>no</td>
<td>yes, [19], [15]</td>
</tr>
<tr>
<td>( \text{SNr}<em>n \text{CA}</em>{n+1}, \text{SRaCA}_3 )</td>
<td>yes</td>
<td>yes, [15]</td>
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<tr>
<td>( \text{SNr}<em>n \text{CA}</em>{t(n)}, \text{SRaCA}_m )</td>
<td>no</td>
<td>no, Thm 3.10</td>
</tr>
</tbody>
</table>

5. Complete representations and non-elementary classes

We next strengthen Theorem 4.7. We first define a game \( H \) that involves certain hypernetworks. A \( \lambda \)-neat hypernetwork is roughly a network endowed with hyperedges of length \( \neq n \) allowed to get arbitrarily long but are of finite length, and such hyperedges get their labels from a non-empty set of labels

\(^2\)In the CA context, the terminology **minimal completion** is misleading because \( \mathfrak{A}^+ \) is another completion of \( \mathfrak{A} \); so supposedly the minimal completion of \( \mathfrak{A} \) should embed into \( \mathfrak{A}^+ \), which is not, as we have already seen in Theorem 3.10, always true. Conversely, for an atomic Boolean algebra \( \mathfrak{B} \), \( \text{CmAt} \mathfrak{B} \) always embeds into \( \mathfrak{B}^+ \) as it should.
\[\lambda;\] such that all so-called short hyperedges are constantly labelled by \(\lambda \in \Lambda\).

The board of the game consists of \(\lambda\)-neat hypernetworks:

**Definition 5.1.** For an \(n\)-dimensional atomic network \(N\) on an atomic \(\text{CA}_n\) and for \(x, y \in \text{nodes}(N)\), set \(x \sim y\) if there exists \(z\) such that \(N(x, y, z) \leq d_{01}\). Define the equivalence relation \(\sim\) over the set of all finite sequences over \(\text{nodes}(N)\) by \(\bar{x} \sim \bar{y}\) iff \(|\bar{x}| = |\bar{y}|\) and \(x_i \sim y_i\) for all \(i < |\bar{x}|\). (It can be easily checked that this indeed an equivalence relation). A hypernetwork \(N = (N^a, N^h)\) over an atomic \(\text{CA}_n\) consists of an \(n\)-dimensional network \(N^a\) together with a labelling function for hyperlabels \(N^h: <\omega\text{nodes}(N) \to \Lambda\) (some arbitrary set of hyperlabels \(\Lambda\)) such that for \(\bar{x} \in <\omega\text{nodes}(N)\) if \(\bar{x} \sim \bar{y}\) then \(N^h(\bar{x}) = N^h(\bar{y})\). If \(|\bar{x}| = k \in \mathbb{N}\) and \(N^h(\bar{x}) = \lambda\), then we say that \(\lambda\) is a \(k\)-ary hyperlabel. \(\bar{x}\) is referred to as a \(k\)-ary hyperedge, or simply a hyperedge. A hyperedge \(\bar{x} \in <\omega\text{nodes}(N)\) is short, if there are \(y_0, \ldots, y_{n-1}\) that are nodes in \(N\), such that \(N(x_i, y_0, z) \leq d_{01}\) or \(N(x_i, y_{n-1}, z) \leq d_{01}\) for all \(i < |\bar{x}|\), for some (equivalently for all) \(z\). Otherwise, it is called long. This game involves, besides the standard cylindrifier move, two new amalgamation moves. Concerning his moves, this game with \(m\) rounds \((m \leq \omega)\), call it \(\text{H}_m\), \(\forall\) can play a cylindrifier move, like before but now played on \(\lambda\)-neat hypernetworks (\(\lambda\) a constant label). Also \(\forall\) can play a transformation move by picking a previously played hypernetwork \(N\) and a partial, finite surjection \(\theta: \omega \to \text{nodes}(N)\), this move is denoted \((N, \theta)\). \(\exists\)'s response is mandatory. She must respond with \(N\theta\). Finally, \(\forall\) can play an amalgamation move by picking previously played hypernetworks \(M, N\) such that \(M \upharpoonright \text{nodes}(M) \cap \text{nodes}(N) = N \upharpoonright \text{nodes}(M) \cap \text{nodes}(N)\), and \(\text{nodes}(M) \cap \text{nodes}(N) \neq \emptyset\). This move is denoted \((M, N)\). To make a legal response, \(\exists\) must play a \(\lambda_0\)-neat hypernetwork \(L\) extending \(M\) and \(N\), where \(\text{nodes}(L) = \text{nodes}(M) \cup \text{nodes}(N)\).

**Theorem 5.2.** Let \(\alpha\) be a countable atom structure. If \(\exists\) has a winning strategy in \(\text{H}_\omega(\alpha)\), then there exists a complete \(\mathfrak{D} \in \text{RCA}_\omega\) such that \(\exists\text{m}_\alpha \equiv \mathfrak{M}_\alpha \mathfrak{D}\) and \(\alpha \equiv \text{AtRCA}_\omega\mathfrak{D}\). In particular, \(\exists\text{m}_\alpha \in \text{Nrc}_\omega\text{CA}_\omega\) and \(\alpha \in \text{AtNrc}_\omega\text{CA}_\omega\).

**Proof.** Fix some \(a \in \alpha\). The game \(\text{H}_\omega\) is designed so that using \(\exists\)'s winning strategy in the game \(\text{H}_\omega(\alpha)\) one can define a nested sequence \(M_0 \subseteq M_1, \ldots\) of \(\lambda\)-neat hypernetworks where \(M_0\) is \(\exists\)'s response to the initial \(\forall\)-move \(a\) such that: If \(M_r\) is in the sequence and \(M_r(\bar{x}) \leq c_i a\) for an atom \(a\) and some \(i < n\), then there is \(s \geq r\) and \(d \in \text{nodes}(M_s)\) such that \(M_s(\bar{y}) = a, \bar{y}_i = d\) and \(\bar{y} \equiv_i \bar{x}\). In addition, if \(M_r\) is in the sequence and \(\theta\) is any partial isomorphism of \(M_r\), then there is \(s \geq r\) and a partial isomorphism \(\theta^+\) of \(M_s\) extending \(\theta\) such that \(\text{rng}(\theta^+) \supseteq \text{nodes}(M_r)\) (This can be done using \(\exists\)'s responses to amalgamation moves). Now let \(\mathfrak{M}_a\) be the limit of this sequence, that is \(\mathfrak{M}_a = \bigcup M_i\), the labelling of \(n - 1\) tuples of nodes by atoms, and hyperedges by hyperlabels done in the obvious way using the fact that the \(M_i\)'s are nested. Let \(L\) be the signature with one \(n\)-ary relation for each \(b \in \alpha\), and one \(k\)-ary predicate symbol for each \(k\)-ary hyperlabel \(\lambda\). Now we
work in $L_{\infty, \omega}$. For fixed $f_a \in \omega$-nodes($\mathcal{M}_a$), let $\mathcal{U}_a = \{ f \in \omega$-nodes($\mathcal{M}_a$) : $i < \omega$: $g(i) \neq f_a(i) \}$ is finite}. We make $\mathcal{U}_a$ into the base of an $L$ relativized structure $\mathcal{M}_a$ like in [12, Theorem 29] except that we allow a clause for infinitary disjunctions. In more detail, for $b \in \alpha$, $l_0, \ldots, l_{n-1}, i_0 \ldots, i_{k-1} < \omega$, $k$-ary hyperlabels $\lambda$, and all $L$-formulas $\phi, \phi_i, \psi$, and $f \in U_a$:

\[
\begin{align*}
\mathcal{M}_a, f \models b(x_0, \ldots, x_{l-1}) & \iff \mathcal{M}_a(f(l_0), \ldots, f(l_{n-1})) = b, \\
\mathcal{M}_a, f \models \lambda(x_0, \ldots, x_{k-1}) & \iff \mathcal{M}_a(f(i_0), \ldots, f(i_{k-1})) = \lambda, \\
\mathcal{M}_a, f \models \neg \phi & \iff \mathcal{M}_a, f \not\models \phi, \\
\mathcal{M}_a, f \models (\bigvee_{i \in I} \phi_i) & \iff (\exists i \in I)(\mathcal{M}_a, f \models \phi_i), \\
\mathcal{M}_a, f \models \exists x_i \phi & \iff \mathcal{M}_a, [i/m] \models \phi, \text{ some } m \in \text{nodes}(\mathcal{M}_a).
\end{align*}
\]

For any such $L$-formula $\phi$, write $\phi^{\mathcal{M}_a}$ for $\{ f \in \mathcal{U}_a : \mathcal{M}_a, f \models \phi \}$. Let $D_a = \{ \phi^{\mathcal{M}_a} : \phi \text{ is an } L\text{-formula} \}$ and $\mathcal{D}_a$ be the weak set algebra with universe $D_a$. Let $\mathcal{D} = \mathcal{P}_{a \in \alpha} \mathcal{D}_a$. Then $\mathcal{D}$ is a generalized complete weak set algebra [11, Definition 3.1.2 (iv)]. Now we show that $\alpha \cong \mathcal{At}_{\mathcal{R}_n} \mathcal{D}$ and $\mathcal{L}_{\alpha} \equiv \mathcal{N}_{\mathcal{R}_n} \mathcal{D}$. Let $x \in \mathcal{D}$. Then $x = (x_a : a \in \alpha)$, where $x_a \in \mathcal{D}_a$. For $b \in \alpha$ let $\pi_b : \mathcal{D} \rightarrow \mathcal{D}_b$ be the projection map defined by $\pi_b(x_a : a \in \alpha) = x_b$. Conversely, let $\iota_a : \mathcal{D}_a \rightarrow \mathcal{D}$ be the embedding defined by $\iota_a(y) = (x_b : b \in \alpha)$, where $x_a = y$ and $x_b = 0$ for $b \neq a$. Suppose $x \in \mathcal{N}_{\mathcal{R}_n} \mathcal{D} \setminus \{0\}$. Since $x \neq 0$, then it has a non-zero component $\pi_a(x) \in \mathcal{D}_a$, for some $a \in \alpha$. Assume that $\emptyset \neq \phi(x_0, \ldots, x_{i_{k-1}})^{\mathcal{D}_a} = \pi_a(x)$, for some $L$-formula $\phi(x_0, \ldots, x_{i_{k-1}})$. We have $\phi(x_0, \ldots, x_{i_{k-1}})^{\mathcal{D}_a} \in \mathcal{N}_{\mathcal{R}_n} \mathcal{D}_a$. Pick $f \in \phi(x_0, \ldots, x_{i_{k-1}})^{\mathcal{D}_a}$ and assume that $\mathcal{M}_a, f \models b(x_0, \ldots x_{n-1})$ for some $b \in \alpha$. We show that $b(x_0, x_1, \ldots, x_{n-1})^{\mathcal{D}_a} \subseteq \phi(x_0, \ldots, x_{i_{k-1}})^{\mathcal{D}_a}$. Take any $g \in b(x_0, x_1, \ldots, x_{n-1})^{\mathcal{D}_a}$, so that $\mathcal{M}_a, g \models b(x_0, \ldots x_{n-1})$. The map $\{(f(i), g(i)) : i < n\}$ is a partial isomorphism of $\mathcal{M}_a$. Here that short hyperedges are constantly labelled by $\lambda$ is used. This map extends to a finite partial isomorphism $\theta$ of $\mathcal{M}_a$ whose domain includes $f(i_0), \ldots, f(i_{k-1})$. Let $g' \in \mathcal{M}_a$ be defined by

\[
g'(i) = \begin{cases} 
\theta(i) & \text{if } i \in \text{dom}(\theta) \\
g(i) & \text{otherwise.}
\end{cases}
\]

We have $\mathcal{M}_a, g' \models \phi(x_0, \ldots, x_{i_{k-1}})$. But $g'(0) = \theta(0) = g(0)$ and similarly $g'(n - 1) = g(n - 1)$, so $g$ is identical to $g'$ over $n$ and it differs from $g'$ on only a finite set. Since $\phi(x_0, \ldots, x_{i_{k-1}})^{\mathcal{D}_a} \in \mathcal{N}_{\mathcal{R}_n} \mathcal{D}_a$, we get that $\mathcal{M}_a, g \models \phi(x_0, \ldots, x_{i_{k-1}})$, so $g \in \phi(x_0, \ldots, x_{i_{k-1}})^{\mathcal{D}_a}$ (this can be proved by induction
on quantifier depth of formulas). This proves that
\[ b(x_0, x_1 \ldots x_{n-1})^{\mathcal{D}} \subseteq \phi(x_{i_0}, \ldots, x_{i_k})^{\mathcal{D}} = \pi_a(x), \]
and so
\[ \iota_a(b(x_0, x_1 \ldots x_{n-1})^{\mathcal{D}}) \leq \iota_a(\phi(x_{i_0}, \ldots, x_{i_k})^{\mathcal{D}}) \leq x \in \mathcal{D}_a \setminus \{0\}. \]

Now every non-zero element \( x \) of \( \mathcal{N}_n \mathcal{D}_a \) is above a non-zero element of the following form \( \iota_a(b(x_0, x_1, \ldots, x_{n-1})^{\mathcal{D}}) \) (some \( a, b \in \alpha \)) and these are the atoms of \( \mathcal{N}_n \mathcal{D}_a \). The map defined via \( b \mapsto (b(x_0, x_1, \ldots, x_{n-1})^{\mathcal{D}} : a \in \alpha) \) is an isomorphism of atom structures, so that \( \alpha \in \text{At}\mathcal{N}_n \mathcal{C}_{\mathcal{A}_\omega} \). Let \( X \subseteq \mathcal{N}_n \mathcal{D} \).

Then by completeness of \( \mathcal{D} \), we get that \( d = \sum_{\mathcal{D}} X \) exists. Assume that \( i \notin n \), then \( c_i d = c_i \sum_{\mathcal{D}} X = \sum_{x \in X} c_i x = \sum_{x \in X} x = d \), because the \( c_i \)'s are completely additive and \( c_i x = x \), for all \( i \notin n \), since \( x \in \mathcal{N}_n \mathcal{D} \). We conclude that \( d \in \mathcal{N}_n \mathcal{D} \), hence \( d \) is an upper bound of \( X \) in \( \mathcal{N}_n \mathcal{D} \). Since \( d = \sum_{x \in X} X \) there can be no \( b \in \mathcal{N}_n \mathcal{D} (\subseteq \mathcal{D}) \) with \( b < d \) such that \( b \) is an upper bound of \( X \) for else it will be an upper bound of \( X \) in \( \mathcal{D} \). Thus \( \sum_{x \in X} X = d \) We have shown that \( \mathcal{N}_n \mathcal{D} \) is complete. Making the legitimate identification \( \mathcal{N}_n \mathcal{D} \mathcal{K} \subseteq \mathcal{D} \mathcal{C}_{\mathcal{A}_\omega} \) by density, we get that \( \mathcal{N}_n \mathcal{D} = \mathcal{C}_{\mathcal{A}_\omega} \) (since \( \mathcal{N}_n \mathcal{D} \) is complete), hence \( \mathcal{C}_{\mathcal{A}_\omega} \subseteq \mathcal{N}_n \mathcal{C}_{\mathcal{A}_\omega} \).

If \( \mathcal{A} \) is a Boolean algebra and \( b \in \mathcal{B} \), then \( \mathcal{A}_b \mathcal{B} \) denotes the Boolean algebra with domain \( \{ x \in B : x \leq b \} \), top element \( b \), and other Boolean operations those of \( \mathcal{B} \) relativized to \( b \).

**Lemma 5.3.** In the following \( \mathcal{A} \) and \( \mathcal{D} \) are Boolean algebras.

1. If \( \mathcal{A} \) is atomic and \( 0 \neq a \in \mathcal{A} \), then \( \mathcal{A}_a \mathcal{A} \) is also atomic. If \( \mathcal{A} \subseteq_{d} \mathcal{D} \), and \( a \in \mathcal{A} \), then \( \mathcal{A}_a \mathcal{A} \subseteq_{d} \mathcal{A}_a \mathcal{D} \).

2. If \( \mathcal{A} \subseteq_{d} \mathcal{D} \) then \( \mathcal{A} \subseteq_{c} \mathcal{D} \). In particular, for any class \( K \) of BAOs, \( K \subseteq \mathcal{S}_d \mathcal{K} \subseteq \mathcal{S}_c \mathcal{K} \). If furthermore \( \mathcal{A} \) and \( \mathcal{D} \) are atomic, then \( \text{At}\mathcal{D} \subseteq \text{At}\mathcal{A} \).

**Proof.** (1): Let \( b \in \mathcal{A}_a \mathcal{D} \) be non-zero. Then \( b \leq a \) and \( b \) is non-zero in \( \mathcal{D} \). By atomicity of \( \mathcal{D} \) there is an atom \( c \) of \( \mathcal{D} \) such that \( c \leq b \). So \( c \leq b \leq a \), thus \( c \in \mathcal{A}_a \mathcal{D} \). Also \( c \) is an atom in \( \mathcal{A}_a \mathcal{D} \) because if not, then it will not be an atom in \( \mathcal{D} \). The second part is similar.

(2): Assume that \( \sum_{\mathcal{A}} S = 1 \) and for contradiction that there exists \( b' \in \mathcal{D} \), \( b' < 1 \) such that \( s \leq b' \) for all \( s \in S \). Let \( b = 1 - b' \) then \( b \neq 0 \), hence by assumption (density) there exists a non-zero \( a \in \mathcal{A} \) such that \( a \leq b \), i.e. \( a \leq (1 - b') \). If \( a \cdot s \neq 0 \) for some \( s \in S \), then \( a \) is not less than \( b' \) which is impossible. So \( a \cdot s = 0 \) for every \( s \in S \), implying that \( a = 0 \), contradiction.

Now we prove the second part. Assume that \( \mathcal{A} \subseteq_{d} \mathcal{D} \) and \( \mathcal{D} \) is atomic. Let \( b \in \mathcal{D} \) be an atom. We show that \( b \in \text{At}\mathcal{A} \). By density there is a non-zero \( a' \in \mathcal{A} \), such that \( a' \leq b \) in \( \mathcal{D} \). Since \( \mathcal{A} \) is atomic, there is an atom \( a \in \mathcal{A} \) such
that \( a \leq a' \leq b \). But \( b \) is an atom of \( \mathcal{D} \), and \( a \) is non-zero in \( \mathcal{D} \), too, so it must be the case that \( a = b \in \text{At}\mathfrak{A} \). Thus \( \text{At}\mathfrak{B} \subseteq \text{At}\mathfrak{A} \) and we are done. 

**Theorem 5.4.** For \( 2 < n < \omega \), any class \( \mathbf{K} \) such that \( \mathbf{S}_{d}N_{r}CA_{\omega} \cap \mathbf{CRCA}_{n} \subseteq \mathbf{K} \subseteq \mathbf{S}_{c}N_{r}CA_{n+3} \), \( \mathbf{K} \) is not elementary.

**Proof.** We use the construction in [35, Theorem 5.12] based on ideas in [12]. The algebra \( \mathfrak{C}_{Z,N}(\in \mathbf{RCA}_{n}) \) based on \( \mathbb{Z} \) (greens) and \( \mathbb{N} \) (reds) denotes the rainbow-like algebra used in *op.cit* which is defined as follows: The reds \( \mathcal{R} \) is the set \( \{ r_{ij} : i < j < \omega(= \mathbb{N}) \} \) and the green colours used constitute the set \( \{ g_{i} : 1 \leq i < n - 1 \} \cup \{ g_{0}^{i} : i \in \mathbb{Z} \} \). In complete coloured graphs the forbidden triples are like the usual rainbow constructions based on \( \mathbb{Z} \) and \( \mathbb{N} \), with a significant addition: First the colours used are:

- greens: \( g_{i} \) (\( 1 \leq i \leq n - 2 \)), \( g_{0}^{i} \), \( i \in \mathbb{Z} \),
- whites: \( w_{i} \), \( i \leq n - 2 \),
- reds: \( r_{ij} \) (\( i, j \in \mathbb{N} \)),
- shades of yellow: \( g_{S}^{i,j,k} : S \) a finite subset of \( \omega \) or \( S = \omega \).

The rainbow algebra depending on \( \mathbb{N} \) and \( \mathbb{Z} \) from the class \( \mathbf{K} \) consisting of all coloured graphs \( \mathcal{M} \) such that:

1. \( \mathcal{M} \) is a complete graph and \( \mathcal{M} \) contains no triangles (called forbidden triples) of the following types:

   - (4) \( (g, g', g^{*}), (g_{i}, g_{i}, w_{i}) \) any \( 1 \leq i \leq n - 2 \),
   - (5) \( (g_{0}^{j}, g_{0}^{k}, w_{0}) \) any \( j, k \in \mathbb{Z} \),
   - (6) \( (r_{ij}, g_{j}^{k}, r_{i}^{k}) \) unless \( i = i^{*}, j = j' \) and \( k' = k^{*} \)

Observe that this 1.7 is not as item 1.3 in the proof of Theorem 3.10. Here inconsistent triples of reds are defined differently.

2. The triple \( (g_{0}^{i}, g_{0}^{j}, r_{kl}) \) is also forbidden if \( \{(i, k), (j, l)\} \) is not an order preserving partial function from \( \mathbb{Z} \rightarrow \mathbb{N} \)

It is proved in *op.cit* that \( \exists \) has a winning strategy in \( G_{k}(\text{At}\mathfrak{C}_{Z,N}) \) for all \( k \in \omega \), so that \( \mathfrak{C}_{Z,N} \in \mathbf{EI}\mathbf{CRCA}_{n} \). With some more effort it can be proved

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3The article [12] claims that it proves that any class \( \mathbf{K} \) such that \( \mathbf{RaCA}_{\omega} \cap \mathbf{CRRA} \subseteq \mathbf{K} \subseteq \mathbf{S}_{c}\mathbf{RaCA}_{5} \) is not elementary. But there a mistake in the proof which is assuming that the implication \( \text{At}\mathfrak{A} \in \text{At}\mathbf{RCA}_{\omega} \implies \mathfrak{A} \in \mathbf{RCA}_{\omega} \) which is not true in general. This implication is also not true for \( \mathbf{RRA} \) and \( \mathbf{RCA}_{n} \) by Theorem 3.10. However, for \( \mathbf{CRRA}, \mathbf{CRCA}_{n}, \mathbf{LCA}_{n} \) and \( \mathbf{S}_{c}N_{r}CA_{m} \) (\( n < m \)) it is true; for e.g. \( \text{At}\mathfrak{A} \in \text{At}\mathbf{CRCA}_{n} \implies \mathfrak{A} \in \mathbf{CRCA}_{n} \). The mistake was corrected in [13] proving the weaker result of the non-first order definability of any class \( \mathbf{K} \) such that \( \mathbf{S}_{c}\mathbf{RaCA}_{\omega} \cap \mathbf{CRRA} \subseteq \mathbf{K} \subseteq \mathbf{S}_{c}\mathbf{RaCA}_{5} \). This was strengthened by the present author by replacing \( \mathbf{S}_{c}\mathbf{RaCA}_{\omega} \) by \( \mathbf{S}_{d}\mathbf{RaCA}_{\omega} \). In the cylindric (present) case adjoining the auxiliary construction in [28], we could prove the CA analogue extremely close to the ‘stronger result’ alleged in [12].
that \( \exists \) has a winning strategy \( \sigma_k \) say in \( H_k(AtC_{Z,N}) \) for all \( k \in \omega \). We can assume that \( \sigma_k \) is deterministic. Let \( \mathcal{D} \) be a non-principal ultrapower of \( C_{Z,N} \). Then \( \exists \) has a winning strategy \( \sigma \) in \( H_\omega(At\mathcal{D}) \) – essentially she uses \( \sigma_k \) in the \( k \)'th component of the ultraproduct so that at each round of \( H_\omega(At\mathcal{D}) \), \( \exists \) is still winning in co-finitely many components, this suffices to show she has still not lost. We can also assume that \( C_{Z,N} \) is countable by replacing it by the term algebra. Now one can use an elementary chain argument to construct countable elementary subalgebras \( C_{Z,N} = A_0 \preceq A_1 \preceq \ldots \preceq A_\omega \) in this manner. One defines \( A_{i+1} \) as a countable elementary subalgebra of \( A_i \) and all elements of \( \mathcal{D} \) that \( \sigma \) selects in a play of \( G(At\mathcal{D}) \) in which \( \forall \) only chooses elements from \( A_i \). Now let \( \mathcal{B} = \bigcup_{i<\omega} A_i \). This is a countable elementary subalgebra of \( \mathcal{D} \), hence necessarily atomic, and \( \exists \) has a winning strategy in \( H_\omega(At\mathcal{B}) \) and \( \mathcal{B} \equiv C_{Z,N} \). Thus by Lemma 5.8 \( \mathcal{B} \subseteq \text{At}_{Nr_i}CA_\omega \) and \( \text{At}C\mathcal{B} \subseteq N_{r_i}CA_\omega \). (This does not imply that \( \mathcal{B} \subseteq N_{r_i}CA_\omega \), cf. example 5.6). Since \( \mathcal{B} \equiv d \text{At}C\mathcal{B} \), \( \mathcal{B} \subseteq S_dN_{r_i}CA_\omega \), so \( \mathcal{B} \subseteq S_eN_{r_i}CA_\omega \). Being countable, it follows by [34, Theorem 5.3.6] that \( \mathcal{B} \in \text{CR}CA_n \). We now show that hat \( \forall \) has a winning strategy in \( G^{n+3}(AtC_{Z,N}) \) (denoted in \( \text{op.cit by } F^{n+3}(AtC_{Z,N}) \)), hence by Lemma 3.7, \( C_{Z,N} \notin S_eN_{r_i}CA_{n+3} \). It can be shown that \( \forall \) has a winning strategy in the graph version of the game \( G^{n+3}(At\mathfrak{C}) \) played on coloured graphs [14]. The rough idea here is, that, as is the case with winning strategy’s of \( \forall \) in rainbow constructions, \( \forall \) bombards \( \exists \) with cones having distinct green tints demanding a red label from \( \exists \) to appexes of successive cones. The number of nodes are limited but \( \forall \) has the option to re-use them, so this process will not end after finitely many rounds. The added order preserving condition relating two greens and a red, forces \( \exists \) to choose red labels, one of whose indices form a decreasing sequence in \( \mathbb{N} \). In \( \omega \) many rounds \( \forall \) forces a win, so by lemma 3.7, \( C_{Z,N} \notin S_eN_{r_i}CA_{n+3} \). More rigorously, \( \forall \) plays as follows: In the initial round \( \forall \) plays a graph \( M \) with nodes \( 0,1,\ldots,n-1 \) such that \( M(i,j) = w_0 \) for \( i < j < n-1 \) and \( M(i,n-1) = g_i \), \( (i = 1,\ldots,n-2) \), \( M(0,n-1) = g_0^0 \) and \( M(0,1,\ldots,n-2) = y_z \). This is a 0 cone. In the following move \( \forall \) chooses the base of the cone \( (0,\ldots,n-2) \) and demands a node \( n \) with \( M_2(i,n) = g_i \) \( (i = 1,\ldots,n-2) \), and \( M_2(0,n) = g_0^{-1} \). \( \exists \) must choose a label for the edge \( (n+1,n) \) of \( M_2 \). It must be a red atom \( r_{mk}, m,k \in \mathbb{N} \). Since \( -1 < 0 \), then by the ‘order preserving’ condition we have \( m < k \). In the next move \( \forall \) plays the face \( (0,\ldots,n-2) \) and demands a node \( n+1 \), with \( M_3(i,n) = g_i \), \( (i = 1,\ldots,n-2) \), such that \( M_3(0,n+2) = g_0^{-2} \). Then \( M_3(n+1,n) \) and \( M_3(n+1,n-1) \) both being red, the indices must match. \( M_3(n+1,n) = r_{kl} \) and \( M_3(n+1,n-1) = r_{km} \) with \( l < m \in \mathbb{N} \). In the next round \( \forall \) plays \( (0,1,\ldots,n-2) \) and re-uses the node 2 such that \( M_4(0,2) = g_0^{-3} \). This time we have \( M_4(n,n-1) = r_{jl} \) for some \( j < l < m \in \mathbb{N} \). Continuing in this manner leads to a decreasing sequence in \( \mathbb{N} \). We have proved the required. Let \( K \) be a class between \( S_dN_{r_i}CA_\omega \cap \text{CRCA}_n \) and \( S_eN_{r_i}CA_{n+3} \). Then \( K \) is not elementary, because \( C_{Z,N} \notin S_dN_{r_i}CA_{n+3} \), \( \mathcal{B} \in S_dN_{r_i}CA_\omega \cap \text{CRCA}_n(\subseteq K) \), and \( C_{Z,N} \equiv \mathcal{B} \).
Theorem 5.5. For $2 < n < \omega$, any class $K$ such that $\text{Nr}_n \text{CA}_\omega \cap \text{CRCA}_n \subseteq K \subseteq S_d \text{Nr}_n \text{CA}_{n+1}$, $K$ is not elementary.

Proof. We use the construction in [28], where two atomic algebras $\mathfrak{A}, \mathfrak{B} \in \text{CA}_n$ are constructed such that $\mathfrak{A} \notin \text{Nr}_n \text{CA}_\omega$, $\mathfrak{B} \notin S_d \text{Nr}_n \text{CA}_{n+1}$ where $S_d$ is the operation of forming dense subalgebras. In op.cit, it is only proved that $\mathfrak{B} \notin \text{Nr}_n \text{CA}_{n+1}$. But it can be shown using the same argument that $\mathfrak{B}$ is actually outside the bigger class $S_d \text{Nr}_n \text{CA}_{n+1}$. To removing the $S_d$, We slightly modify the construction in [34, Lemma 5.1.3, Theorem 5.1.4].

Using the same notation, the algebras $\mathfrak{A}$ and $\mathfrak{B}$ constructed in op.cit satisfy $\mathfrak{A} \in \text{Nr}_n \text{CA}_\omega$, $\mathfrak{B} \notin \text{Nr}_n \text{CA}_{n+1}$ and $\mathfrak{A} \equiv \mathfrak{B}$. As they stand, $\mathfrak{A}$ and $\mathfrak{B}$ are not atomic, but it can be fixed that they are atomic, giving the same result with the rest of the proof unaltered. This is done by interpreting the uncountably many tenary relations in the signature of $M$ defined in [34, Lemma 5.1.3], which is the base of $\mathfrak{A}$ and $\mathfrak{B}$ to be disjoint in $M$, not just distinct. The construction is presented this way in [28], where (the equivalent of) $M$ is built in a more basic step-by-step fashion. We work with $2 < n < \omega$ instead of only $n = 3$. The proof presented in op.cit lift verbatim to any such $n$. Let $u \in ^n \omega$. Write $1_u$ for $\chi^M_u$ (denoted by $1_u$ for $n = 3$ in [34, Theorem 5.1.4].) We denote by $\mathfrak{A}_u$ the Boolean algebra $\mathfrak{A}_{1,u} = \{ x \in \mathfrak{A} : x \leq 1_u \}$ and similarly for $\mathfrak{B}_u$ writing $\mathfrak{B}_u$ short hand for the Boolean algebra $\mathfrak{A}_{1,u} = \{ x \in \mathfrak{B} : x \leq 1_u \}$. Then exactly like in [34], it can be proved that $\mathfrak{A} \equiv \mathfrak{B}$. Using that $M$ has quantifier elimination we get, using the same argument in op.cit that $\mathfrak{A} \in \text{Nr}_n \text{CA}_\omega$. The property that $\mathfrak{B} \notin \text{Nr}_n \text{CA}_{n+1}$ is also still maintained. To see why, consider the substitution operator $n_s(0,1)$ (using one spare dimension) as defined in the proof of [34, Theorem 5.1.4]. Assume for contradiction that $\mathfrak{B} = \text{Nr}_n \mathfrak{C}$, with $\mathfrak{C} \in \text{CA}_{n+1}$. Let $u = (1,0,2,\ldots ,n-1)$. Then $\mathfrak{A}_u = \mathfrak{B}_u$ and so $|\mathfrak{B}_u| > \omega$. The term $n_s(0,1)$ acts like a substitution operator corresponding to the transposition $\{0,1\}$: it ‘swaps’ the first two co-ordinates. Now one can show that $n_s(0,1)^\mathfrak{B}_u \subseteq S_d \mathfrak{B}_u = \mathfrak{B}_{1d}$, so $|n_s(0,1)^\mathfrak{B}_u|$ is countable because $\mathfrak{B}_{1d}$ was forced by construction to be countable. But $n_s(0,1)$ is a Boolean automorphism with inverse $n_s(1,0)$, so that $|\mathfrak{B}_u| = |n_s(0,1)^\mathfrak{B}_u| > \omega$, contradiction. It can be proved exactly like in [34] that the property $\mathfrak{A} \equiv \mathfrak{B}$ is also still maintained after making the atoms disjoint. In fact, this change offers more for it can be proved that $\mathfrak{A} \equiv_{\infty} \mathfrak{B}$. We show that $\exists \text{ has a winning strategy in an Ehrenfeucht–Fra"issé-game over } (\mathfrak{A}, \mathfrak{B})$ concluding that $\mathfrak{A} \equiv_{\infty} \mathfrak{B}$.

At any stage of the game, if $\forall$ places a pebble on one of $\mathfrak{A}$ or $\mathfrak{B}$, $\exists$ must place a matching pebble, on the other algebra. Let $\bar{a} = \langle a_0, a_1, \ldots , a_{n-1} \rangle$ be the position of the pebbles played so far (by either player) on $\mathfrak{A}$ and let $\bar{b} = \langle b_0, \ldots , b_{n-1} \rangle$ be the the position of the pebbles played on $\mathfrak{B}$. $\exists$ maintains the following properties throughout the game: For any atom $x$ (of either algebra) with $x \cdot \mathbf{1}_{1d} = 0$ then $x \in a_i \iff x \in b_i$ and $\bar{a}$ induces a finite partition of $\mathbf{1}_{1d}$ in $\mathfrak{A}$ of $2^n$ (possibly empty) parts $p_i : i < 2^n$ and $\bar{b}$ induces a partition of $\mathbf{1}_{1d}$ in $\mathfrak{B}$ of parts $q_i : i < 2^n$. Furthermore, $p_i$ is finite $\iff q_i$ is finite and,
in this case, \(|p_i| = |q_i|\). That such properties can be maintained is fairly easy to show. Now because \(\mathfrak{A} \in \text{Nr}_n \text{CA}_\omega \cap \text{CRCA}_n\), it suffices to show (since \(\mathfrak{B}\) is atomic) that \(\mathfrak{B}\) is in fact outside \(S_d \text{Nr}_n \text{CA}_{n+1} \cap \text{At}\). Take \(\kappa\) the signature of \(\mathcal{M}\); more specifically, the number of \(n\)-ary relation symbols to be \(2^{2^\omega}\), and assume for contradiction that \(\mathfrak{B} \in S_d \text{Nr}_n \text{CA}_{n+1} \cap \text{At}\). Then \(\mathfrak{B} \subseteq_d \text{Nr}_n \mathfrak{D}\), for some \(\mathfrak{D} \in \text{CA}_{n+1}\) and \(\text{Nr}_n \mathfrak{D}\) is atomic. For brevity, let \(\mathcal{C} = \text{Nr}_n \mathfrak{D}\). Then by item (1) of Lemma 5.3 \(\mathcal{R}_{Id} \mathfrak{B} \subseteq_d \mathcal{R}_{Id} \mathcal{C}\). Since \(\mathcal{C}\) is atomic, then by item (1) of the same Lemma \(\mathcal{R}_{Id} \mathcal{C}\) is also atomic. Using the same reasoning as above, we get that \(|\mathcal{R}_{Id} \mathcal{C}| > 2^\omega\) (since \(\mathcal{C} \in \text{Nr}_n \text{CA}_{n+1}\)) By the choice of \(\kappa\), we get that \(|\text{At} \mathcal{R}_{Id} \mathcal{C}| > \omega\). By density, we get from item (2) of Lemma 5.3, that \(\text{At} \mathcal{R}_{Id} \mathcal{C} \subseteq \text{At} \mathcal{R}_{Id} \mathcal{B}\). Hence \(|\text{At} \mathcal{R}_{Id} \mathcal{B}| \geq |\text{At} \mathcal{R}_{Id} \mathcal{C}| > \omega\). But by the construction of \(\mathfrak{B}\), \(|\mathcal{R}_{Id} \mathfrak{B}| = |\mathcal{R}_{Id} \mathcal{C}| = \omega\), which is a contradiction and we are done. Thus \(\mathfrak{B} \in \text{EL}(\text{Nr}_n \text{CA}_\omega \cap \text{CRCA}_n) \sim S_d \text{Nr}_n \text{CA}_\omega\). \(\square\)

**Example 5.6.** Assume that \(1 < n < \omega\). Let \(V = n^\omega\) and let \(\mathfrak{A} \in \text{Cs}_n\) have universe \(\varphi(V)\). Then \(\mathfrak{A} \in \text{Nr}_n \text{CA}_\omega\). Let \(y = \{s \in V: s_0 + 1 = \sum_{i > 0} s_i\}\) and \(\mathfrak{E} = g^3(\{y\} \cup X)\), where \(X = \{\{s\}: s \in V\}\). Now \(\mathfrak{E}\) and \(\mathfrak{A}\) having same top element \(\varphi\), share the same atom structure, namely, the singletons, so \(\text{CmAt} \mathfrak{E} = \mathfrak{A}\). Thus \(\text{At} \mathfrak{E} \in \text{At} \text{Nr}_n \text{CA}_\omega\) and \(\mathfrak{A} = \text{CmAt} \mathfrak{E} \in \text{Nr}_n \text{CA}_\omega\). Since \(\mathfrak{E} \subseteq_d \mathfrak{A}\), so \(\mathfrak{E} \in S_d \text{Nr}_n \text{CA}_\omega \subseteq S_c \text{Nr}_n \text{CA}_\omega\), but as proved in [37] \(\mathfrak{E} \notin \text{EL} \text{Nr}_m \text{CA}_{n+1} \subseteq \text{Nr}_m \text{CA}_{n+1} \supseteq \text{Nr}_n \text{CA}_\omega\).

Fix finite \(k > 2\). Then \(V_k = \text{Str} (\text{SNr}_n \text{CA}_{n+k})\) is not elementary \(\implies \) \(V_k\) is not-atom canonical. But the converse implication does not hold because (arguing contrapositively) in the case of atom-canonicity, we get that \(\text{Str} (\text{SNr}_n \text{CA}_{n+k}) = \text{At} (\text{SNr}_n \text{CA}_{n+k})\), and the last class is elementary, cf. [15, Theorem 2.84]. In particular, we do not know whether \(\text{Str} (\text{SNr}_n \text{CA}_{n+k})\), for a particular finite \(k \geq 3\), is elementary or not. Nevertheless, it is easy to show that there has to be a finite \(k = \omega\) such that \(V_j\) is not elementary for all \(j \geq k\):

**Theorem 5.7.** 1. There is a finite \(k \geq 2\), such that for all \(m \geq n + k\) the class of frames \(\text{Str} (\text{SNr}_n \text{CA}_m) = \{\mathfrak{F}: \text{Cm} \mathfrak{F} \in \text{SNr}_n \text{CA}_m\}\) is not elementary. An entirely analogous result holds for RA,

2. Let \(\mathcal{O} \in \{S_c, S_d, I\}\) and \(k \geq 3\). Then the class of frames \(K_k = \{\mathfrak{F}: \text{Cm} \mathfrak{F} \in \text{ONr}_n \text{CA}_{n+k}\}\) is not elementary.

**Proof.** (1): We show that \(\text{Str} (\text{SNr}_n \text{CA}_m)\) is not elementary for some finite \(m \geq n + 2\). By [3] \(m\) cannot be equal to \(n + 1\). Let \((\mathfrak{A}_i: i \in \omega)\) be a sequence of (strongly) representable \(\text{CA}_n\)s with \(\text{CmAt} \mathfrak{A}_i = \mathfrak{A}_i\) and \(\mathfrak{A} = \Pi_{i/\omega} \mathfrak{A}_i\) is not strongly representable with respect to any non-principal ultrafilter \(U\) on \(\omega\). Such algebras exist [17]. Hence \(\text{CmAt} \mathfrak{A} \notin \text{SNr}_n \text{CA}_\omega = \bigcap_{i/\omega} \text{SNr}_n \text{CA}_{n+i}\), so \(\text{CmAt} \mathfrak{A} \notin \text{SNr}_n \text{CA}_l\) for all \(l > m\), for some \(m \in \omega\), \(m \geq n + 2\). But for each such \(l\), \(\mathfrak{A}_i \in \text{SNr}_n \text{CA}_l(\subseteq \text{RC}_n)\), so \((\mathfrak{A}_i: i \in \omega)\) is a sequence of algebras such that \(\text{CmAt} (\mathfrak{A}_i) \in \text{SNr}_n \text{CA}_l\) \((i \in I)\), but \(\text{Cm}(\text{At}(\Pi_{i/\omega} \mathfrak{A}_i)) = \text{CmAt} (\mathfrak{A}) \notin \text{SNr}_n \text{CA}_l\) for all \(l \geq m\).
(2): We use the same construction (and notation) in the last item of Theorem ???. It suffices to show that the class of algebras $K_k = \{ \mathfrak{A} \in CA_n \cap \mathfrak{At}: \operatorname{EmAt} \mathfrak{A} \in ON_{r_n} CA_k \}$ is not elementary. $\exists$ has a winning strategy in $H_\omega (\alpha)$ for some countable atom structure $\alpha$, $\operatorname{Tm} \alpha \subseteq d \operatorname{Em} \alpha \in Nr_n CA_\omega$ and $\operatorname{Tm} \alpha \in CRCA_n$. Since $\mathcal{E}_{Z,N} \notin S_d Nr_n CA_{n+3}$, then $\mathcal{E}_{Z,N} = \operatorname{EmAt} \mathcal{E}_{Z,N} \notin K_k$, $\mathcal{E}_{Z,N} \equiv CRCA_\alpha$ and $\operatorname{Tm} \alpha \in K_k$ because $\operatorname{Em} \alpha \in Nr_n CA_\omega \subseteq S_d Nr_n CA_\omega \subseteq S_c Nr_n CA_\omega$. We have shown that $\mathcal{E}_{Z,N} \in \operatorname{El} K_k \sim K_k$, proving the required. □

To obtain the RA analogue of item (2) of Theorem 5.7, we need to strengthen [12, Theorem 39]. We prove more by allowing infinite conjunctions in constructing a certain model (denoted by $M_\alpha$) as clarified below. The $k$ rounded game $H_k (k \leq \omega)$ is defined for relation algebras in [12, Definition 28]. $\mathcal{G}_{\omega \beta}$ denotes the class of generalized weak set algebras of dimension $\beta$ in the sense of [11, Definition 3.1.2].

**Theorem 5.8.** Let $\alpha$ be a countable atom structure. If $\exists$ has a winning strategy in $H_\omega (\alpha)$, then there exists a complete $\mathcal{D} \in CRCA_\omega$ such that $\operatorname{Em} \alpha \cong Ra \mathcal{D}$ and $\alpha \cong At Ra \mathcal{D}$. In particular, $\operatorname{Em} \alpha \in Ra CA_\omega$, $\alpha \in At Ra CA_\omega$ and $\alpha$ is completely representable.

**Proof.** Fix some $a \in \alpha$. As shown in [12], the game $H_\omega$ is designed so that using $\exists$'s winning strategy in the game $H_\omega (\alpha)$ one can define a nested sequence $M_0 \subseteq M_1, \ldots$ of $\lambda$-neat networks where $M_0$ is $\exists$'s response to the initial $\forall$-move $a$ such that: If $M_r$ is in the sequence and $M_r (x,y) \leq a; b$ for an atoms $a$ and $b$ then there is $s \geq r$ and a witness $z \in \operatorname{nodes} (M_s)$ such that $M_s (x,z) = a$ and $M_s (z,y) = b$. In addition, if $M_r$ is in the sequence and $\theta$ is any partial isomorphism of $M_r$, then there is $s \geq r$ and a partial isomorphism $\theta^+$ of $M_s$ extending $\theta$ such that $\operatorname{rng} (\theta^+) \supseteq \operatorname{nodes} (M_r)$. Now let $M_a$ be the limit of this sequence as defined in [12]. Let $L$ be the signature with one binary relation for each $b \in \alpha$, and one $k$-ary predicate symbol for each $k$-ary hyperlabel $\lambda$. We work in $L_{\infty, \omega}$. For fixed $f_a \in \omega \operatorname{nodes} (M_a)$, let $\mathcal{U}_a = \{ f \in \omega \operatorname{nodes} (M_a): \{ i < \omega: g(i) \neq f_a (i) \} \text{ is finite} \}$. One makes $\mathcal{U}_a$ into the base of an $L$ relativized structure $M_a$ like in [12, Theorem 29] except that we allow a clause for infinitary disjunctions. We are now working with (weak) set algebras whose semantics are induced by $L_{\infty, \omega}$ formulas in the signature $L$, instead of first order ones. For any such $L$-formula $\phi$, write $\phi^{M_a}$ for $\{ f \in \mathcal{U}_a: M_a, f \models \phi \}$. Let $D_a = \{ \phi^{M_a}: \phi \text{ is an } L\text{-formula} \}$ and $\mathcal{D}_a$ be the weak set algebra with universe $D_a$. Let $\mathcal{D} = \coprod_{a \in \alpha} \mathcal{D}_a$. Then $\mathcal{D} \in \mathcal{G}_{\omega \beta}$ and furthermore $\mathcal{D}$ is complete. Suprema exists in $\mathcal{D}$ because we chose to work with $L_{\infty, \omega}$ while forming the dilations $\mathcal{D}_a (a \in \alpha)$. Each $\mathcal{D}_a$ is complete, hence so is their product $\mathcal{D}$. Now $\alpha \cong At Ra \mathcal{D}$ as proved in [12, Theorem 39]. We show that $\operatorname{Em} \alpha \cong Ra \mathcal{D}$. Since $\mathcal{D}$ is complete, then $Ra \mathcal{D}$ is complete. Making the legitamite identification $Ra \mathcal{D} \subseteq d \operatorname{Em} \alpha$, by density we get that $Ra \mathcal{D} \cong \operatorname{Em} \alpha$ because $Ra \mathcal{D}$ is complete, so $\operatorname{Em} \alpha \in Ra CA_\omega$. Now $\operatorname{Em} \alpha \in S_d Ra CA_\omega (\subseteq S_c Ra CA_\omega)$ and $\alpha$ is countable, so $\alpha$ is completely representable. □
COROLLARY 5.9. Let $\mathcal{O} \in \{S_c, S_d, I\}$ and $m \geq 5$. Then the class of frames $L_m = \{\mathfrak{F} : \mathfrak{F} \in \mathcal{O}RaCA_m\}$ is not elementary. Furthermore, any class $K$ such that $S_dRaCA_\omega \cap CRRA \subseteq K \subseteq S_cRaCA_6$, $K$ is not elementary.

PROOF. Let $L_m = \{\mathcal{A} \in RA_n \cap \text{At} : \mathcal{CmAt}\mathcal{A} \in \mathcal{O}RaCA_m\}$. Take the relation algebra atom structure $\beta$ based on $\mathbb{N}$ and $\mathbb{Z}$ as defined in [12], for which $\exists$ has a winning strategy in $H_k(\text{At}\mathcal{A}_{\mathbb{Z},n})$, where $\mathcal{A}_{\mathbb{Z},n} = \mathcal{Cm}\beta$ for all $k < \omega$, and $\forall$ has a winning strategy in $F^5(\beta)$ with $F^5$ as in [12, Definition 28]. By [12, Lemma 2.6], we get that $\mathcal{A}_{\mathbb{Z},n} \notin S_cRaCA_5$. The usual argument of taking ultrapowers followed by an elementary chain argument, one gets a countable atom structure $\alpha$, such that $\mathcal{Tm}\alpha \equiv \mathcal{A}_{\mathbb{Z},n}$ and $\exists$ has a winning strategy in $H(\alpha)$. By Theorem 5.8, there is $\mathcal{D} \in CA_\omega$, such that $\alpha \equiv \beta$, $\alpha \equiv \text{At}\mathcal{D}$ and $\mathcal{Tm}\alpha \equiv \text{At}\mathcal{D}$, so that $\mathcal{Tm}\alpha \in S_dRaCA_\omega$. Since $\mathcal{A}_{\mathbb{Z},n} \notin S_cRaCA_5$, then $\mathcal{A}_{\mathbb{Z},n} = \mathcal{CmAt}\mathcal{C}_{\mathbb{Z},n} \notin L_m$, $\mathcal{A}_{\mathbb{Z},n} \equiv \mathcal{Tm}\alpha$ and $\mathcal{Tm}\alpha \in L_m$ because $\mathcal{Tm}\alpha \in RaCA_\omega$. For the second required one uses the same elementary equivalent algebras $\mathcal{Tm}\alpha \in S_dRaCA_\omega$ and $\mathcal{A}_{\mathbb{Z},n} \notin S_cRaCA_5$. \hfill $\Box$

In the next table we summarize the results hitherto obtained on first order definability:

<table>
<thead>
<tr>
<th>Algebras</th>
<th>Elementary</th>
<th>Citation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_dNr_nCA_\omega \cap CRCA_n \subseteq K \subseteq S_dNr_nCA_{n+1}$</td>
<td>no</td>
<td>Theorem 5.5</td>
</tr>
<tr>
<td>$S_dNr_nCA_\omega \cap CRCA_n \subseteq K \subseteq S_cNr_nCA_{n+3}$</td>
<td>no</td>
<td>Theorem 5.4</td>
</tr>
<tr>
<td>$\text{At}(Nr_nCA_\omega \cap CRCA_n) \subseteq K \subseteq \text{At}S_cNr_nCA_{n+3}$</td>
<td>no</td>
<td>From result in previous row</td>
</tr>
<tr>
<td>$Nr_nCA_\omega \subseteq K \subseteq Nr_nCA_{n+1}$</td>
<td>no</td>
<td>[34, Theorem 5.4.2]</td>
</tr>
<tr>
<td>$Nr_nCA_\omega \cap CRCA_n \subseteq K \subseteq S_dNr_nCA_{n+1}$</td>
<td>no</td>
<td>Theorem 5.5</td>
</tr>
<tr>
<td>$S_dRaCA_\omega \cap CRRA \subseteq K \subseteq S_cRaCA_6$</td>
<td>no</td>
<td>[13]</td>
</tr>
<tr>
<td>$S_dRaCA_\omega \cap CRRA \subseteq K \subseteq S_cRaCA_6$</td>
<td>no</td>
<td>Corollary 5.9</td>
</tr>
<tr>
<td>$\text{RaCA}_\omega \cap CRRA_n \subseteq K \subseteq S_cRaCA_6$</td>
<td>no</td>
<td>[12, 13, 32]</td>
</tr>
<tr>
<td>$\text{At}(RaCA_\omega \cap CRRA) \subseteq K \subseteq \text{At}S_cRaCA_6$</td>
<td>no</td>
<td>Corollary 5.9, cf.[12]</td>
</tr>
</tbody>
</table>

The result in row seven is stronger than the result proved in [13] because $S_d \leq S_c$. Neither of the results in the second and third row is superfluous for the two classes $S_cNr_nCA_{n+3}$ and $S_dNr_nCA_{n+1}$ are mutually distinct: The algebra $\mathfrak{B}$ in the last part of Theorem 5.4 is completely representable, so $\mathfrak{B} \in S_cNr_nCA_{n+3} \sim S_dNr_nCA_{n+1}$. Conversely, in [18, §2], a finite algebra $\mathfrak{A} \in Nr_nCA_{n+1} \sim SNr_nCA_{n+2}(\subseteq S_dNr_nCA_{n+1} \sim S_cNr_nCA_{n+3})$ is constructed. From the no’s in the sixth and last row we cannot answer (negatively) the only question mark in the table. This would be a far too hasty decision (taken mistakenly in [12] and corrected in [13]). The implication $\text{At}\mathfrak{A} \in \text{At}RaCA_\omega$ and $\mathcal{CmAt}\mathfrak{A} \in RaCA_\omega \implies \mathfrak{A} \in RaCA_\omega$ may not be
valid. Indeed, the algebra \( \mathcal{B} \) used in example 5.6 confirms our doubts, since \( \text{At}\mathcal{B} \in \text{At}\text{Nr}_n\mathcal{C}_A \), \( \text{CmAt}\mathcal{B} \in \text{Nr}_n\mathcal{C}_A \), but \( \mathcal{B} \notin \text{Nr}_n\mathcal{C}_A \).

Unless otherwise explicitly indicated, fix \( 2 < n < \omega \): The last example motivates:

**Definition 5.10.** The class \( \mathcal{K} \) is **gripped by its atom structures** or simply **gripped**, if for \( \mathfrak{A} \in \mathcal{C}_A \), whenever \( \text{At}\mathfrak{A} \in \mathcal{K} \), then \( \mathfrak{A} \in \mathcal{K} \). An \( \omega \)-rounded game \( \mathcal{H} \) **grips** \( \mathcal{K} \), if whenever \( \mathfrak{A} \in \mathcal{C}_A \) is atomic with countably many atoms and \( \exists \) has a winning strategy in \( \mathcal{H}(\mathfrak{A}) \), then \( \mathfrak{A} \in \mathcal{K} \). The game \( \mathcal{H} \) **weakly grips** \( \mathcal{K} \), if whenever \( \mathfrak{A} \in \mathcal{C}_A \) is atomic with countably many atoms and \( \exists \) has a winning strategy in \( \mathcal{H}(\text{At}\mathfrak{A}) \), then \( \text{At}\mathfrak{A} \in \mathcal{K} \). The game \( \mathcal{H} \) **densely grips** \( \mathcal{K} \), if whenever \( \mathfrak{A} \in \mathcal{C}_A \) is atomic with countably many atoms and \( \exists \) has a winning strategy in \( \mathcal{H}(\text{At}\mathfrak{A}) \), then \( \text{At}\mathfrak{A} \in \mathcal{K} \) and \( \text{CmAt}\mathfrak{A} \in \mathcal{K} \).

**Example 5.11.** The classes \( \text{RCA}_n \) and \( \text{Nr}_n\mathcal{C}_A \) are not gripped, by [19] and example 5.6. In [19] a weakly representable \( \mathcal{C}_A \) atom structure that is not strongly representable is constructed, showing that \( \text{RCA}_n \) is not closed under Dedekind-MacNeille completions, because \( \text{CmAt} \) is the Dedekind-MacNeille completion of \( \text{CmAt} \) in the case of completely additive varieties of BAOs. But we can go even further: For \( m \geq t(n) \), with \( t(n) = n(n + 1)/2 + 1 \) the variety \( \text{SNr}_n\mathcal{C}_A \) is not gripped by Theorem 3.10. On the other hand, the class \( \text{Sr}_n\mathcal{C}_A \) is gripped. For any \( n < \omega \), the class \( \text{CRCA}_n \), and its elementary closure, namely, the class of algebras satisfying the Lyndon conditions as defined in [17] is gripped. The usual atomic game \( \mathcal{G} \) weakly grips, densely grips and grips \( \text{RCA}_n \).

**Theorem 5.12.** The game \( \mathcal{H}_\omega \) densely grips \( \text{Nr}_n\mathcal{C}_A \). but does not grip \( \text{Nr}_n\mathcal{C}_A \).

### 5.8 correct?

**Proof.** The first part follows from Theorem 5.8. For the second more tricky part. Take \( \mathcal{E} \in \mathcal{C}_s_n \) to be the algebra in example 5.6. We know that \( \text{At}\mathcal{E} \in \text{At}\text{Nr}_n\mathcal{C}_A \) and \( \text{CmAt}\mathcal{E} \in \text{Nr}_n\mathcal{C}_A \) but \( \mathcal{E} \notin \text{Nr}_n\mathcal{C}_{A+1} \). We show that \( \exists \) has a winning strategy in \( \mathcal{H}_\omega(\text{At}\mathcal{E}) \). First a piece of notation. Let \( m \) be a finite ordinal \( > 0 \). An \( s \) word is a finite string of substitutions \( (s^j_i) \) \( (i, j < m) \), a \( c \) word is a finite string of cylindrifications \( (c_i) \), \( i < m \); an \( sc \) word \( w \), is a finite string of both, namely, of substitutions and cylindrifications. An \( sc \) word induces a partial map \( \tilde{w} : m \to m : \hat{c} = \text{Id}, \tilde{w}^j_i = \tilde{w} \circ [i,j] \) and \( \tilde{wc}_i = \tilde{w} \upharpoonright (m \setminus \{i\}) \). If \( \tilde{a} \in m^{-1}m \), we write \( \tilde{s}_a \), or \( s_{a_0...a_{k-1}} \), where \( k = |\tilde{a}| \), for an arbitrary chosen \( sc \) word \( w \) such that \( \tilde{w} = \tilde{a} \). Such a \( w \) exists by [15, Definition 5.23 Lemma 13.29].

Fix \( 2 < n < m \). Assume that \( \mathcal{E} \in \mathcal{C}_A \), \( \mathfrak{A} \subseteq \mathcal{E} \) \( \text{Nr}_n\mathcal{E} \) is an atomic \( \mathcal{C}_A \) and \( N \) is an \( \mathfrak{A} \)-network with nodes\( (N) \subseteq m \). Define \( N^+ \in \mathcal{E} \) by

\[
N^+ = \prod_{i_0,...,i_{n-1} \in \text{nodes}(N)} s_{i_0,...,i_{n-1}} N(i_0, \ldots, i_{n-1}).
\]
For a network \( N \) and function \( \theta \), the network \( N\theta \) is the complete labelled graph with nodes \( \theta^{-1}(\text{nodes}(N)) = \{ x \in \text{dom}(\theta): \theta(x) \in \text{nodes}(N) \} \), and labelling defined by

\[
(N\theta)(i_0, \ldots, i_{n-1}) = N(\theta(i_0), \theta(i_1), \ldots, \theta(i_{n-1})),
\]

for \( i_0, \ldots, i_{n-1} \in \theta^{-1}(\text{nodes}(N)) \). Then the following hold:

(1): for all \( x \in \mathcal{C} \setminus \{0\} \) and all \( i_0, \ldots, i_{n-1} < m \), there is \( a \in \text{At} \mathfrak{A} \), such that \( s_{i_0,\ldots,i_{n-1}}a \cdot x \neq 0 \),

(2): for any \( x \in \mathcal{C} \setminus \{0\} \) and any finite set \( I \subseteq m \), there is a network \( N \) such that \( \text{nodes}(N) = I \) and \( x \cdot N^+ \neq 0 \). Furthermore, for any networks \( M, N \) if \( M^+ \cdot N^+ \neq 0 \), then \( M \upharpoonright \text{nodes}(M) \cap \text{nodes}(N) = N \upharpoonright \text{nodes}(M) \cap \text{nodes}(N) \).

(3): if \( \theta \) is any partial, finite map \( m \rightarrow m \) and if \( \text{nodes}(N) \) is a proper subset of \( m \), then \( N^+ \neq 0 \Rightarrow (N\theta)^+ \neq 0 \). If \( i \notin \text{nodes}(N) \), then \( c_iN^+ = N^+ \).

\( \exists \)’s strategy dealing with \( \lambda \)-neat hypernetworks, where \( \lambda \) is a constant label kept on short hyperedges is exactly like in the proof of Theorem 5.8. The rest of her winning strategy is to play \( \lambda \)-neat hypernetworks \( (N^a, N^h) \) with \( \text{nodes}(N_a) \subseteq \omega \) such that \( (N^a)^+ \neq 0 \) (recall that \( (N^a)^+ \) is as defined in the proof of lemma 3.7). In the initial round, let \( \forall \) play \( a \in \text{At} \). \( \exists \) plays a network \( N \) with \( N^a(0, 1, \ldots, n - 1) = a \). Then \( (N^a)^+ = a \neq 0 \). The response to the cylindrifier move is exactly like in the first part of Lemma 3.7 because \( \mathcal{E} \) is completely representable so \( \mathcal{E} \in S_{\text{cN}n\text{CA}} \) by [34, Theorem 5.3.6]. Since \( \mathfrak{A} \subseteq \mathfrak{A} \mathfrak{N}_n \mathcal{C} \), then \( \sum \mathcal{E} \text{At} \mathfrak{A} = 1 \). For (1), \( s^i_j \) is a completely additive operator (any \( i, j < m \), hence \( s_{i_0,\ldots,i_{n-1}} \) is, too. So \( \sum \mathcal{E} \{s_{i_0,\ldots,i_{n-1}}a: a \in \text{At} \mathfrak{A}\} = s_{i_0,\ldots,i_{n-1}} \sum \mathcal{E} \text{At} \mathfrak{A} = s_{i_0,\ldots,i_{n-1}}1 = 1 \) for any \( i_0, \ldots, i_{n-1} < m \). Let \( x \in \mathcal{C} \setminus \{0\} \). Assume for contradiction that \( s_{i_0,\ldots,i_{n-1}}a \cdot x = 0 \) for all \( a \in \text{At} \mathfrak{A} \). Then \( 1 - x \) will be an upper bound for \( \{s_{i_0,\ldots,i_{n-1}}a: a \in \text{At} \mathfrak{A}\} \). But this is impossible because \( \sum \mathcal{E} \{s_{i_0,\ldots,i_{n-1}}a: a \in \text{At} \mathfrak{A}\} = 1 \).

To prove the first part of (2), we repeatedly use (1). We define the edge labelling of \( N \) one edge at a time. Initially, no hyperedges are labelled. Suppose \( E \subseteq \text{nodes}(N) \times \text{nodes}(N) \ldots \times \text{nodes}(N) \) is the set of labelled hyperedges of \( N \) (initially \( E = \emptyset \)) and \( x \cdot \prod_{\bar{e} \in E} s_{\bar{e}}N(\bar{e}) \neq 0 \). Pick \( \bar{d} \) such that \( \bar{d} \notin E \). Then by (1) there is \( a \in \text{At} \mathfrak{A} \) such that \( x \cdot \prod_{\bar{e} \in E} s_{\bar{e}}N(\bar{e}) \cdot s_{\bar{d}}a \neq 0 \). Include the hyperedge \( \bar{d} \) in \( E \). We keep on doing this until eventually all hyperedges will be labelled, so we obtain a completely labelled graph \( N \) with \( N^+ \neq 0 \), it is easily checked that \( N \) is a network.

For the second part of (2), we proceed contrapositively. Assume that there is \( \bar{e} \in \text{nodes}(M) \cap \text{nodes}(N) \) such that \( M(\bar{e}) \neq N(\bar{e}) \). Since edges are labelled by atoms, we have \( M(\bar{e}) \cdot N(\bar{e}) = 0 \), so \( 0 = s_{\bar{e}}0 = s_{\bar{e}}M(\bar{e}) \cdot s_{\bar{e}}N(\bar{e}) \geq M^+ \cdot N^+ \). A piece of notation. For \( i < m \), let \( I_{d-i} \) be the partial map \( \{(k, k): k \in m \setminus \{i\}\} \). For the first part of (3) (cf. [15, Lemma 13.29] using the notation in op.cit), since there is \( k \in m \setminus \text{nodes}(N) \), \( \theta \) can be expressed as a product \( \sigma_0\sigma_1 \ldots \sigma_t \) of maps such that, for \( s \leq t \), we have ei-
ther $\sigma_s = Id_{-i}$ for some $i < m$ or $\sigma_s = [i/j]$ for some $i, j < m$ and where $i \not\in \text{nodes}(N_{\sigma_0} \ldots \sigma_{s-1})$. But clearly $(N_{Id})^+ \geq N^+$ and if $i \not\in \text{nodes}(N)$ and $j \in \text{nodes}(N)$, then $N^+ \neq 0 \rightarrow (N[i/j])^+ \neq 0$. The required now follows.

The last part is straightforward.

Using the above proven facts, we are now ready to show that $\exists$ has a winning strategy in $G^m$. She can always play a network $N$ with $\text{nodes}(N) \subseteq m$, such that $N^+ \neq 0$.

In the initial round, let $\forall$ play $a \in \text{At}A$. $\exists$ plays a network $N$ with $N(0, \ldots, n-1) = a$. Then $N^+ = a \neq 0$. Recall that here $\forall$ is offered only one (cylindrifier) move. At a later stage, suppose $\forall$ plays the cylindrifier move, which we denote by $(N, f_i \in \text{nodes}(N)$, $l < n$. Then for all $N$ the cylindrifier move, which we denote by $(N, G)$ winning strategy in $\gamma$. Hence $\forall$ plays a network $N$ such that $c_i N^+ \cdot s_b \neq 0$ and so by first part of (2), there is a network $M$ such that $M^+ \cdot c_i N^+ \cdot s_b \neq 0$. Hence $M(f_0, \ldots, f_i - 1, k, f_i + 1, \ldots, f_n - 2) = b$, $\text{nodes}(M) = \text{nodes}(N) \cup \{k\}$, and $M^+ \neq 0$, so this property is maintained.

For transformation moves: if $\forall$ plays $(M, \theta)$, then it is easy to see that we have $(M^\theta \theta)^+ \neq 0$, so this response is maintained in the next round. For each $J \subseteq \omega$, $|J| = n$ say, for $\mathfrak{A} \in \text{CA}_\omega$, let $\text{Nr}_J \mathfrak{A} = \{x \in \mathfrak{A}: g \in \mathfrak{A}, \forall \in \omega \}$. We have $\text{At} \mathfrak{E} \in \text{At} \mathfrak{N}_J \text{CA}_\omega$, so assume that $\text{At} \mathfrak{E} = \text{At} \mathfrak{N}_J \mathfrak{F}$ with $\mathfrak{F} \in \text{CA}_\omega$. Then for all $y \in \text{Nr}_J \mathfrak{F}$, where $J = \{i_0, i_1, \ldots, i_{n-1}\}, i_0, \ldots, i_{n-1} \in \omega$, the following holds for $a \in \text{At} \mathfrak{E}$: $s_{i_0 i_1 \ldots i_{n-1}} \cdot y = 0 \Rightarrow s_{i_0 i_1 \ldots i_{n-1}} \cdot a \leq y$. Now we are ready to describe $\exists$'s strategy in response to amalgamation moves. For better readability, we write $\bar{i}$ for $\{i_0, i_1, \ldots, i_{n-1}\}$, if it occurs as a set, and we write $s_i$ short for $s_{i_0 i_1 \ldots i_{n-1}}$. Also we only deal with the network part of the game. Now suppose that $\forall$ plays the amalgamation move $(M, N)$ where $\text{nodes}(M) \cap \text{nodes}(N) = \{\bar{i}\}$, then $M(\bar{i}) = N(\bar{i})$. Let $\mu = \text{nodes}(M) \sim \bar{i}$ and $\nu = \text{nodes}(N) \sim \bar{i}$. Then $c_{\bar{i}} M^+ = M^+$ and $c_{\bar{i}} N^+ = M^+$. Hence using (*), we have; $c_{\bar{i}} M^+ = s_i M(\bar{i}) = s_i N(\bar{i}) = c_{\bar{i}} N^+$ so $c_{\bar{i}} M^+ = M^+ \leq c_{\bar{i}} N^+ = c_{\bar{i}} N^+ \cdot N^+ = x \neq 0$. So there is $L$ with $\text{nodes}(L) = \text{nodes}(M) \cup \text{nodes}(N)$, and $L^+ \cdot x \neq 0$, thus $L^+ \cdot M^+ \neq 0$ and consequently $L \upharpoonright_{\text{nodes}(M)} = M \upharpoonright_{\text{nodes}(M)}$, hence $M \subseteq L$ and similarly $N \subseteq L$, so that $L$ is the required amalgam.

**Example 5.13.** Fix $2 < n < \omega$. The game $H$ weakly and densely grips $\text{Nr}_n \text{CA}_\omega$ but $H$ does not grip $\text{Nr}_n \text{CA}_\omega$. We devise an $\omega$-rounded non-atomic game $G$ gripping $\text{Nr}_n \text{CA}_\omega$. By non-atomic, we mean that arbitrary elements of the algebra not necessarily atoms are allowed during the play. By the example in item 1 of Theorem ??, $G$ is strictly stronger than $H$. That is to say, $\exists$ has a winning strategy in $G \implies \exists$ has a winning strategy in $H$, but the converse implication is false as, using the notation in op.cit, $\exists$ has a winning strategy in $H(\text{At} \mathfrak{E})$ but does not have a winning strategy in $G(\mathfrak{E})$. The game $G$ is played on both $\lambda$-neat hypernetworks as defined for
H, and complete labelled graphs (possibly by non-atoms) with no consistency conditions. The play at a certain point, like in H, will be a λ-neat hypernetwork, call its network part X, and we write X(x) for the atom the edge x. By network part we mean forgetting k-hypedges getting non-atomic labels. An n-matrix is a finite complete graph with nodes including 0, . . . , n − 1 with all edges labelled by arbitrary elements of B. No consistency properties are assumed. ∀ can play an arbitrary n-matrix N, ∃ must replace N(0, . . . , n − 1), by some element a ∈ B; this is a non-atomic move. The final move is that ∀ can pick a previously played n-matrix N, and pick any tuple x = (x0, . . . , xn−1) whose atomic label is below N(0, . . . , n − 1). ∃ must respond by extending X to X′ such that there is an embedding θ of N into X′ such that θ(0) = x0 . . . , θ(n − 1) = xn−1 and for all i0, . . . , i n−1 ∈ N, we have

\[ X(\theta(i_0), . . . , \theta(i_{n-1})) \leq N(i_0, . . . , i_{n-1}). \]

This ensures that in the limit, the constraints in N really define the element a. Assume that B ∈ RCAω is atomic and has countably many atoms. If ∃ has a winning strategy in G(B), then the extra move involving non-atoms labelling matrices, ensures that that every n-dimensional element generated by B in a dilation D ∈ RCAω having base M, constructed from a winning strategy in G as the limit of the λ-neat hypernetworks played during the game (and further assuming without loss that ∀ plays every possible move) is already an element of B. For k < ω, let Gk be the game G truncated to k rounds, and let Gkra and Gkra be the relation algebra analogue of the game obtained by adding the non-atomic move replacing n-matrices by 2-matrices, to the game H as defined for relation algebras in [12, Definition 28].

Using the argument in the proof of the Theorem 5.8 replacing H. If there exists a countable atom structure α such that ∃ has a winning strategy in Gk(α) for all k ∈ ω and ∀ has a winning strategy in Fm, then any class K, such that NrαCAω ⊆ K ⊆ ScNrnCAm, is not elementary. We have already proved the last result. The relation algebra case is more interesting. Undefined notation can be found in [12]; detailed citation is given in the proof.

**Theorem 5.14.** Assume that 2 < m < ω. If there exists a countable atom structure α such that ∃ has a winning strategy in Gkra(α) for all k ∈ ω and ∀ has a winning strategy in Fm, then any class K, such that RaCAω ⊆ K ⊆ Sc RaCA5, K is not elementary.

**Proof.** The analogous result can be obtained for relation algebras for 2 < m < ω obtained by replacing Nr by Ra. One uses the arguments in [12, Theorem 39, 45], but resorting to the game Gkra in place of Hk (k < ω), as defined for relation algebras in [12, Definition 28]. Now by assumption we have a countable relation algebra atom structure α, for which ∃ has a winning strategy in Gkra(α), for all k < ω, and ∀ has a winning strategy in Fm(α) with Fm as defined in [12, Definition 28]. By the RA analogue of
lemma 3.7 proved in [12, Theorem 33], we get that \( Cm \alpha \not\in S_cRaCA_m \). The usual argument of taking an ultrapower of \( Cm \alpha \), followed by a downward elementary chain argument, one gets a countable \( \mathcal{B} \in RA \), such that \( \mathcal{B} \equiv Cm \alpha \) and \( \exists \) has a winning strategy in \( G^{ra}(\mathcal{B}) \), so \( \mathcal{B} \in RaCA_\omega \) because \( G^{ra} \) grips \( RaCA_\omega \). Hence for any \( K \), such that \( RaCA_\omega \subseteq K \subseteq S_cRaCA_5 \), we have \( Cm \alpha \not\in K \), \( \mathcal{B} \in K \) and \( Cm \alpha \equiv \mathcal{B} \). \( \square \)

To obtain the required result (generalizing Theorem 5.4 to any class of algebras between \( Sc \) and \( QEA \) (to be formulated in a while as Theorem 5.15) the following changes are needed:

- modifying the game \( H \) to the \( QEA \) case, call the new game \( H^q \). Here one has to modify only the ‘network part’ of \( \lambda \)-neat hypernetworks by adding a symmetry condition \( ((s_{i,j}]N(\bar{x}) = N(\bar{x} \circ [i,j])) \),
- working with \( C_{Z,N} \) now expanded with the unary operations \( s_{[i,j]} \) \( (i < j < n) \), call the resulting rainbow algebra \( C^q_{Z,N} \in RQEA_n \),
- generalizing winning strategy’s of \( \forall \) given in Theorem 5.4 in \( H_\omega(AtC_{Z,N}) \) to the game \( H^q_k(AtRd_{sc}C^q_{Z,N}) \) and \( \forall \)’s winning strategy in \( H^q_k(AtC^q_{Z,N}) \) for each \( k < \omega \); the last is given in the proof of the same theorem. Then one uses the (easy) modification of Lemma 5.8 to the \( QEA_n \) case (in the modification, for example in defining the weak model \( M_a \) \( (a \in \alpha) \) one adds a clause to formulas for satisfiability of the unary connectives \( s_{[i,j]} \) \( (i < j < n) \) interpreted as swapping the \( i \)th and \( j \)th variables). Using lemma 3.7, one shows that \( Rd_{sc}C^q_{Z,N} \not\in S_cNr_nSc_{n+3} \).

Exactly the CA case, we get that any class \( K \) between \( Sc \) nd \( QEA \), and any class between the two classes \( S_dNr_nK_\omega \cap CRK_n \) and \( S_cNr_nK_{n+3} \), is not elementary,

- to obtain the analogue of Theorem 5.5 like was done in the CA case, one adjoins the (splitting) construction in [30] for classes between \( Sc \) and \( QEA \) in place of the modification of the construction in [34, Theorem 5.1.4] (as appeared [28]) addressing only the CA case. The construction in [37] shows that this last item is necessary. In opcit, an atomic \( \mathfrak{C} \in RQEA_n \) such that \( CmAt\mathfrak{C} \in Nr_nQA_\omega \) and \( At\mathfrak{C} \in AtNr_nQEA_\omega \), but \( Rd_{sc}\mathfrak{A} \not\in Nr_nSc_{n+1}(\supseteq Nr_nSc_\omega) \), is constructed. In fact, this \( \mathfrak{C} \) is simple (has no proper ideals), so that it is a set algebra of dimension \( n \) which is the \( QEA_n \) generated by the same generators of \( \mathfrak{E} \) in example 5.6, but now in the full quasipolyadic equality set algebra with top element \( n^Q \). Like in Theorem 5.4, it can be shown that \( \exists \) has a winning strategy in \( H^q_\omega(At\mathfrak{C}) \).

So for diagonal free reducts of \( QEA \), namely, \( Sc \) and \( QA \), we obtain the weaker result which we formulate only for \( Scs \). The Theorem is true for any class between \( Sc \) and \( QA \). Recall that \( CRSc_n \) denotes the class of completely representable \( Sc_n \)s.
THEOREM 5.15. Any class between $\text{CRSc}_n \cap \text{SdNr}_n \text{Sc}_\omega$ and $\text{Sc}_n \text{Nr}_n \text{Sc}_n^{ad}$ and any class between $\text{Nr}_n \text{Sc}_\omega$ and $\text{SdNr}_n \text{Sc}_{n+1}$ are not elementary.

Complete additivity appears on the left hand side (giving a smaller class than the class $\text{Sc}_n \text{Nr}_n \text{Sc}_n^{ad+3}$) due to the intrusion of lemma 3.7. A discrepancy that deserves to be highlighted here is that in the case of non-atom canonicity, proved in Theorem 5.12, though the same lemma 3.7 was also used, additivity did not interfere at all for diagonal free reducts (like $\text{Sc}_s$) because algebras, more specifically dilations of algebras used, were finite.

COROLLARY 5.16. Let $2 < n < \omega$, $K$ any class between $\text{Sc}$ and $\text{QEA}$, $m > n$, and $k \geq 3$. Then the following classes are not elementary: $\text{CRK}_n$, cf. [14, 21], $\text{Nr}_n K_m$, $\text{Nr}_2 K_m$, cf. [28–31] $\text{SdNr}_n K_{n+k}$ and $\text{Sc}_n \text{Nr}_n K_{n+k}$.

COROLLARY 5.17. Let $k \geq 5$. Then the classes $\text{CRRA}$, $\text{RaCA}_k$, $\text{SdRaCA}_k$ and $\text{ScRaCA}_k$ are not elementary, cf. [12, 14, 32]. The first two classes are not closed under $\equiv_\omega$, but are closed under ultraproducts. Furthermore, $\text{RaCA}_\omega \subseteq \text{SdRaCA}_\omega \subsetneq \text{ScRaCA}_\omega$.

PROOF. The first two classes are closed under ultraproducts because they are psuedo-elementary (reducts of elementary classes), cf. [15, Item (2), p. 279], [12, Theorem 21]. Proving the strictness of the last inclusion can be easily distilled from the proof of [12, Theorem 36]. □

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