Logical Equivalence

<u>Definition 5.1:</u> A compound statement is called a tautology if it is true for all truth value assignments for its component statements. If a compound statement is false for all such assignments, then it is called a contradiction.

Example 5.1: $\boldsymbol{P} \vee \neg \boldsymbol{P}$ is a tautology since:

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

Example 5.2: $P \wedge \neg P$ is a contradiction since:

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

Example5.3: Show that $(P \wedge Q) \rightarrow P$ is tautology.

Call "$P \wedge Q$" A and "$(P \wedge Q) \rightarrow P$" B

P	Q	A	B
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Definition 5.2 : Two statements s_1, s_2 are said to be logically equivalent and we write $s_1 \Leftrightarrow s_2$ when the statement s_1 is true (respectively, false) if and only if the statement s_2 is true (respectively, false).

Remark 5.1: When $s_1 \Leftrightarrow s_2$ the statements s_1 and s_2 provide the same truth tables because s_1, s_2 have the same truth values for all choices of truth values for their primitive components. This means that $s_1 \leftrightarrow s_2$ is a tautology.

Example5.4: show that $P \to Q \ \Leftrightarrow \neg P \vee Q$.

Solution: write A for $P \to Q$ and B for $\neg P \vee Q$, we get:

P	Q	A	$\neg P$	B
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

This means that the statements A, B have the same truth values for all truth values of their components. Hence, they are logically equivalent.

Another solution: we can solve this problem by proving that the Biconditional $(P \to Q) \leftrightarrow (\neg P \vee Q)$ is a tautology.

Therefore: write C for

$$"(P \to Q) \leftrightarrow (\neg P \vee Q)",$$

We have:

P	Q	$\neg P$	A	B	C
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

As required.

Example5.5: If Q has the truth value T, determine all truth value assignments for the primitive statements $P, r, and\ s$ for which the truth value of the statement

(1) $(\boldsymbol{Q} \rightarrow [(\neg \boldsymbol{P} \vee \boldsymbol{r}) \wedge \neg \boldsymbol{s}]) \wedge [\neg \boldsymbol{s} \rightarrow (\neg \boldsymbol{r} \wedge \boldsymbol{Q})]$ is T.

Solution: Since (1) is true, it follows (from the properties of the" and" statement) that :

(2) $Q \rightarrow [((\neg P \vee r) \wedge \neg s)]$ is true, and

(3) $\neg s \rightarrow (\neg r \wedge Q)$ is true.

Since Q is true, therefore from (2) we must conclude that: $(\neg P \vee r) \wedge \neg s$ is also true. Hence,

(4) $\neg P \vee r$ is true .

(5) $\neg s$ is true .

From (3) and (5),one gets:

(6) $\neg r \wedge Q$ is true.

Since Q is assumed to be true, it follows that:

(7) $\neg r$ is true. Hence, r is false. Hence, from (4), we get:

(8) $\neg P$ is true. This means that P is false.

Finally, from (5), We conclude that s is false.

Homework 5.1: Solve the example if Q has the truth value F.

Definition 5.3 : Let s be a statement. If s contains no logical connectives other than $\neg$, $\wedge$, and $\vee$, then the dual of s, denoted by s^d, is the statement obtained from s by replacing each of the occurrence of ∧ and ∨ by ∨ and ∧ ,respectively, and each occurrence of T and F by F and T, respectively.

Example 5.6: given the primitive statements P, Q, r and the compound statement: s : $(P \wedge \neg Q) \vee (r \wedge T)$, we find that the dual of s is:

$$s^d: (P \vee \neg Q) \wedge (r \vee F) .$$

Remark: $\neg Q$ is unchaned as we go from s to s^d.

Theorem 5.1 (The Principal of Duality) : Let A and B be statements as described in definition 5.3. If $A \Leftrightarrow B, then\ A^d \Leftrightarrow B^d$.

The Laws of Logic

For any primitive statements P, Q, r, any tautology T and any contradiction F,

(1) $P \vee \neg P \Leftrightarrow T$.

(2) $P \wedge \neg P \Leftrightarrow F$.

(3) $\neg\neg P \Leftrightarrow P$. **Law of double negation.**

(4) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$. **DeMorgan's first Law .**

(5) $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$. **DeMorgan's second Law.**

(6) $P \vee Q \Leftrightarrow Q \vee P$. **Commutative Law for disjunction.**

(7) $P \wedge Q \Leftrightarrow Q \wedge P$. **Commutative Law for conjunction.**

(8) $P \vee (Q \vee r) \Leftrightarrow (P \vee Q) \vee r$. **Associative Law of Disjunction.**

(9) $P \wedge (Q \wedge r) \Leftrightarrow (P \wedge Q) \wedge r$. **Associative Law of conjuction.**

(10) $P \vee (Q \wedge r) \Leftrightarrow (P \vee Q) \wedge (P \vee r)$ **First Distributive Law .**

(11) $P \wedge (Q \vee r) \Leftrightarrow (P \wedge Q) \vee (P \wedge r)$ **second Distributive Law.**

(12) $P \vee P \Leftrightarrow P$ **Idempotent Law for disjunction.**

(13) $P \wedge P \Leftrightarrow P$ **Idempotent Law for conjunction.**

(14)	$\boldsymbol{P \vee F \Leftrightarrow P}$	**F is the identity element of disjunction.**
(15)	$\boldsymbol{P \wedge T \Leftrightarrow P}$	**T is the identity element of conjunction.**
(16)	$\boldsymbol{P \vee T \Leftrightarrow T}$	**T is a dominant element for disjunction.**
(17)	$\boldsymbol{P \wedge} \mathbf{F} \boldsymbol{\Leftrightarrow F}$	**F is a dominant element for conjunction.**
(18)	$\boldsymbol{P \vee (P \wedge Q) \Leftrightarrow P}$	**First Absorption Law.**
(19)	$\boldsymbol{P \wedge (P \vee Q) \Leftrightarrow P}$	**Second Absorption Law.**

Remark 5.4: These rules can be proved using truth table and Theorem 5.1.

Example 5.7: Prove the first absorption law.

Solution: the result is proven if we show that:
$[P \vee (P \wedge Q)] \leftrightarrow$ P is tautology.
Write A for "$P \wedge Q$", B for " $P \vee (P \wedge Q)$", and
C for "$[P \vee (P \wedge Q)] \leftrightarrow P$". Therefore,

P	Q	A	B	C
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	T	F	F	T

Hence, the compound statement C is a tautology.

Remark 5.5: it is clear that the dual of $P \vee (P \wedge Q)$ is:

$$P \wedge (P \vee Q).$$

Therefore, recalling that P is the dual of itself, from Theorem 5.1**,** we conclude that: $P \wedge (P \vee Q) \Leftrightarrow P$. (this is the second absorption law).

Now, we shall give two important rules that can be used in proving logical equivalence compound statements.

(1) The First Substitution Rule .

Suppose that a compound statement $\boldsymbol{P}$ is a tautology. If p is a primitive statement that appears in P and we replace each occurrence of p by the statement q, then the resulting compound statement P_1 is also a tautology.

Example 5.8 : we know that for any primitive statements p, q
$(\boldsymbol{p} \vee \boldsymbol{q}) \leftrightarrow (\boldsymbol{q} \vee \boldsymbol{p})$ is a tautology. **(since $\boldsymbol{p} \vee \boldsymbol{q} \Leftrightarrow \boldsymbol{q} \vee \boldsymbol{p}$).**
Replace p by "$(r \wedge s)$", we get: $[(r \wedge s) \vee q] \leftrightarrow [q \vee (r \wedge s)]$ is also a tautology.

Example 5.9 : from DeMorgan's first law, we know that for any primitive statements p, q, the compound statement
P: $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ is a tautology (since $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$).

Replace p by "$(r \wedge s)$", we get:

$P_{1:}$ $\neg[(r \wedge s) \vee q] \leftrightarrow [\neg(r \wedge s) \wedge \neg q]$ is a tautology.

Remark 5.6. this means that:

$\neg[(r \wedge s) \vee q] \Leftrightarrow [\neg(r \wedge s) \wedge \neg q]$.

The Second Substitution Law:

Let P be a compound statement where p is arbitrary statement that appears in P, and let q be a statement such that $p \Leftrightarrow q$. Suppose that in P we replace one or more occurrence of p by q. Then this replacement yields the compound statement P_1 such that $P_1 \Leftrightarrow P$.

We give now some illustrating examples

Example 5.11: Let P: $(p \rightarrow q) \rightarrow r$.Since $(p \rightarrow q) \Leftrightarrow (\neg p \vee q)$. Using the second substitution law, replacing "$(p \rightarrow q)$" by "$(\neg p \vee q)$" in P yields the compound statement P_1: $(\neg p \vee q) \rightarrow r$ such that $P_1 \Leftrightarrow P$.

Remark 5.7. Note that:

$$[(\boldsymbol{p} \to \boldsymbol{q}) \to \boldsymbol{r}] \Leftrightarrow [(\neg \boldsymbol{p} \vee \boldsymbol{q}) \to \boldsymbol{r}].$$

Example5.12:

Let$P$: $p \to (p \vee q)$. We know that: $\neg\neg p \Leftrightarrow p$.

Using the second substitution law we deduce that:

(1) $[p \to (p \vee q)] \Leftrightarrow [\neg\neg p \to (p \vee q)]$.

(2) $[p \to (p \vee q)] \leftrightarrow [\neg\neg p \to (p \vee q)]$ is a tautology.

(3) $[p \to (p \vee q)] \Leftrightarrow [p \to (\neg\neg p \vee q)]$.

(4) $[p \to (p \vee q)] \leftrightarrow [p \to (\neg\neg p \vee q)]$ is a tautology.

(5) $[p \to (p \vee q)] \Leftrightarrow [\neg\neg p \to (\neg\neg p \vee q)]$.

(6) $[p \to (p \vee q)] \leftrightarrow [\neg\neg p \to (\neg\neg p \vee q)]$ is a tautology.

Negation of a compound statement

How can we negate a compound statement?

We have two cases:

(1) The statement contains only the operators: $\wedge, \vee, \neg$.
In this case we apply the operator $\neg$ on the statement and using some laws of logic to simplify the statement.

(2) The statement contains operators other than Conjunction, Disjunction, and Negation.
In this case, we transform the statement into another statement that is logically equivalent to it and contains only the operators $\wedge, \vee, \neg$ then apply the negation operator then simplify the statement as possible using the laws of logic. The following examples illustrate this concept.

Example 5.12 : Negate and simplify the statement $(p \vee q) \rightarrow r$.

Solution: The first step is to transform the above statement into a logical equivalent statement that contains only conjunction, disjunction, and negation. It is known that:

$\neg[(p \vee q \rightarrow r)] \Leftrightarrow \neg[\neg(p \vee q) \vee r] \Leftrightarrow$ (using DeMorgan's rule)

$$\neg\big(\neg(p \vee q)\big) \wedge \neg r \Leftrightarrow (p \vee q) \wedge \neg r.$$

Example 5.13: Negate the following statements:

$\neg p \rightarrow (\neg q \wedge r)$.

Solution:

$$\begin{aligned}
&\neg[\neg p \rightarrow (\neg q \wedge r)] \\
&\quad \Leftrightarrow \neg[\neg\neg p \vee (\neg q \wedge r)] \\
&\quad \Leftrightarrow \neg[p \vee (\neg q \wedge r)] \\
&\quad \Leftrightarrow [\neg p \wedge \neg(\neg q \wedge r)] \Leftrightarrow [\neg p \wedge (\neg\neg q \vee \neg r)] \\
&\quad \Leftrightarrow [\neg p \wedge (q \vee \neg r)]
\end{aligned}$$

Example 5.14:

Negate and simplify the following statements:

(a) $p \rightarrow (\neg q \wedge r)$.

(b) $p \vee q \vee (\neg p \wedge \neg q \wedge r)$.

Solution:

(a) $\neg[\boldsymbol{p} \rightarrow (\neg \boldsymbol{q} \wedge \boldsymbol{r})]$

$$\begin{aligned}
&\Leftrightarrow \neg[\neg p \vee (\neg q \wedge r)] \\
&\Leftrightarrow [\neg\neg p \wedge \neg(\neg q \wedge r)] \\
&\Leftrightarrow [p \wedge (\neg\neg q \vee \neg r)] \Leftrightarrow [p \wedge (q \vee \neg r)].
\end{aligned}$$

(b) $\neg[\boldsymbol{p} \vee \boldsymbol{q} \vee (\neg \boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \boldsymbol{r})]$

$$\Leftrightarrow \neg[\boldsymbol{p} \vee \boldsymbol{q} \vee (\neg(\boldsymbol{p} \vee \boldsymbol{q}) \wedge \boldsymbol{r})]$$
$$\Leftrightarrow \neg\big[\big((\boldsymbol{p} \vee \boldsymbol{q}) \vee \neg(\boldsymbol{p} \vee \boldsymbol{q})\big) \wedge (\boldsymbol{p} \vee \boldsymbol{q} \vee \boldsymbol{r})\big]$$
$$\Leftrightarrow \neg[\boldsymbol{T} \wedge (\boldsymbol{p} \vee \boldsymbol{q} \vee \boldsymbol{r})]$$
$$\Leftrightarrow \neg[\mathrm{p} \vee \mathrm{q} \vee \mathrm{r}] \Leftrightarrow \neg\mathrm{p} \wedge \neg\mathrm{q} \wedge \neg\mathrm{r}\ .$$

Example 5.15: without using truth table method show that the following statements are tautology

(a) $p \to [q \to (p \wedge q)]$.

(b) $(\boldsymbol{p} \vee \boldsymbol{q}) \to (\boldsymbol{q} \to \boldsymbol{q})$.

Solution:

We shall solve the problems using some laws of logic

(a) $$\big[p \to [q \to (p \wedge q)]\big]$$
$$\Leftrightarrow \big[p \to \big(\neg q \vee (p \wedge q)\big)\big]$$
$$\Leftrightarrow \big[\neg p \vee [\neg q \vee (p \wedge q)]\big]$$
$$\Leftrightarrow [\neg p \vee \neg q \vee (p \wedge q)] \Leftrightarrow [\neg(p \wedge q) \vee (p \wedge q)]$$
$$\Leftrightarrow (\neg s \vee s) \Leftrightarrow T.$$

We replaced " $(p \wedge q)$" by s and used the substitution rules.

(b) $$[(p \vee q) \to (q \to q)]$$
$$\Leftrightarrow [(p \vee q) \to (\neg q \vee q)]$$
$$\Leftrightarrow [(p \vee q) \to T] \Leftrightarrow [\neg(p \vee q) \vee T]$$
$$\Leftrightarrow T\ .$$

<u>**Definition**</u> **5.4:** consider the statement S: $p \to q$.

(1) The contrapositive of$S$ is L: $\neg q \rightarrow \neg p$.

(2) The converse of S is A: $q \rightarrow p$.

(3) The inverse of S is B: $\neg p \rightarrow \neg q$.

Theorem 5.2

(1) $(\boldsymbol{p} \rightarrow \boldsymbol{q}) \Leftrightarrow (\neg \boldsymbol{q} \rightarrow \neg \boldsymbol{p})$**.**

(2) $(q \rightarrow p) \Leftrightarrow (\neg p \rightarrow \neg q)$.

Proof

(1) $(p \rightarrow q) \Leftrightarrow (\neg p \vee q) \Leftrightarrow (q \vee \neg p) \Leftrightarrow (\neg q \rightarrow \neg p)$.

(2) $(q \rightarrow p) \Leftrightarrow (\neg q \vee p) \Leftrightarrow (p \vee \neg q) \Leftrightarrow (\neg p \rightarrow \neg q)$.