Fundamental Principals of Counting

There are two important basic rules of counting, namely, the rule of Sum and the rule of Product.

(1) **The Rule of Sum** : If a first task can be performed in m ways, while a second task be performed in n ways ,and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m+n$ ways.

Example 1.1 . During a local campaign, eight Republican and five Democratic candidates are nominated for president of the school board. If the president is to be one of these candidates, how many possibilities are there for the eventual winner?

Solution: We have two tasks, the first (choosing one from the Republican) can be performed by 8 ways, while the second(choosing one from Democratic) can be performed by 5 ways. Since these tasks cannot performed simultaneously, we use the rule of Sum ,therefore, we have 8+5=13 ways for the eventual winner.

Example 1.2: There are 18 mathematics majors and 325 computer science majors at a college. How many ways are there to pick one representative who is either a mathematics major or a computer science major?

Solution: Again , we have two disjoint tasks, and conditions of the Sum rule are satisfied. Therefore, the number of ways is 18+325=343.

Remark 1.1: The sum rule can be extended to more than two tasks.

Example 1.3: A representative to the College Personnel Committee is to be chosen from the Biology Department, the Chemistry Department, the Geology Department, or the Physics Department. How many different choices are there for this representative if there are 47 members of the

Biology Department, 52 members of the Chemistry Department ,31 members of the Geology Department, and 38 members of the Physics Department, and no faculty member belongs to more than one of these departments?

Solution: It is clear that we have 4 pairwise disjoint tasks (i.e., any two of them cannot be performed simultaneously). So conditions of the sum rule are satisfied

Hence the number of ways to choose one representative is
47+52+31+38=168

(2) **The Rule of Product:** If a procedure can be broken down into first and second stages, and if there are m possible ways of performing the first stage and if for each of these ways, there are n ways of performing the second stage then the total procedure can be carried out, in the designed order, in mn ways.

Example 2.1 In reference to example 1.1 above, how many possibilities exists for a pair of candidates (one from each party) to oppose each other for the eventual election?

Solution : We have a procedure consists of two tasks, the first one (choosing one from Republican) can be performed by 8 ways and for each of these 8 ways, the second task (choosing one of the Democratic) can be achieved by 5 ways. Hence, by the product rule, the procedure can be achieved by $8 \times 5 = 40$ ways.

Remark2.2: the product rule can be extended to more than two tasks.

Example 2.2 : Buick automobile come in 4 models,12 colors, 3 engine sizes, and 2 transmission types.

(a) How many distinct Buicks can be manufactured ?

(b) If one of available colors is blue, how many different blue Buicks can be manufactured ?

Solution: **(a**) We have a procedure consists of 4 tasks, the first one (choosing the model) can performed by 4 ways, and for each of these 4 ways the second task (choosing the color) can be accomplished by 12 ways, and for each of these ways (4 and 12) the third task (choosing the engine size) can be performed by 3 ways and for each of these ways (4, 12,and 3) the fourth task (choosing transmission type) can be performed by 2 ways . Hence, by the product rule, the procedure can be achieved by $4 \times 12 \times 3 \times 2 = 288$ ways.

(b) In this case, we still have a procedure consists of 4 stages as before, the only difference is that the number of ways to choose color is 1, therefore, the procedure will be achieved by $4 \times 1 \times 3 \times 2 = 24$ ways.

Example 2.3 : In a certain implementation of the programing language Pascal, an identifier consists of a single letter or a letter followed by up to 7 symbols, which may be letters or digits. (We assume that the computer does not distinguish between capital and lowercase letters). Certain keywords, however, are reversed for commands, consequently, these keywords may not be used as identifiers. If this implementation has 36 reserved words, how many distinct identifiers are possible in this version of Pascal?

Solution: In this problem we shall use the sum rule and the product rule. We have

(1) 26 identifier of length 1. (2) 26x36 identifier of length 2.

(3) $26 \times (36)^2$ identifier of length 3. (4) $26 \times (36)^3$ identifier of length 4.

(5) $26 \times (36)^4$ identifier of length 5. (6) $26 \times (36)^5$ identifier of length 6.

(7) $26 \times (36)^6$ identifier of Length 7. (8) $26 \times (36)^7$ identifier of length 8.

So by the sum rule, the total number of identifiers =

$26 + 26(36) + 26(36)^2 + 26(36)^3 + 26(36)^4 + 26(36)^5 + 26(36)^6 + 26(36)^7$= 2,095,681,645,538 .

Now 36 of these implantation are reserved for commands, therefore the actual number of Pascal identifier= 2,095,681,645,538 - 36 = 2,095,681,502.

Some important Concepts :

Let Σ be a finite set of characters or alphabet .

A string over Σ is an ordered n-tuple of characters of Σ, written without parentheses or commas, or the null string ε (sometimes called space or empty word).

Formal language over Σ is a set of strings over Σ .

Let n be any non-negative integer, then

$\Sigma^* =$ the set of all strings of finite length over Σ .

$\Sigma^n =$ the set of all strings of length n over Σ .

Example : Let $\Sigma = \{a, b\}$.

(a) Define a language $\boldsymbol{L_1}$ over Σ to be the set of all strings that begin with the character $\boldsymbol{a}$ and have length less than or equal to 3. Find L_1.
(b) Define another language L_2 over Σ to be the set of strings of finite length that look the same whether written forward or backward (such strings are called palindromes). Write 10 elements of L_2.
(c) Write $\Sigma^0, \Sigma^1, \Sigma^2, \Sigma^3$
(d) How many words (strings) of length less than 5?

Solution:

(a) $\boldsymbol{L_1 = \{a, aa, ab, aaa, aab, aba, abb\}}$.

(b) L_2 contains the following 10 strings:

ε , $a, b, aa, bb, aaa, bab, abba, baab$, baabbbbaab **.**

(c) $\boldsymbol{\Sigma^0 = \{\varepsilon\}}$, $\boldsymbol{\Sigma^1 = \{a, b\}}$ **,**

$\boldsymbol{\Sigma^2 = \{aa, ab, ba, bb\}}$**,**

$\boldsymbol{\Sigma^3 = \{aaa, aab, aba, abb, baa, bab, bba, bbb\}}$

(d) Using the product rule, we have :

(1) one word of length 0 **,**

(2) 2 words of length1 ,

(3) 4 words of length 2 ,

(4) 8 words of length 3 ,

(5) 16 words of length 4

Hence by the sum rule, the number of words of length less than 5 = 1+2+4+8+16= 31