Cairo University **Discrete Structures (Ms521)**

FGSSR **Lecture 17**

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Equivalence Relations

In this lecture, we will study relations that have a particularly useful combination of properties, used to relate objects that are similar in some way.

Definition 17.1: A relation *R* on a set *A is called a*n equivalence relation if it is reflexive, symmetric, and transitive.

Remarks 17.1:

(1) Two elements that are related by an equivalence relation are called equivalent.
(2) In any equivalence relation *R* every element is equivalent to itself (Note that *R* is reflexive).
(3) In any equivalence relation R, if *a* and *b* are equivalent and *b* and *c* are equivalent, then *a* and *c* are also equivalent (note that *R* is transitive).

Example 17.1: Let *A* be the set of strings of English letters and $R \subseteq A \times A$ be such that: $(a, b) \in R$ iff $l(a) = l(b)$, where :

$l: A \to N$,
$l(a)$ =number of letters in *a*=length of *a* , for every $a \in A$.

Show that R is an equivalence relation on *A*.

Solution: (1) *R* is a reflexive relation.

Proof: $\forall a \in A$, $l(a) = l(a)$. Therefore, $(a, a) \in R, \forall a \in A$.

(2) *R* is a symmetric relation on *A* .

Proof: $\forall \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{A}$, let $(a, b) \in R$. Then, $l(a) = l(b)$. This implies that $l(b) = l(a)$.

This means that $(b, a) \in R$. Hence *R* is a symmetric relation.

(3) *R* is a transitive relation on *A*

Proof: $\forall \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{A}$, let $(a, b) \in R$ and $(b, c) \in R$.

Therefore**,** $\boldsymbol{l}(\boldsymbol{a}) = \boldsymbol{l}(\boldsymbol{b})$ and $l(b) = l(c)$

Hence, $l(a) = l(c)$. This means that $(a, c) \in R$.

Thus, *R* is a transitive relation on *A.*

From (1), (2), and (3) it follows that *R* is an equivalence relation on *A*.

<u>Example</u> 17.2: Let $R \subseteq Z \times Z$ be such that $(a, b) \in R$ iff $a = b$ or $a = -b$. Then, it is easy to show that *R* is an equivalence relation on *Z*.

(1) *R* is a reflexive relation on *Z.*

Proof: $\forall a \in Z$**,** $\boldsymbol{a} = a$. Hence $(a, a) \in R$.

(2) *R* is a symmetric relation on *Z.*

Proof: $\forall \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{Z}$**,** let $(a, b) \in R$. This means that either $a = b$ or $a = -b$.

Hence ,either $b = a$ or $b = -a$. Thus $(b, a) \in R$.

(3) *R* is a transitive relation on *Z.*

Proof: $\forall \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{Z}$**,** let $(a, b) \in R$ and $(b, c) \in R$.

Hence, either $a = b$ or $a = -b$ (1) and

Either $b = c$ or $b = -c$ (2)

From (1) and (2) it follows that either $a = c$ or $a = -c$

Thus, $(a, c) \in R$.

<u>Example</u> 17.3: Let $R \subseteq \mathbb{R} \times \mathbb{R}$ be such that: $(a, b) \in R$ iff $a - b \in Z$, where, $\mathbb{R}$ is the set of real numbers and *Z* is the set of integers.

R is an equivalence relation on $\mathbb{R}$.

Proof:

(1) *R* is a reflexive relation.

$\forall a \in \mathbb{R}$, let $(a, a) \in R$ since $a - a = 0 \in Z$.

(2) *R* is a symmetric relation.

$\forall a, b \in \mathbb{R}$, let$(a, b) \in \mathbb{R}$. Hence $a - b \in Z$, therefore, $b - a \in Z$. This means that $(b, a) \in R$.

(3) R is a transitive relation on $\mathbb{R}$.

Proof: $\forall a, b, c \in \mathbb{R}$, let $(a, b) \in R$ and $(b, c) \in R$

Then

$$a - b \in Z \qquad (1)$$

And $$b - c \in Z \qquad (2)$$

Adding (1) and (2) , one gets: $a - c \in Z$. Hence, $(a, c) \in R$.

<u>Remark</u> 17.2: the equivalence relation on a set classifies the elements of this set into groups of equivalent elements. The following definition explains this concept.

<u>Definition</u> 17.2: Let *R* be an equivalence relation on a set *A*. The equivalence class of the element $a \in A$, denoted by $[a]$, is defined as follows:

$[a] = \{b \in A : (a, b) \in R\}$.

If $s \in [a]$, *s* is called a representative of this equivalence class.

<u>Example</u> 17.4: In example 17.2, the equivalent class of the integer a is :

$$[a] = \{a, -a\}, \quad \forall a \in Z .$$

<u>Example</u> 17.5: Let *m* be a positive integer greater than 1. Let $R \subseteq Z \times Z$ be defined as follows: $R = \{(a, b): a \equiv b(mod\ m)\}$. Show that *R* is an equivalence relation on *Z*. Write the equivalence classes induced by *R*.

Before solving example 17.5, let us explain what is meant by $a \equiv b(mod\ m)$.

<u>Remark</u> 17.3: Note that the expression, $a \equiv b(mod\ m)$ means that:

(1) Both *a*,*b* has the same reminder when divided by *m.*
(2) *m* exactly divides *b-a ,i.e.,* $a - b = km$ for some $k \in Z$.

Solution: (1) *R* is a reflexive relation on *Z*

Proof: $\forall a \in Z, a - a = 0m = 0$. Hence, $(a, a) \in R$.

(2)*R* is a symmetric relation on Z

Proof: $\forall \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{Z}$, let $(a, b) \in R$. Then $a \equiv b(mod\ m)$.

This means that: $a - b = km$, *k* is an integer. Therefore,

$b - a = (-k)m$, where $(-k) \in Z$. Hence, $b \equiv a(mod\ m)$,

which means that $(b, a) \in R$.

(3) *R* is a transitive relation on *Z*.

Proof: $\forall a, b, c \in Z$, let $(a, b) \in R$ and $(b, c) \in R$. Therefore,

$a \equiv b(mod\ m) \Longrightarrow \quad a - b = km,$ for some $k \in Z$. (1)

And $b \equiv c(mod\ m) \Longrightarrow \quad b - c = lm,$ for some $l \in Z$ (2)

Adding (1) and (2), one gets: $a - c = (k + l)m,$ for some $(k + l) \in Z$.

This means that $a \equiv c(mod\ m)$, i.e., $(a, c) \in R$.

Now, we have the following equivalence classes:

$[0] = \{\cdots, -3m, -2m, -m, 0, m, 2m, 3m, \cdots\}$.

$[1] = \{\cdots, (-3m + 1), (-2m + 1), (-m + 1), 1, (m + 1), (2m + 1), (3m + 1), \cdots\}$.

$[2] = \{\cdots, (-3m + 2), (-2m + 2), (-m + 2), 2, (m + 2), (2m + 2), (3m + 2), \cdots\}$.

$\vdots\vdots$

$[m - 1] = \{\cdots, -(3m + 1), -(2m + 1), (m - 1), (2m - 1), (3m - 1), \cdots\}$.

<u>Remarks</u> 17.4:

(1) $[0] = [m] = [2m] = \cdots$
(2) Either $[k] \cap [m] = \emptyset$, or $[k] = [m]$. For every $k, m \in Z$.
(3) $Z = [0] \cup [1] \cup [2] \cup \cdots \cup [m - 1]$.

<u>Homework</u> : the student is advised to solve this problem for specified values of *m*, for example put m=2,3,4 to help himself obtaining higher understanding.

Example 17.5 gives us an idea about an important concept in mathematics called "partition of the set". The following definition concerns with this concept.

<u>Definition</u> 17.3: Let *A* be a non-empty set, the collection of non-empty subsets of *A,* say, B_1, B_2 , $B_3, \cdots, B_n$ is called a partition of A provided that:

(1) $B_i \cap B_j = \emptyset,\ for\ i \neq j.$ (2) $A = \bigcup_{i=1}^{n} B_i$

<u>Example</u> 17.6: which of the following collections of subsets are partitions of the set $A = \{1,2,3,4,5,6\}$.

(a) $\{1,2\}, \{2,3,4\}, \{4,5,6\}$.
(b) $\{1\}, \{2,3,6\}, \{4\}, \{5\}$.
(c) $\{2,4,6\}, \{1,3,5\}$.
(d) $\{1,4,5\}, \{2,6\}$.

Solution

(a) It is not a partition since sets are not mutually disjoint.
(b) It is a partition since sets are mutually disjoint and their union equals *A*.
(c) It is a partition.(conditions are satisfied).
(d) It is not a partition since their union does not equal *A*.

<u>Theorem</u> 17.1: Let *R* be an equivalence relation on a set *A*. the following statements are equivalent:

(1) $(a,b) \in R.$ (2) $[a] = [b].$ (3) $[a] \cap [b] \neq \emptyset.$

<u>Theorem</u> 17.2: Let *R be an e*quivalence relation on a set *A*. Then the equivalence relation *R* form a partition of *A*. Conversely, given a partition $\{A_i : i \in I\}$ of sets of *A*, there is an equivalence relation *R* that has the sets $A_i\ i \in I$ as its equivalence classes.

<u>Example</u> 17.7: In example 17.6, find the equivalence relations associated with the partition given in (a) and (b).

Solution: Note that $R = \big\{\{1\} \times \{1\}\big\} \cup \big\{\{2,3,6\} \times \{2,3,6\}\big\} \cup \big\{\{4\} \times \{4\}\big\} \cup \big\{\{5\} \times \{5\}\big\}$

$\therefore R = \{(1,1), (2,2), (2,3), (3,2), (2,6), (3,6), (6,3), (3,3), (6,2), (6,6), (4,4), (5,5)\}.$

(c) Note that $R = \big\{\{2,4,6\} \times \{2,4,6\}\big\} \cup \big\{\{1,3,5\} \times \{1,3,5\}\big\}$

$$\therefore R = \{(2,2), (2,4), (4,2), (4,4), (2,6), (6,2), (6,6), (4,6), (6,4),\} \cup$$
$$\{(1,1), (1,3), (3,1), (3,3), (1,5), (5,1), (5,5), (3,5), (5,3)\}$$

<u>Example</u> 17.7: consider the relation $R \subseteq Z \times Z$, defined as follows:

$R = \{(a, b): a \equiv b(mod\ 5)\}$. Show that this relation is an equivalence relation and find the corresponding partition.

Solution : the proof that this relation is an equivalence relation is similar to that of example 17.5. (replace m by 5).

The corresponding partition is :

$A_1 = [0] = \{\cdots, -10, -5,0,5,10, \cdots\}$,

$A_2 = [1] = \{\cdots, -9, -4,1,6,11, \cdots\}$,

$A_3 = [2] = \{\cdots, -8, -3,2,7, \cdots\}$,

$A_4 = [3] = \{\cdots, -7, -2,3,8, \cdots\}$, and

$A_5 = [4] = \{\cdots, -6, -1,4,9,4, \cdots\}$.