Cairo University. **Discrete Structures (Ms521).**

FGSSR. **Lecture 16.**

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Partial Orderings

Introduction : We often use relations to order some or of the elements of sets. For instance, we order words using the relation containing pairs of words (x, y) where *x* comes before *y* in the dictionary. We schedule projects using the relation consisting of pairs (x, y) where *x* and *y* are tasks in a project such that *x* must be completed before *y* begins. The set of integers can be ordered using the relation containing the pairs (x, y) where *x* is less than *y*

Definition 16.1. A relation *R* on a set *A* is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive. A set *A together* with a partial ordering *R* is called a partially ordered set, or poset , and is denoted by (A, R).

Example 16.1: Show that $(Z, \geq)$ is a poset, where *Z* is the set of integers and $\geq$ is the " greater than or equal relation".

Solution: $\forall x \in Z$ we have $x \geq x$. This implies that " $\geq$ "

is a reflexive relation on *Z*. (1) .

$\forall x, y \in Z$, let $(x, y) \in \geq$ and $(y, x) \in \geq$. This means that $x \geq y$ and $y \geq x$. this implies that $x = y$. Hence " $\geq$ " is an antisymmetric relation on *Z* . (2)

$\forall x, y, z \in Z$ let $(x, y) \in \geq$ and $(y, z) \in \geq$. This means that $x \geq y$ and $y \geq z$. This implies that $x \geq z$. Hence, $(x, z) \in \geq$. Hence, " $\geq$ "

is a transitive relation on *Z.* (3) .

From (1), (2), and (3) it follows that $(Z, \geq)$ is a poset.

Example 16.2: Show that (N, R) is a poset, where *N* is the set of natural numbers and *R* is defined as follows: $R: N \times N \to N$,

$(x, y) \in R \; iff \; y = mx$, m is a positive integer. (this is divisibility relation, i.e., $x \; divides \; y$ or $y \; can \; be \; divided \; on \; x \; without \; reminder$).

Solution: $\forall x \in N$, we have $x = x$ (note that $m = 1$). This means that *x* divides *x*. hence $(x, x) \in R$. Hence, *R* is a reflexive relation on *N.* (1).

$\forall x, y \in N$, let $(x, y) \in R$ and $(y, x) \in R$. Hence :

$y = mx$ and $x = ny$, for some positive integers *m,n.* this implies that $x = mnx$.

This means that $mn = 1$. Hence, $m = n = 1$. Hence $x = y$. Therefore *R* is an antisymmetric relation on *N.* *(2)*

$\forall x, y, z \in N$, let $(x, y) \in R$ and $(y, z) \in R$. Then $y = mx$ and $z = ny$, for some natural numbers *m,n* . Hence, $z = mnx = kx$, $k \in N$. This means that $(x, z) \in R$.

Therefore, *R* is a transitive relation on *N* (3)

From (1), (2), and (3) it follows that (N, R) is poset.

Example 16.3: Let $A = \{1,2\}$ and $R \subseteq P(A) \times P(A)$, be defined as follows: for any $B, C \in P(A)$, BRC iff $B \subseteq C$. (Note that this relation is called the inclusion relation on the power set of *A*). Show that this relation is a partial order. Write *R*.

Solution: It is clear that for every $B \in P(A)$, we have $B \subseteq B$. Hence,$(B, B) \in R$. This means that *R* ia a reflexive relation. (1).

(The student must understand that the statement " BRC " is the same as " (B, C) ".

Now, we shall show that *R* is an antisymmetric relation. To do this, let

$(B, C) \in R$ and $(C, B) \in R$. This means that $B \subseteq C$ and $C \subseteq B$. This implies that $B = C$. Therefore R is an antisymmetric relation. (2).

Finally, *R* is *a* transitive relation. To show this, for any $B, C, and\ D \in P(A)$,

let $(B, C) \in R$ and $(C, D) \in R$. This means that $B \subseteq C$ and $C \subseteq D$.

Hence,$B \subseteq D$. Therefore, $(B, D) \in R$ and *R* is a transitive relation on $P(A)$. (3).

From (1), (2), and (3) it follows that *R* is a partial order relation on $P(A)$.

$R = \{(\emptyset, \emptyset), (\emptyset, \{1\}), (\emptyset, \{2\}), (\emptyset, \{1,2\}), (\{1\}, \{1\}), (\{1\}, \{1,2\}), (\{2\}, \{2\}), (\{2\}, \{1,2\}), (\{1,2\}, \{1,2\})\}$.

Definition 16.2: The elements *a* ,*b* of a poset (A, R) are called comparable if either $(a, b) \in R$ or $(b, a) \in R$. When a and b are elements of *A* such that neither $(a, b) \in R$ nor $(b, a) \in R$, *a* and *b* are called incomparable.

Example 16.4 : In example 16.3, $\emptyset$ (and also $\{1,2\}$) is comparable with every element in $P(\{1,2\})$,while $\{1\}$ and $\{2\}$ are incomparable.

Remark 16.2 : The adjective "partial" is used to describe partial orderings since pairs of elements may be incomparable. When every two elements in the set are comparable, the relation is called a total ordering (or linear).

Definition 16.3 : If (A, R) is a poset and every two elements of *A* are comparable, *A* is called a totally ordered or linearly ordered set, and *R* is called a total order or a linear order. A totally ordered set is also called a chain.

Remarks 16.3 :

(1) The poset (Z, R), where *Z* is the set of integers and $R \subseteq Z \times Z$ is defined as: $(a, b) \in R$ iff $a \leq b$ is a totally ordered, since for every $a, b \in Z$, either $a \leq b$ or $b \leq a$.

(2) The poset $(P(A), R)$defined in example 16.3 is a partial order since $\{1\}$ and $\{2\}$ are incomparable.

Definition 16.4 : Let (A, R) be a poset. An element $a \in A$ is called maximal if , for every $x \in A, (a, x) \in R \Rightarrow (x = a)$. In words, if *a* is compared with an element *b* of *A, it will* appear as a second coordinate in the order pairs of *R.* The only case in which *a* appears as a first coordinate in the ordered pairs of *R* occurs when it is comparable with itself.

Similarly, an element *a* of a poset is called minimal if , for every $x \in A$, $(x, a) \in R \Rightarrow (x = a)$. (note that if *a* is compared with any element in *A* (different from *a*) it must be the first coordinate in the order pairs

Example 16.6: Let $A = \{1,2,3, \cdots 10\}$. Consider the poset (A, R), where for any *x,y*$\in A$, $(a, b) \in R$ iff a divides *b.* Then $R =$

$$\left\{\begin{matrix}(1,1),(1,2),\cdots(1,10),(2,2),(2,4),(2.6),(2,8),(2,10),(3,3),(3,6),(3,9),(4,4),(4,8),(8,8),\\(5,5),(5,10),(6,6),(7,7),(8,8),(9,9),(10,10)\end{matrix}\right\}$$

It is easy to see that the set of maximal elements is $\{6,7,8,9,10\}$ since :

(1) $(6, x) \in R \Rightarrow (x = 6)$.

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(2) $(7, x) \in R \Longrightarrow (x = 7)$.
(3) $(8, x) \in R \Longrightarrow (x = 8)$.
(4) $(9, x) \in R \Longrightarrow (x = 9)$.
(5) $(10, x) \in R \Longrightarrow (x = 10)$.

Also, the element 1, is the only minimal of this relation since $(x, 1) \in R \Longrightarrow (x = 1)$.

Definintion 16.5 : Let (A, R) be a poset. An element $a \in A$ is called maximum or greatest element of *A* if $(x, a) \in R$ for every $x \in A$.

Similarly an element $a \in A$ is called minimum of *A* if $(a, x) \in R$ for every $x \in A$.

Remark 16.4 :

(1) maximal (minimal) elements may be not unique .
(2) Maximum (minimum) element, is unique when it exists.

Example 16.7:

(1) In example 16.6, 1 is the minimum but there is no maximum
(2) In example 16.3, $\emptyset$ is the minimum, while $\{1,2\}$ is the maximum.
(3) In the poset $(N, \leq)$, *N* is the set of natural number and " $\leq$ " is the "ordinary less than or equal relation" the minimum is 1. It is easy to see that :
(1) There is no maximum (2) there is no maximal.

Definition 16.6 : Let (A, R) be a poset and $B \subseteq A$. An element $u \in A$ is called an upper bound of *B* if the following condition is satisfied:

If $b \in B$ then $(b, u) \in R$.

(The student must note that the upper bound of a set may not belong to it) .

Definition 16.7: Let (A, R) be a poset and let $B \subseteq A$. An element $u \in A$ is called least upper bound if the following conditions hold:

(1) If $b \in B$ $then$ $(b, u) \in R$.
(2) If $k \in A$ be such that $(b, k) \in R$ (i.,e., *k* is an upper bound of *B*) then $(u, k) \in R$.

<u>Definition</u> 16.8: Let (A, R) be a poset and $B \subseteq A$.

(1) An element $l \in A$ is called a lower bound of *B* if the following condition is satisfied:

If $b \in B$ then$(l, b) \in R$.

(2) An element $l \in A$ is called a greatest lower bound of B provided that :
   (a) If $b \in B$ then $(l, b) \in R$.
   (b) For $b \in B$,If $k \in A$ be such that $(k, b) \in R$ then $(k, l) \in R$.

<u>Example</u> 16.8 : Let $A = \{3,5,9,15,24,45\}$ and $R \subseteq A \times A$ be defined as follows: $(a, b) \in R$ iff $b = ma$ for some natural number. Find:

(1) *R*
(2) The maximal elements.
(3) The minimal elements.
(4) All upper bounds of $\{3,5\}$.
(5) The least upper bound of $\{3,5\}$ if it exists.
(6) All lower bounds of $\{15,45\}$.
(7) The greatest lower bound of $\{15,45\}$ if it exists.
(8) Is there a greatest element?
(9) Is there a least element?

<u>Solution</u> :

(1) $R = \{(3,3), (3,9), (3,15), (3,24), (3,45), (5,5), (5,15), (5,45), (9,9), (9,45), (15,15), (15,45), (24,24), (45,45)\}$.
(2) The set of maximal elements= $\{24,45\}$.
(3) The set of minimal elements is $\{3,5\}$.
(4) The set of upper bounds of $\{3,5\} = \{15,45\}$.
(5) The least upper bounds of $\{3,5\} = 15$.
(6) The set of lower bounds of $\{15,45\} = \{3,5,15\}$.
(7) The greatest lower bound of $\{15,45\} = 15$.
(8) There is no greatest element since there is no $a \in A$ such that $(x, a) \in R$ for all $x \in A$. (note that 24 and 45 are not comparable so that 45 can not be considered greatest element).
(9) There is no least element. (note that 3 and 5 are not comparable so that 3 can not be considered least element).