Cairo University. **Discrete Structures (Msc521).**

FGSSR. **Lecture 14.**

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Closures of Relations

Reflexive closure: Let A be a nonempty set and $R \subseteq A \times A$ be a (not reflexive) relation on A. *Then the reflexive closure of R (if exist) is* defined *to be the smallest* reflexive relation on A containing R.

Remark 14.1 : If S is the reflexive closure of R, then the following conditions must be satisfied:

[1] S is reflexive. [2] $R \subseteq S$.

[3] if K is any reflexive relation containing R , then $S \subseteq K$.

To **explain this concept,** Let $\boldsymbol{A} = \{1,2,3\}$. The relation $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on A is not reflexive. We can produce a reflexive relation on A containing R such that it is contained in every other reflexive relation on A containing R. (i.,e., it is the smallest reflexive relation on A containing R). To do this, added to R the elements $(2,2)$ and $(3,3)$ therefore we get: $S = \{(1,1),(1,2),(2,1),(2,2),(3,2),(3,3)\}$. Note that S is the smallest reflexive relation containing R. Then S is called reflexive clousuer of R.

Example 14.1: In example 13.4, the reflexive closure of the relations R_1, R_2, R_4 and R_5 $((S_i , i = 1,2,4,5)$ can be given as follows:

$$S_i = R_i \cup MD \ , \quad i = 1,2,3,4 \text{ , where } MD = \{(i,i): i = 1,2,3,4\}.$$

Therefore,

$S_1 = R_1 \cup MD = \{(1,1),(1,2),(2,1),(2,2),(3,4),(3,3),(4,1),(4,4)\}$.

$S_2 = R_2 \cup MD = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$.

$S_4 = R_4 \cup MD = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(4,4)\}$.

$S_5 = R_5 \cup MD = \{(1,1),(2,2),(3,3),(3,4)\}$.

Definition14.1: Let *A* be a nonempty set and $R \subseteq A \times A$, the inverse of *R* (denoted by R^{-1}) is defined to be $R^{-1} = \{(b,a): (a,b) \in R\}$.

Example 14.2 : Consider the relations defined in example 13.4,then

$R_1^{-1} = \{(1,1),(1,2),(2,1),(2,2),(4,3),(1,4),(4,4)\}$.

$R_2^{-1} = \{(1,1),(2,1),(1,2)\} = R_2$.

$R_3^{-1} = \{(1,1),(2,1),(4,1),(1,2),(2,2),(3,3),(1,4),(4,4)\} = R_3$.

$R_4^{-1} = \{(1,2),(1,3),(2,3),(1,4),(2,4),(3,4)\}$.

$R_5^{-1} = \{(4,3)\}$.

Example 14.3: Let Z^+be the set of non-negative integers, and $R \subseteq Z^+ \times Z^+$such that: $R = \{(a,b): a < b\}$. Then $R^{-1} = \{(b,a): a < b\}$. Note that $(1,2) \in R$ but $(2,1) \in R^{-1}$.

Symmetric Clousure: Let A be a nonempty set and R⊆A×A be a (not symmetric) relation on A. Then the symmetric closure of R (if exist) is defined to be the smallest symmetric relation on A containing R.

Remark 14.2: If *S* is the symmetric closure of *R*, then the following conditions must be satisfied:

[1] *S* is symmetric. [2] $R \subseteq S$

[3] if K is any symmetric relation containing R , then S⊆K.

Remark 14.3: To the symmetric closure, *S*, of a relation *R* we use the following equation: $S = R \cup R^{-1}$.

Example 14.4: Consider the relation *R* defined in example 14.3, it is easy to see that the symmetric closure *S* of *R* is :

$S = R \cup R^{-1} = \{(a,b): a < b, a,b \in Z^+\} \cup \{(a,b): a > b, a,b \in Z^+\} =$

$\{(a,b): a \neq b \text{ }, a,b \in Z^+\}$.

Example 14.5: Consider the relations defined in example 13.4 it is easy to see that the symmetric clousure of these relations are:

$S_1 = R_1 \cup R_1^{-1} = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(4,1),(1,4),(4,4)\}$.

$S_2 = R_2 \cup R_2^{-1} = R_2$. (Note that $R_2 = R_2^{-1}$) .

$S_3 = R_3 \cup R_3^{-1} = R_3$.

$S_4 = R_4 \cup R_4^{-1} =$
$\{(2,1),(1,2),(3,1),(1,3),(4,1),(1,4),(3,2),(2,3),(4,2),(2,4),(4,3),(3,4)\}$.

$S_5 = \{(3,4),(4,3)\}$.

Transitive Clousure: Let *A* be a nonempty set and $R \subseteq A \times A$ be a not transitive relation on *A.* Then the transitive closure , R^t of R is the smallest transitive relation containing *R.*

Remark 14.4 : If R^t is the transitive closure of R, then the following conditions must be hold:

[1] R^t is transitive. [2] $R \subseteq R^t$.

[3] If *S* is any transitive relation containing *R*, then $R^t \subseteq S$.

Example 14.6 : Consider the relations defined in example 13.4. Note that R_1 is not transitive since it contains $(4,1)$ and $(1,2)$ but it does not contain $(4,2)$, also it contains $(3,4)$ and $(4,1)$ but it does not contain $(3,1)$. We are to find R_1^t.

At first, we note that every ordered pair in *R* must be in R_1^t. Therefore $R_1 \subseteq R_1^t$

Since $(3,4) \in R_1$ and $(4,1) \in R_1$,it follows that $(3,1) \in R_1^t$.

Now ,$(3,1) \in R_1^t$ and $(1,2) \in R_1^t$ implies that $(3,2) \in R_1^t$.

Now $(4,1) \in R_1^t$ and $(1,2) \in R_1^t$ implies that $(4,2) \in R_1^t$. Hence,

$R_1^t = \{(1,1),(1,2),(2,1),(2,2),(3,4)(4,1),(4,4),(3,2),(3,1),(4,2)\}$.

To find R_2^t, we note that $R_2 \subseteq R_2^t$,hence $\{(1,1),(1,2),(2,1)\} \subseteq R_1^t$. Now, $(2,1) \in R_2^t$ and $(1,2) \in R_1^t$ implies that $(2.2) \in R_2^t$. Hence

$R_2^t = \{(1,1),(1,2),(2,1),(2,2)\}$.