CairoUniversity. Discrete Structures (Msc521).

FGSSR. Lecture 14.

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Some solved Examples

(1) Let $R \subseteq Z \times Z$ (where Z is the set of integers).

If $R = \{(a,b): a < b\}$. Discuss the properties of *R.*

Solution:

(a) It is not reflexive, since $a \nless a$,i.e., $(a,a) \notin R\ \forall a \in A$.

(b)It is irreflexive, since $a \nless a$,i.e.,$(a,a) \notin R, \forall a \in A$

(c) It is not symmetric since if $(a,b) \in R$ then $a < b$, hence,

$b \nless a$ this means that $(b,a) \notin R$.

(d) It is antisymmetric since the statement:

" if $(a,b) \in R \wedge (b,a) \in R$ then $a = b$" is tautology.
Note that if $a < b$ then $(a,b) \in R$ is true but $(b,a) \in R$ is false.
Also $a = b$ s false. Therefore the hypothesis is false and the conclusion is false (hence,the "if then" statement) is true .

(e) It is transitive to prove this let $(a,b) \in R$ and $(b,c) \in R$.

We are to show that $(a,c) \in R$

$(a,b) \in R$ implies that $a < b$. (1)
$(b,c) \in R$ implies that $b < c$. (2)

From (1) and (2) it follows that $a < c$, hence, $(a,c) \in R$.

(2) Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$, define
$R_1 \subseteq A \times B$ and $R_2 \subseteq A \times B$

such that $R_1 = \{(1,1),(2,2),(3,3)\}$ and

$R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$.

(a) Discuss the properties of these relations. Then find:
(b) $R_1 \cup R_2$
(c) $R_1 \cap R_2$
(d) $R_1 - R_2$
(e) $R_2 - R_1$
(f) $R_1 \oplus R_2$
(g) $\overline{R_1 \cup R_2}$

Solution: part is left as an exercise we shall solve only the other parts.

(a) $\boldsymbol{R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}}$.
(b) $\boldsymbol{R_1 \cap R_2 = \{(1,1)\}}$.
(c) $\boldsymbol{R_1 - R_2 = \{(2,2),(3,3)\}}$.
(d) $\boldsymbol{R_2 - R_1 = \{(1,2),(1,3),(1,4)\}}$.
(e) $\boldsymbol{R_1 \oplus R_{2=} \quad \{(2,2),(3,3),(1,2),(1,3),(1,4)\}}$.
(f) Note the universal set heir is the set $\boldsymbol{A \times A}$
$\boldsymbol{\overline{R_1 \cup R_2}}$
$\boldsymbol{= \{(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3),(4,4)\}}$.

(3) Let *A* be any non- empty set

(a) Prove that the intersection of a two reflexive relations on A is reflexive.

(b) Prove that the union of a reflexive relation on *A* with any relation on *A is a reflexive relation on A.*

(c) Prove that the intersection of two transitive relations on A is transitive. What can you say about their union?

(d) Prove the intersection of an antisymmetric relation with any relation must be antisymmetric.

Solution:

(a) **Let** R and S be two reflexive relations on A.

Then by definition

$\forall a \in A, (a, a) \in R$ (1)

$\forall a \in A, (a, a) \in S$ (2)

Hence ,$\forall a \in A, \quad (a, a) \in R \cap S$.

This means that $R \cap S$ is reflexive.

(b) Let R be a reflexive relation on *A* and S be any relation on *A*. Then $\forall a \in A$, we have $(a, a) \in R$ (since *R* is reflexive) this implies that $(a, a) \in R \cup S, \forall a \in A$. Hence, $R \cup S$ is reflexive.

(c) Let R and S be two transitive relations on A.

$\forall a, b, c \in A$, let $(a, b), (b, c) \in R \cap S$.

Then $(a, b) \in R \wedge (b, c) \in R$ (1),

$(a, b) \in S \wedge (b, c) \in S$ (2).

From (1), since R is transitive, we get $(a, c) \in R$ (3).

From (2), since S is transitive one gets: $(a, c \in) S$ (4).

From (3) and (4), it follows that $(a, c) \in R \cap S$.

This means that $R \cap S$ is transitive.

The union of R and S need not be transitive to show this we the following example: let $A = \{1,2\}$, $R \subseteq A \times A$, and S$\subseteq A \times A$ be defined as follows : $R = \{(1,2)\}$ and $S = \{(2,1)\}$.

Then R and S are transitive relations on A.(try to show this using mathematical logic).

Now $R \cup S = \{(1,2) \cup (2,1)\}$ which is not transitive since $(1,1) \notin R \cup S$.

(C) let R be an antisymmetric relation on A and let S be any

relation on A (need not be antisymmetric).

For all $a, b \in A$, let $(a, b) \in R \cap S$ and $(b, a) \in R \cap S$.

Hence, $(a, b) \in R$ (1).

$(a, b) \in S$ (2).

$(b, a) \in R$ (3).

$(b, a) \in S$ (4).

From (1) and (3),since R is assumed to be antisymmetric, it follows that $a = b$. This means that $R \cap S$ is antisymmetric relation on A.