Cairo University. **Discrete Structures (Msc521).**

FGSSR. **Lecture. 13**

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Relations and Their Properties

Definition 13.1: Let *A* and *B* be sets. A binary relation from *A* to *B* is a subset of $A \times B$.

Example 13.1: Let $A = \{a, b, c\}$ and $B = \{1,2\}$.

$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$. (1)

Now, by definition, any subset of (1) will be a relation from *A* to *B*. Since $|A \times B| = 6$, therefore we have 2^6= 64 relations from *A* to B. Some examples of these relations are ∅ [of size zero] and *AxB* [of size 6] .

How many relations from *A* to *B* of size one? We have $\mathrm{C}(6,1) = 6$ relations from *A* to B, some examples are $\{(a, 1)\}$,$\{(a, 2)\}$, $\{(b, 1)\}$, and$\{(b, 2)\}$.

How many relations from A to B of size 2? We have $C(6,2)$= $\frac{6\times5}{2} = 15$ relations. Some examples are:$\{(a, 1), (a, 2)\}$,$\{(a, 1), (b, 1)\}$, $\{(a, 1), (b, 2)\}$,and$\{(a, 2), (b, 2)\}$.

How many relations from A to B of size three? We have $\mathrm{C}(6,3) = \frac{6\times5\times4}{3\times2\times1}$=20 relations. Some examples are: $\{(a, 1), (a, 2), (b, 1)\}$, $\{(a, 1), (a, 2), (b, 2)\}$, $\{(a, 1), (a, 2), (c, 1)\}$ and $\{(a, 1), (a, 2), (c, 2)\}$.

How many relations from A to B of size four? We have $C(6,4) = \frac{6\times5\times4\times3}{4\times3\times2\times1} = 15$.

Some examples are : $\{(a, 1), (a, 2), (b, 1), (b, 2)\}$,$\{(a, 1), (a, 2), (b, 1), (c, 1)\}$,

$\{(a, 1), (a, 2)(b, 1), (c, 2)\}$, and $\{(a, 1), (a, 2), (b, 2), (c, 1)\}$.

How many relations from A to B of size five? We have $C(6,5) = \frac{6\times5\times4\times3\times2\times1}{5!} = 6$.

some examples are: $\{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1)\}$ and $\{(a, 1), (c, 2), (a, 2), (b, 2), (c, 1)\}$.

Definition 13.2: A relation on the set A is a relation from A to A. i.e., it is a subset of $A \times A$.

Example 13.2: Let $A = \{1,2,3,4\}$, Let $R \subseteq A \times A$ defined by

$R = \{(a, b): a\ divides\ b\}$. Write R.

Solution: $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$.

Example 13.3: Let $A = \{a_i: 1 \leq i \leq n\}$. How many relations on *A?*

Since a relation on A is a subset of $A \times A$, we conclude that the number of relations on *A* equals to the number of subsets of $A \times A$

$= 2^{|A\times A|} = 2^{n^2}$.

Definition 13.3: A relation R on a set *A* is called reflexive if $(a, a) \in R$ for every *a* $\in A$. This means the statement : "$(a \in A) \to (a, a) \in R$"is tauto;ogy.

Example 13.4: consider the following relations on A= $\{1,2,3,4\}$:

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$.

$R_2 = \{(1,1), (1,2), (2,1)\}$.

$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$.

$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$.

$R_5 = \{(3,4)\}$.

Which of these relations are reflexive?

$R_1$is not reflexive since 3 $\in A$ but $(3,3) \notin R_1$. Note that the statement

"if $a \in A$ then $(a, a) \in R_1$" is not tautology (Chick this using truth table method).

R_2 is not reflexive since 2 , 3 $\in A$ but $(2,2) \notin R_2$ and $(3,3) \notin R_2$. Note that the statement " if $a \in A$ then $(a, a) \in R_2$" is not tautology. (Chick this using truth table method).

R_3 is reflexive since " if $a \in A \to (a, a) \in R_3$ "is tautology.If *a=1,* $1 \in A$ *is true and* $(1,1) \in R_3$ is true, hence the "if statement" is true. Repeat this for all values of *a.*

R_4 is not reflexive since the statement "if $a \in A \to (a,a) \in R_4$" is not tautology.(for example at a=1,1$\in A$ is true, $(1,1) \in R_4$ is false) .

R_5 is not reflexive since 1$\in A$ but $(1,1) \notin R_5$.

Lemma (13.1). Let *A* be a set of size n. Then there are $2^{(n^2-n)}$reflexive relations on *A*.

Proof: Let $A = \{a_1, a_2, a_3, \cdots, a_n\}$ and $R \subseteq A \times A$ be any relation on *A*.

In order that *R* be a reflexive relation on *A, it must* be of the form:

$R = MD \cup B$, where $MD = \{(a_1,a_1),(a_2,a_2),(a_3,a_3),\cdots,(a_n,a_n)\}$

B$= any\ subset\ of\ [(A \times A) - MD]$.

The MD can be chosen with one way only while *B* can be chosen with $2^{(n^2-n)}$ ways. Therefore by the product rule of counting, the number of reflexive relations on A will be $1 \times 2^{(n^2-n)} = 2^{(n^2-n)}$.

Definition 13.4: Let R be a relation on A. R is called symmetric if for all a,$b \in A$, $(a,b) \in R$ if and only if $(b,a) \in R$.(i.e., the statement, "$\forall a,b \in A, (a,b) \in R \leftrightarrow (b,a) \in R$") is a tautology.)

Example 13.5: in the previous example, R_1 is not symmetric since

$(3,4) \in R_1$ but $(4,3) \notin R_1$

R_2 is symmetric since the statement " $\forall a,b \in A,\ (a,b) \in R_2 \leftrightarrow (b,a) \in R_2$" is tautology.

R_3 is also symmetric for the same reason.

R_4 is not symmetric since $(2,1) \in R_4$ but $(1,2) \notin R_4$.

Therefore the statement " for all $a,b \in A$, $(a,b) \in R_4 \leftrightarrow (b,a) \in R_4$" is not a tautology.

R_5 is a symmetric relation. Why?

Lemma 13.2: Let *A* be any set of size *n.* There are $2^{(n^2+n)/2}$ symmetric relations on *A.*

Proof: Let $MD = \{(a_i, a_i): i \in N\}$,where *N* is the set of natural numbers.

And $B = \{(a_i, a_j): 1 \leq i < j \leq N,\ i, j \in N\}$. Note that any subset of *MD* is symmetric. Also, any subset K of *B* such that $(a_i, a_j) \in K$ iff $(a_j, a_i) \in K$, is symmetric. Therefore, for any relation on *A* to be symmetric, it must be the union of subsets one from *MD* and the other is a subset of *K*. It is easy to see that there are 2^n subsets of *MD,* and $2^{\left(2^{n^2-n}\right)/2}$ subsets of *K*. Hence, by the product rule of counting, it follows that there are $2^n \times 2^{(n^2-n)/2} = 2^{(n^2+n)/2}$ symmetric relations on *A.*

Lemma 13.3: Let *A* be any set of size *n.* There are $2^{(n^2-n)/2}$ relations on *A* which is reflexive and symmetric.

Proof: Note that the *MD* set is symmetric and reflexive. Therefore, for a relation, R, on A to be symmetric and reflexive it must contain *MD*. This means that: $R = MD \cup K$. Hence, by the product rule of counting, the number of symmetric and reflexive relations on *A* $= 1 \times 2^{(n^2-n)/2} = 2^{(n^2-n)/2}$.

Example 13.6: Let $A = \{1,2\}$. There are $2^4 = 16$ relations on *A*. There are $2^2 = 4$ reflexive relations on *A*, namely, they are :

$R_1 = MD = \{(1.1), (2,2)\}$, $R_2 = MD \cup \{(1,2)\}$, $R_3 = MD \cup \{(2,1)\}$, and $R_4 = MD \cup \{(1,2), (2,1)\}$.

There are $2^3 = 8$, symmetric relations on *A,* namely, $R_5 = \emptyset$, $R_6 = A \times A$, $R_7 = MD$, $R_8 = \{(1,2), (2,1), (1,1)\}$, $R_9 = \{(1,2), (2,1), (2,2)\}$ and $R_{10} = \{(1,2), (2,1)\}$.

There are $2^1 = 2$ relations on *A* which are reflexive and symmetric , namely, $R_{11} = MD$, $R_{12} = A \times A$.

Definition 13.5: A relation $R \subseteq A \times A$ is called antisymmetric if the statement " for all $a, b \in A$, if $[(a, b) \in R \wedge (b, a) \in R] \rightarrow (a = b)$ " is a tautology.

Example 13.7: in example 13.4, $R_1, R_2, and\ R_3$ are not antisymmetric since :

(1) $(1,2) \in R_1 \wedge (2,1) \in R_1 but\ 1 \neq 2$. [note that the truth value of "$[(1,2) \in R_1 \wedge (2,1) \in R_1] \rightarrow (1 = 2)$ is F

(2) $(2,1) \in R_2 \wedge (1,2) \in R_2$ but $1 \neq 2$.

(3) $(1,4) \in R_3 \wedge (4,1) \in R_3$ but $1 \neq 4$.

The relations R_4, R_5 are antisymmetric since the statement

" $\forall a, b \in A \quad [(a,b) \in R_4 \wedge (b,a) \in R_4] \rightarrow (a = b)$" is tautology.

Similar condition concerning R_5 can be written .

Definition 13.6: A relation $R \subseteq A \times A$ is called transitive if the following statement "$\forall a, b, c \in A, [(a,b) \in R \wedge (b,c) \in R] \rightarrow (a,c) \in R$" is a tautology.

Example 13.8: in example 13.4

R_2 is not transitive since $(2,1) \in R_2 \wedge (1,2) \in R_2$ but $(2,2) \notin R_2$ .All other relations stated in example 13.4 are transitive.

Homework : write the truth table for statement in definition 13.6 to prove that these relations are transitive.

Definition 13.7: A relation $R \subseteq A \times A$ is called an irreflexive relation on *A* iff the following statement is tautology: " if $a \in A$ then $(a,a) \notin R$" .

Lemma 13.4: Let *A* be any set of size *n*. there are $2^{(n^2-n)}$ irreflexive relations on *A.*

Proof: Let $R \subseteq A \times A$, $|A| = n$. For *R* to be an irreflexive relation it must be free from (a,a) for every $a \in A$. Therefore, it will be any subset of $(A \times A) - MD$. But $|(A \times A) - MD| = n^2 - n$. Hence, there are $2^{(n^2-n)}$ irreflexive relations on *A* .

Example 13.9: Let *A*= $\{1,2\}$. There are $2^2 = 4$ irreflexive relations on *A*, namely, $R_1 = \emptyset$, $R_2 = \{(1,2)\}$, $R_2 = \{(2,1)\}$, and $R_4 = \{(1,2),(2,1)\}$**.**