Cairo University. **Discrete Structures (Msc521).**

FGSSR. **Lecture 12.**

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Functions

Definition 12.1: Let A and B be sets. A function f from A to B is an assignment of a unique element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f: A \to B$.

Example 12.1: Let $f: A \to B$

$A = \{i: is\ a \text{ positive } integer\ < 6\}$, $B = \{1,2,3,4,5\}$**,** and

$$f(i) = \begin{cases} 1 & i \in \{1,5\} \\ 2 & i \in \{2,4\} \\ 3 & i \in \{3\} \end{cases}$$

Remark 12.1: *f* is a function since the following conditions are satisfied:

(1) Each element in *A* has an image in *B*.
(2) This image is unique.

Remark 12.2: The function *f* can be written as follows:

$f = \{(1,1), (5,1), (2,2), (4,2), (3,3)\} \subseteq A \times B$.

If we have an expression written in this formula, then in order that it represents a function, we must check that each element in *A* appears in f as a first component only one time .

Definition 12.2: Let $f:\ A \to B$. We say that *A* is the domain of *f* and *B* is the codomain of *f*. If $f(a) = b$, we say that *b* is the image of *a* and *a* is a preimage of *b* . The range of *f* is the set of all images of *A*. Also if *f* a function from *A* into *B*, we say that *f* maps *A* to *B*.

Example 12.2: In the previous example,

Domain of f is $A = \{1,2,3,4,5\}$,

Codomain of f is B=$\{1,2,3,4,5\}$, Range $f = \{1,2,3\}$, where,

The image of both 1 and 5 is 1, the image of both 2 and 4 is 2, and the image of 3 is 3.

Example 12.3: Let Z be the set of integers, i.e.,

$Z = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$, define $f: Z \rightarrow Z$ by:

$$f(z) = z^2$$

Domain *f= codomain f* = Z

Range *f*=$\{0,1,4,9,16,\cdots\} = \{z^2: z \in Z\}$.

The image of "-2"=the image of "2" = 4.

-2 ∈ codomain *f* has no preimage since there no $x \in Z$ (domain) such that: $f(a) = a^2 = -2$.

Also, 2 has no preimage since there is no *a* belonging to Z such that $f(a) = a^2 = 2$. (note that √2 is not an integer).

Definition 12.3: A real valued function is a function whose codomain is a subset of the real numbers (i.e., $B \subseteq \mathbb{R}$).

Definition 12.4: Let f_1 **and** f_2 **be two real valued functions having the same domain. Then** $f_1 + f_2$ **and** $f_1 f_2$ **are two real valued functions defined by :**

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \quad \text{and}$$

$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

Example 12.4:

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ **and** $g: \mathbb{R} \rightarrow \mathbb{R}$ **be defined as follows:**

$f(x) = x^2$ and $g(x) = x - x^2$ **find** $f + g$, **f-g, and** fg

Solution: $f + g: \mathbb{R} \to \mathbb{R}$ **, where**

$(f + g)(x) = f(x) + g(x) = x^2 + x - x^2 = x.$

$(f - g): \mathbb{R} \to \mathbb{R}$, where,

$(f - g)(x) = f(x) - g(x) = x^2 - (x - x^2) = 2x^2 - x.$

$fg: \mathbb{R} \to \mathbb{R}$**, where**

$(fg)(x) = f(x)g(x) = x^2(x - x^2) = x^3 - x^4.$

Definition 12.5: Let $f: A \to B$ be a function and $S \subseteq A$. The image of *S* under *f* is defined to be the set of all images of its elements, i.e., $f(S) = \{f(x): x \in S\}$.

Example 12.5: in example 12.1, we have:

$f(\{1, 5\}) = \{1\}$, $f(\{2, 4\}) = \{2\}$, $f(\{3\}) = \{3\}$, **and**

$f(A) = B.$

Remark 12.2: The student must note the difference between:

"$f(3) = 3$" and "$f(\{3\}) = \{3\}$" **.**

Each statement is true but they have different meanings.

Example12.6: Let $A = \{1,2,3,4,5\}$ and $B = \{x, y, z, w\}$ and let $f: A \longrightarrow B$ be defined by:

$f = \{(1, w), (2, x), (3, x), (4, y), (5, y)\}.$

Verify that f is a function and find the images of the following sets:

$A_1 = \{1\}$ and $= \{1,2\}$, $A_3 = \{1,2,3\}$, $A_4 = \{2,3\}$, $A_5 = \{2,3,4,5\}$.

SOLUTION: f is a function since every element in *A* appears as a first coordinate in *f* only one time.

$f(A_1) = \{w\}$, $f(A_2) = \{x, w\}$, $f(A_3) = \{x, w\}$,

$f(A_4) = \{x\}$, and $f(A_5) = \{x, y\}$.

Example 12.7: Let $g: \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^2$.Find

(a) Range of *g, where* $\mathbb{R}$ is the set of real numbers ,
(b) image of Z (the set of integers)
(c) image of $A_1 = [-2,1]$.

Solution:

since for any $x \epsilon \mathbb{R}$, $0 \le x^2$ it follows that:

(a) range of g $= [0, \infty)$.
(b) Image of Z= $\{, 0,1,4,9, 16, \cdots\}$.
(c) Note that $[-2,1] = [-2,0) \cup [0,1]$

If $-2 \le x < 0$, **then** $0 < x^2 \le 4$.

If $0 \le x \le 1$, **then** $0 \le x^2 \le 1$ **, hence** $g([-2,1]) = [0,4]$.

Types of Functions

(a) One –to- one (injective) Functions

Definition 12.6: A function $f: A \to B$ *is said to be one -to-one* or injective if and only if different elements in domain have different images in codomain. In mathematical form we write *f is* injective iff $f(x) = f(y)$ implies that $x = y$ for all $x, y \in A$.
Remark 12.3: this condition is equivalent to the contrapositive statement: if $x \neq y$ then $f(x) \neq f(y)$ for all $x, y \in A$.
Example 12.8: Let $f: Z \to Z$, defined by $f(x) = x^2$, where *Z* is the set of integers. Show that *f* is not injection and find the image of *Z.*

Solution: Note that $f(-1) = f(1) = 1$, hence f is not one -t-one.
Now $\boldsymbol{f(Z) = \{f(x): x \in Z\} = \{x^2: x \in Z\} = \{0, 1, 4, 9, 16, \cdots\}}$.

Example 12.9: Let $f: Z \to Z$ be such that $f(x) = 2x + 1$.
Prove that $f(x)$ is injection.
Proof:
Assume $f(x) = f(y)$ then $2x + 1 = 2y + 1$adding -1 to both sides and dividing the result by 2, we get: x=y.

(b) Increasing (decreasing) functions:
Definition 12.7: Let $f: A \to B$, where $A, B \subseteq \mathbb{R}$. Then

(1) " If $x < y$ then $f(x) < f(y)\ for\ all\ x, y\ in\ A$ " then f

Is called strictly increasing function.

(2 **) " if** $x < y$ *then* $f(x) \leq f(y)$ *for all x,y in A* then f is called

monotone Increasing"

(3) " if $x < y$ then $f(x) > f(y)$ for x,y in A" then f is called strictly decreasing function.
(4) " if $x < y$ then $f(x) \geq f(y)$ for all x, y in A" then f is called monotone decreasing.

Example 12.10: (1) $\boldsymbol{f(x) = \frac{1}{x}}$ $, \boldsymbol{x} > 1$is a strictly decreasing function since for any $x, y \in (1, \infty)$, we have $1 < x < y < \infty \Rightarrow$ $0 < \frac{1}{y} < \frac{1}{x} < 1 \Longrightarrow f(y) < f(x)$.

(2) $f(x) = \frac{1}{x}$, $x \in (0,1)$ is a strictly decreasing function since

For every $x, y \in (0,1)$, we have $0 < x < y < 1 \Longrightarrow$

$1 < \frac{1}{y} < \frac{1}{x} < \infty$ i.,e., $f(y) < f(x)$.

Example 12.11: Let $\boldsymbol{f}$: $\mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$.

Is f increasing? Find range f and the image of $(-3,2)$, $\{-3,2\}$, $[-3.2)$, and $(-1,5]$

Solution:

Note that $\mathbb{R} = (-\infty, 0) \cup [0, \infty)$. Let $x, y \in (-\infty, 0)$ such that: $x < y$. Then $x^2 > y^2$. This means $x < y \Longrightarrow f(x) > f(y)$. Hence $f(x)$ is a strictly decreasing function in $(-\infty, 0)$. Let $x, y \in [0, \infty)$ such that: $x < y$. Then $x^2 < y^2$.

This means $x < y \Longrightarrow f(x) < f(y)$. Hence $f(x)$ is a strictly increasing function in $[0, \infty)$.

For the image of $(-3,2) = (-3,0) \cup [0,2)$.

-3$< x < 0 \Longrightarrow 0 < x^2 < 9 \Longrightarrow f(x) \in (0,9)$.

$0 \le x < 2 \Longrightarrow 0 \le x^2 < 4 \Longrightarrow f(x) \in [0,4)$.

Hence $f\big((-3,2)\big) = [0,9)$.

Now $f(-3) = 9$ and $f(2) = 4$. Therefore $f(\{-3,2\})$.

Homework: Find $f([-3,2))$ and $f((0,5])$.

(c) Surjective (onto) function.

Definition 12.8: A function *f* from A to B is called onto ,or surjective if and only if every $b \in B$ has at least one preimage x in A such that $f(x) = b$.

Example 12.12: in example 12.8, f is not onto since "-2" has no preimage .

Example 12.13: Let $f: \mathbb{R} \to \mathbb{R}$, given by $f(x) = 5x - 1$.

Then *f* is onto since every *y* in the codomain has preimage $\frac{y+1}{5}$

In the domain.

(d) A one -to- one onto (bijection)function.

Let $f: A \to B$. then f is called bijection or one-to-one onto if and only if it is one-to-one and onto.

. Also the function *f* is called 1-1 correspondence

Example 12.14 : the function defined in example 12.12 is one-to-one and onto therefore it is bijection.

Example 12.15: Let *f*: *A*→ B, $A = \{a, b, c, d\}$, and $B = \{1,2,3,4\}$ such that : $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$.

At first, *f* is a function since every $x \in A$ has a unique image in *B*.

In addition *f* is an injection since different elements in domain has different images in codomain. Finally, f is a surjection every element $y \in B$ has a preimage $x \in A$.

Definition 12.9: Let *A* be a set. The identity function on *A* is the function $i_A: A \to A$, where $i_A(x) = x, \forall x \in A$.

Remarks:

(1) The identity function assigns each element to itself.

(2) The identity function is a bijection.

Example 12.16: Let $f: \mathbb{R} \to \mathbb{R}$ **,** defined by $f(x) = \boldsymbol{x}$, $\forall x \in \mathbb{R}$,

then *f* is an identity function on $\mathbb{R}$.

Definition 12.10 :Inverse Functions and Compositions of Functions.

Assume that *f* is a bijection from *A* to *B.* We note that each element *b* in *B* has exactly one preimage *a* in *A*. therefore, we can define new function, f^{-1}, from *B* to *A* as follows : $f^{-1}(b) = a$, where $f(a) = b$.

Example 12.17: Find the inverse function of the function defined in example 12.14.

Solution: Define $\boldsymbol{f^{-1}}: \boldsymbol{B} \to \boldsymbol{A}$ as follows:

$f^{-1}(4) = a,\ f^{-1}(2) = b,\ f^{-1}(c) = 1$, and $f^{-1}(3) = d$.

Example 12.18: Let $f: Z \to Z$, be defined by $f(x) = x + 1$.

Show that *f* is invertible and find its inverse.

Solution:

(1) *f* is one to one

Proof: let $f(x) = f(y)$. Then $x + 1 = y + 1$.subtracting 1 from each side, we get *x*=*y* .

(2) To prove that *f* is onto let *y* ∈ Z. If *y* is an image for some *x* in *Z,* it must be of the form *y=x+1,* hence, *x*=y-1∈ Z for all $y \in Z$.

Hence f is a bijection so it has an inverse.

$$f^{-1}(y) = y - 1, \qquad \forall y \in Z .$$

Definition 12.10 (Composition of functions).

Let $g: A \to B$, let $f: B \to C$. The composition of the functions f and g, denoted by $f^\circ g$, is defined by $f^\circ g: A \to C$

$$f^\circ g(a) = f\big(g(a)\big)$$

Example 12.19: Let $g: A \to A$, $\quad f: A \to C$,where

$A = \{a, b, c\}$, $\quad C = \{1,2,3\}$ such that $g(a) = b$, $g(b) =$ c,

and $\quad g(c) = a$, $f(a) = 3$, $f(b) = 2$ and $f(c) = 1$. Then

$f^\circ g: A \to C$, $\quad$ such that:

$(f^\circ g)(a) = f\big(g(a)\big) = f(b) = 2$,

$(f^\circ g)(b) = f\big(g(b)\big) = f(c) = 1$, and

$(f^\circ g)(c) = f\big(g(c)\big) = f(a) = 3$.

Example 12.20: Let $f: Z \to Z$, defined by $f(x) = 2x + 3$,

$g: Z \to Z$, defined by $g(x) = 3x + 2$.

Find $\quad f^\circ g$ and $g^\circ f$

Solution:

$$f^\circ g: Z \to Z$$

$$(f^\circ g)(x) = f\big(g(x)\big) = f(3x + 2) =$$

$$2(3x + 2) + 3 = 6x + 7.$$

$$g^\circ f: Z \to Z ,$$

$$(g^\circ f)(x) = g\big(f(x)\big) = g(2x + 3) = 3(2x + 3) + 2$$

$$= 6x + 11.$$

Example 12.21: Let $f: \mathbb{R} \to \mathbb{R}$, defined by $f(x) = 2x + 3$.

Show that f is invertible and find f^{-1}, $f^\circ f^{-1}$, and $f^{-1\circ} f$.

Solution: we are to show that f is an injection and a surjection.

To prove that f is injection , let $f(x) = f(y)$, then

2x+3= 2y+3. Subtracting 3 from both sides and dividing the result by 2, one gets: x=y.

To prove that f is a surjection, let $y \in \mathbb{R}$,write $y = 2x + 3$. Solving this equation for x , we get: $x = \frac{y-3}{2} \in \mathbb{R}$. Hence for every $y \in \mathbb{R}$ (the codomain) there exists a preimage $\frac{y-3}{2} \in \mathbb{R}$ (the domain).

Hence f is a bijection and it is invertible.

$f^{-1}: \mathbb{R} \to \mathbb{R}$,

$f^{-1}(x) = \frac{x-3}{2}$, $x \in \mathbb{R}$.

$f^\circ f^{-1}: \mathbb{R} \to \mathbb{R}$,

$(f^\circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{x-3}{2}\right) = 2\left(\frac{x-3}{2}\right) + 3 = x.$

$$f^\circ f^{-1}: \mathbb{R} \to \mathbb{R}$$

$(f^{-1\circ} f)\ (x) = f^{-1}(f(x)) = f^{-1}(2x+3) = \frac{2x+3-3}{2} = x.$

Important Problem : Let A and B be two sets such that $|A| = m$ and n.

(1) How many functions can be defined from A into B.
(2) How many one to one functions can be defined from A into B.
(3) How many bijections can be defined from A onto B.

Solution:

Let $A = \{a_1, a_2, a_3, \cdots, a_m\}$, and $B = \{b_1, b_2, b_3, \cdots, b_n\}$.

(1) Any function *f* from *A* into *B* can be written as follows:

$f = \{(a_1, x_1), (a_2, x_2), (a_3, x_3), \cdots, (a_m, x_m)\}$, where each $a_i, i = 1,2, \cdots m$, will appear only one time as a first component in *f. Now* x_1 can be chosen from *B* by n ways. Also, *y* can be an onto multiplication rule of counting, we conclude that the number of functions from *A* i*nto B* will be $n.n.\cdots n = n^m$.

(2) Note that if *f* is a one to one function from *A* into *B* then $m \leq n$. Therefore the image x_1 of a_1 can be chosen by *n* ways. Therefore the image x_2 of a_2 can be chosen by *n-1* ways (since it must differ from the image of a_1). In the same manner the image x_3 of a_3 can be chosen by *n*-2 ways. Continuing in this manner we see that the last image x_m of a_m can be chosen by *m-m+1* ways. So using the multiplication rule of counting we see that the number of one to one functions from *A* into *B will be :*

$$n.(n-1).(n-2)\cdots(n-m+1) = P(n,m).$$

(3) Note that if *f* is a bijection from *A* into *B* then *m=n*. therefore using the equation deduced in (2) with *m=n,* we conclude that the number of bijections from *A* onto *B* will be $P(n,n) = n.(n-1).(n-2)\cdots 2.1 = n!$