Cairo University — Discrete Structures (Msc521)

FGSSR — Lecture 4

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Fundamentals of Logic

Definition 4.1 A proposition or a statement is a declarative sentence that is either true or false but not both.

Example 4.1: The following sentences are propositions:

P: Combinatorics is a required course for probability.

Q: 2+3=7.

R: Margarete Mitchel wrote Gone with Mind.

Remarks 4.1:

(1) Each statement is a primitive statement since it cannot be broken down into simpler statement.

(2) Each statement has one truth value (true or false) but not both ,i.e., statement P is true, statement Q is false, and statement R is true.

New statements can be obtained from existing ones in some ways as follows:

(1)Transform a given statement P into the statement $\neg P$ which denotes its negation and is read as "Not P".

Example 4.2:

$\neg P$: Combinatorics is not a required course for Probability.

$\neg Q$: $2+3\neq 7$.

(2)Combine two or more statements into a compound statements the following logical connectives:

(a) **Conjunction**: The conjunction of any statements P, Q is denoted by P$\wedge Q$ and read as $P\ and\ Q$, For example, "combinatorics is required course for probability and Margarete wrote Gone with Wind".

(b) **Disjunction**: The expression $P \vee Q$ denote the disjunction of any two statements P, Q and is read as " "$P\ or\ Q$". For example, "Combinatorics is a required course for probability or Margarete wrote Gone with Wind" .

(c) **Implication** : We say that " $P\ implies\ Q$" and write $P \rightarrow Q$ to designate the sentence, " if $P\ then\ Q$". For example," if combinatorics is a required course for probability then Margarete wrote Gone with Mind".P is called premise (antecedent),or hypothesis of the implication while Q is called consequent or conclusion of the implication. Here are some of the ways that used to express that $P \rightarrow Q$ is true

(1) $Q\ if\ P$

(2) If P is true then Q is true

(3) P is true only if $Q$is true.

(4) For P to be true it is necessary that Q be true.

(5) For Q to be true, it is sufficient that P be true.

Remark 4.2

Note that when statements are combined in this manner, there need not be any causal relationship between the statements for the implication to be true.

(d)Bicoditional: The biconditional of two statements P, Q is denoted by $P \leftrightarrow Q$, which is read " P if and only if Q"or "P is necessary and sufficient for Q". For example, Combinatorics is a required course for if and only if $2 + 3 = 7$. We can write "P if and only if Q" as "P iff Q".

Remarks 4.3 :

(1) A sentence such as "The number x is an integer" is not a statement because its truth value (true or false) cannot be determined until a numerical value is assigned for x.

(2) Sentences such as the exclamation " What a beautiful evening" are not statement.

(3) Sentences like the command " Get up and do your exercises" are not statement.

The following tables give the possible truth values of the compound statements:

(a)Negation:

P	$\neg P$
T	F
F	T

(b) Conjunction :

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Note that the compound statement will be true only when each component is true.

(c) Inclusive Disjunction:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Note that the compound statement will be false only when each component is false.

(d) Exclusive Disjunction:

P	Q	$P \underline{\vee} Q$
T	T	F
T	F	T
F	T	T
F	F	F

Note that the compound statement is true only when only one of its components is true.

(e) Conditional" If then":

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Note that the compound statement is false only when the hypothesis is true and the conclusion is false.

(f) Biconditional "iff":

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Note that the compound statement is true only when both components have the same truth value.

<u>Example 4.3:</u> Let P be the proposition "$2+3=5$" and Q be the proposition " the moon is made of cheese" Find the symbolic form of the following statements:

(a) $2+3 \neq 5$.
(b) $2+3 \neq 5$ and the moon is not made of cheese.
(c) 2+3$\neq$ 5 or the moon is made of cheese.
(d) If 2+3$\neq$ 5 then the moon is made of cheese.

(e) That the moon is made of cheese implies that $2 + 3 \neq 5$.

Solution:

(a) $\neg P$.
(b) $\neg P \wedge \neg Q$.
(c) $\neg P \vee Q$.
(d) $\neg P \rightarrow Q$.
(e) $Q \rightarrow \neg P$.

Example 4.4: Let P, Q be primitive statements for which the implication $P \rightarrow Q$ is false. Determine the truth values for :

(a) $P \wedge Q$ (b) $\neg P \vee Q$. (c) $\neg Q \rightarrow \neg P$

Solution : Since $P \rightarrow$Q is false, we conclude that P is true ,and Q is false,

Hence, $\neg P$ is false and $\neg Q$ is true, therefore (a) is false, (b) is false, and (c) is true .

Example 4.5 :

Construct a truth table for each of the following compound statements , P, Q, and r denote primitive statements.

(a) $\neg(P \vee \neg Q) \rightarrow \neg P$. **(b)** $P \rightarrow (Q \rightarrow r)$. **(c)** $(\boldsymbol{P} \rightarrow \boldsymbol{Q}) \rightarrow \boldsymbol{r}$.

(d) $(\boldsymbol{P} \rightarrow \mathrm{Q}) \rightarrow (\mathrm{Q} \rightarrow \mathrm{P})$. $(\boldsymbol{e})$ $[P \wedge (P \rightarrow Q)] \rightarrow Q$.

(f) $(P \wedge Q) \rightarrow P$.

(g) $\boldsymbol{Q} \leftrightarrow (\neg \boldsymbol{P} \vee \neg \boldsymbol{Q})$.

(h) $[(\boldsymbol{P} \rightarrow \boldsymbol{Q}) \wedge (\boldsymbol{Q} \rightarrow \boldsymbol{r})] \rightarrow (\boldsymbol{P} \rightarrow \boldsymbol{r})$.

We shall solve (a) and (b) as examples and leave the rest for the students.

Solution: (a)

P	Q	$\grave{P}$	$\grave{Q}$	A	$\grave{A}$	C
T	T	F	F	T	F	T
T	F	T	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	T	F	T

Where we write $\neg P$ as $\grave{P}$, $\neg Q$ as $\grave{Q}$, $P \vee \neg Q$ as A,$\neg(P \vee \neg Q)$ as $\grave{A}$,and $\neg(P \vee \neg Q) \rightarrow \neg P$ as C .

(b)

P	Q	r	A	B
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

Where, we write , $Q \rightarrow r$ as A, and $P \rightarrow (Q \rightarrow r)$ as B.