Cairo University **Msc521 (Discrete Structures)**

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Permutations with Repetition

(3.1) Introduction.

Suppose that we have $A = \{b, a, c\}$ and we want to find some 4-permutations of A in which any element of $\{a, b\}$ appears only one time while c appears two times. In this case, we have permutations in the form: $abcc$, $acbc$, $accb$ $\cdots$etc.

The question now is: how can we find the number of such permutations. To achieve this work manually, let $A = \{a, b, c_1, c_2\}$,then the number of 4-permutations (using the product rule) is $4! = 24$. Write all this 24 permutations, then remove the indices 1,2 so that the permutations abc_1c_2 , and abc_2c_1 will be $abcc$. Continuing in this manner, we shall find that the 4-permutations will be 12.

In the same manner, one can think of 5-permutations of A in which any element of $\{a, b\}$appears only one time while c appears three times. In this case, it is easy to see that we have permutations like: $abccc$, $acbcc$, $accbc$ and so on.

It is clear that, we can think of permutations in which any element of$A$ appears more than one time. The general rule to find the number permutations in case of repetition is given in the following theorem.

Theorem(3.1)

Let $A=\{a_1,a_2,\cdots,a_n\}$. The number of n-permutations in which a_1 appears n_1 times, a_2, appears n_2 times,$\cdots,a_r$ appears r times such that $n_1+n_2+\cdots+n_r=n$ will be:

$$\frac{n!}{n_1!\times n_2!\times\cdots\times n_r!}=\frac{n!}{\prod_{i=1}^{r}n_i!}\quad .$$

Example (3.1): How many arrangements of all of six letters in pepper.

Solution: we have $n=6$, p is repeated 3 times, so $n_1=3$, e is repeated 2 times, so $n_2=2$, and r is repeated one time, so $n_3=1$ such that $n_1+n_2+n_3=6$, hence the number of permutations=$\frac{6!}{3!\times2!\times1!}=\frac{6\times5\times4\times(3!)}{3!\times2!}$=60.

Example (3.2): H0w many arrangements of all letters in the word MASSASAUGA.

Solution: we have m is repeated 1 time, a 4 times, s 3 times, g 1 time, and g 1 time, therefore the number of permutations =

$$\frac{10!}{1!\times4!\times3!\times1!\times1!}=\frac{10\times9\times8\times7\times6\times5\times4!}{1!\times4!\times3!\times1!\times1!}=25200.$$

Example (3.3): Determine the number of (staircase)paths in the x y plane from (2,1) to (7,4) where path is made up of individual steps going one unit to the right (R) or one unit upward (U).

Solution: Note that to move from (2,1) to (7,4) under these conditions you have to move 5 units (7-2=5)along the X-axis and3 units (4-1=3) along the Y-axis, .i.e., we have sequence of steps in the form UUURRRRR. Therefore our question now is "haw many arrangements (8-permutation) in which R is repeated 5 times and U is repeated 5 times?", hence

The number of paths $= \frac{8!}{3!\times 5!} = 56$ paths.

Combination

Let$A = \{a_1, a_2, \cdots a_n\}$. We are interested in the number of subsets of$A$. Clearly the empty set is a subset of$A$. Also, there are n subsets of A of size one namely,$\{a_1\}, \{a_2\}, \cdots, \{a_n\}$. Moreover, there are some subsets of size 2, namely, $\{a_{\mathrm{i}}, a_{\mathrm{j}}\}$, for $i \neq$j and i, j $\in \{1,2, \cdots n\}$, there are some subsets of size 3,namely,$\{a_i, a_j, a_k\}$,for$i \neq j \neq k$ and $i, j, k \in \{1,2 \cdots, n\}$ and so on. Finally, we have one subset of size n, namely A itself.In general, let C be a subset of A of size r and our question now is " how many subsets of A of size r".To answer this question, denote by $C(n,r)$the number of subsets of size r. Note that for have $r!$ arrangements (r-permutation). Therefore, we have the following equation:

$$(r!) \times C(n,r) = P(n,r).$$

Hence,

$$C(n,r) = \frac{P(n,r)}{r!} \quad , \qquad 0 \leq r \leq n.$$

$$= \frac{n!}{r!\times(n-r!)} \quad , \qquad 0 \leq r \leq n.$$

$$= \frac{n\times(n-1)\times(n-2)\times\cdots(n-r+1)}{r!} \quad , \qquad 0 \leq r \leq n.$$

Some special cases:

(1) $C(n,0) = 1$

(2) $C(n,1) = n$

(3) $C(n,2) = \frac{n(n-1)}{2}$

(4) $C(n,n) = 1$

Remarks(3.1)

(1) one must recall that $C(n,r)$means the number of ways to choose r elements from n elements without repetition and neglecting order of choice.

(2) One must recall that $P(n,r)$ means the number of ways to choose r elements from n elements without repetition but taking into account the order of choice.

Example(3.4) :

In how many ways can set of 5 letters be selected from the English letters.

Solution: The word "select" means choosing without order and without repetition. Therefore, number of selection will be $C(26,5) = \frac{P(26,5)}{5!} = \frac{26\times25\times24\times23\times22}{5\times4\times3\times2\times1} =$

$$26 \times 5 \times 23 \times 22$$

Example(3.5):

A club has 25 members.

(a) How many ways are there to choose 4 members of the club to serve on an executive committee?

(b) How many ways are there to choose president, vice president, secretary, and treasurer of the club?

Solution:

(a) Number **of ways** $\frac{\boldsymbol{P(25,4}}{\mathbf{4!}}$**=** $\frac{\mathbf{25\times24\times23\times22}}{\mathbf{4\times3\times2\times1}} = \mathbf{12650}$

In The 2nd part the order is important. therefore,

$$\boldsymbol{P}(\mathbf{25},\mathbf{4}) = \mathbf{25} \times \mathbf{24} \times \mathbf{23} \times\times \mathbf{22} = \mathbf{303600}$$

Example(3.6):

How many license plates consisting of three letters followed by three digits contain no letter or digit twice?

Solution :

Number of license $P(26,3) \times P(10,3) = 11232000$.