FARM MANAGEMENT
Lecture.3
Sign, Slope and Curvature

By

Dr. Mahmoud Arafa

Lecturer of Agricultural Economic, Cairo Un.

Contacts:

E-Mail: mahmoud.arafa@agr.cu.edu.eg

W.S: http://scholar.cu.edu.eg/mahmoudarafa

Sign, Slope and Curvature

By repeatedly differentiating a production function, it is possible to determine accurately the shape of the corresponding *MPP* function. For the production function

$$y = f(x)$$

the first derivative represents the corresponding *MPP* function

$$dy/dx = f'(x) = f1 = MPP$$

Insert a value for x into the function $f'(x)$ equation $dy/dx = f'(x) = f1 = MPP$ If $f'(x)$ (or dy/dx or *MPP*) is positive, then incremental units of input produce additional output.

MPP is negative sign after the production function is down sloping, having already achieved its maximum point.

A negative sign on $f'(x)$ indicates that the underlying production function has a negative slope and has achieved a maximum point.

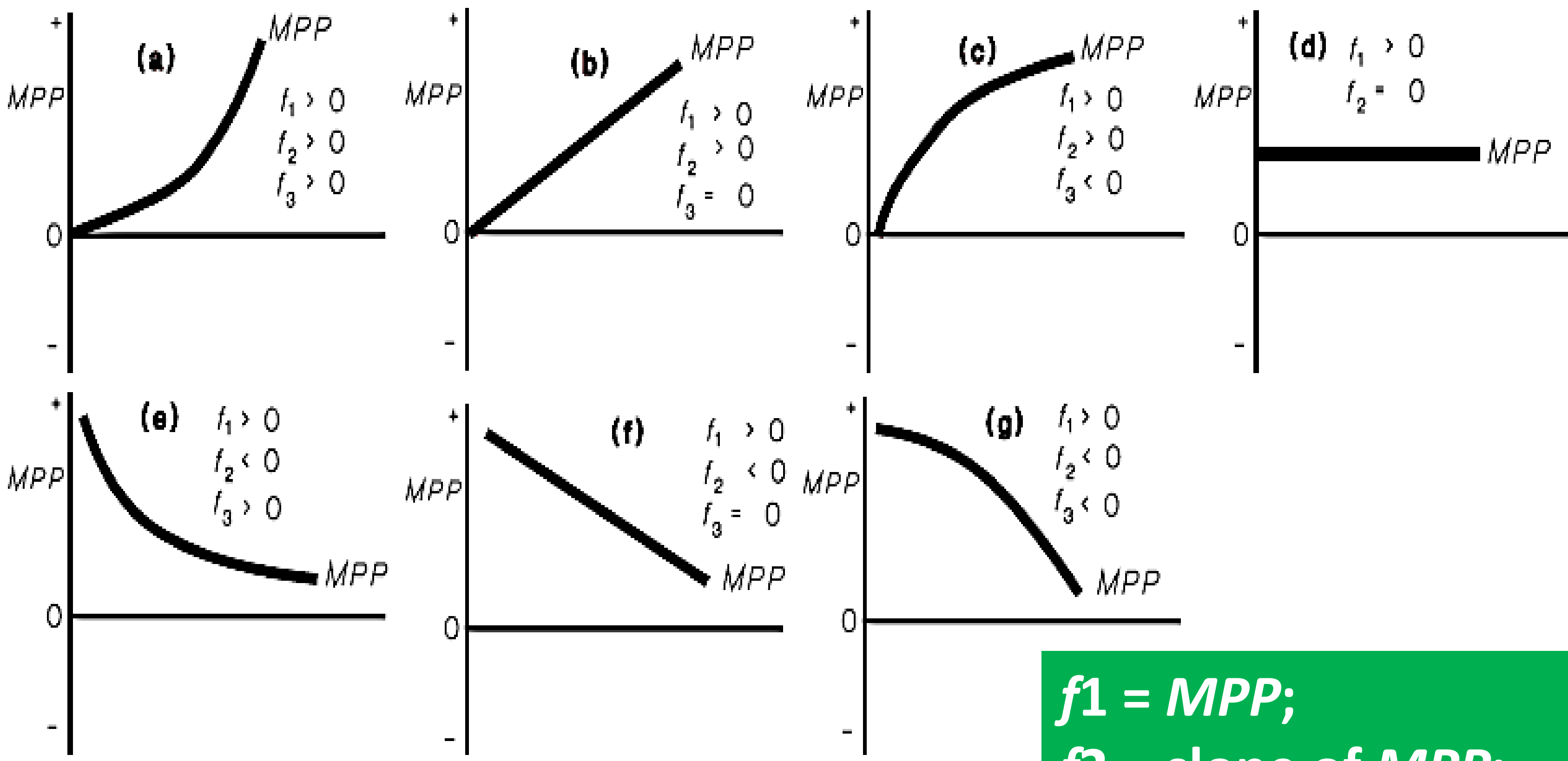
The sign on the first derivative of the production function indicates if

(1) the slope of the production function is positive or negative and,

(2) if *MPP* lies above or below the horizontal axis.

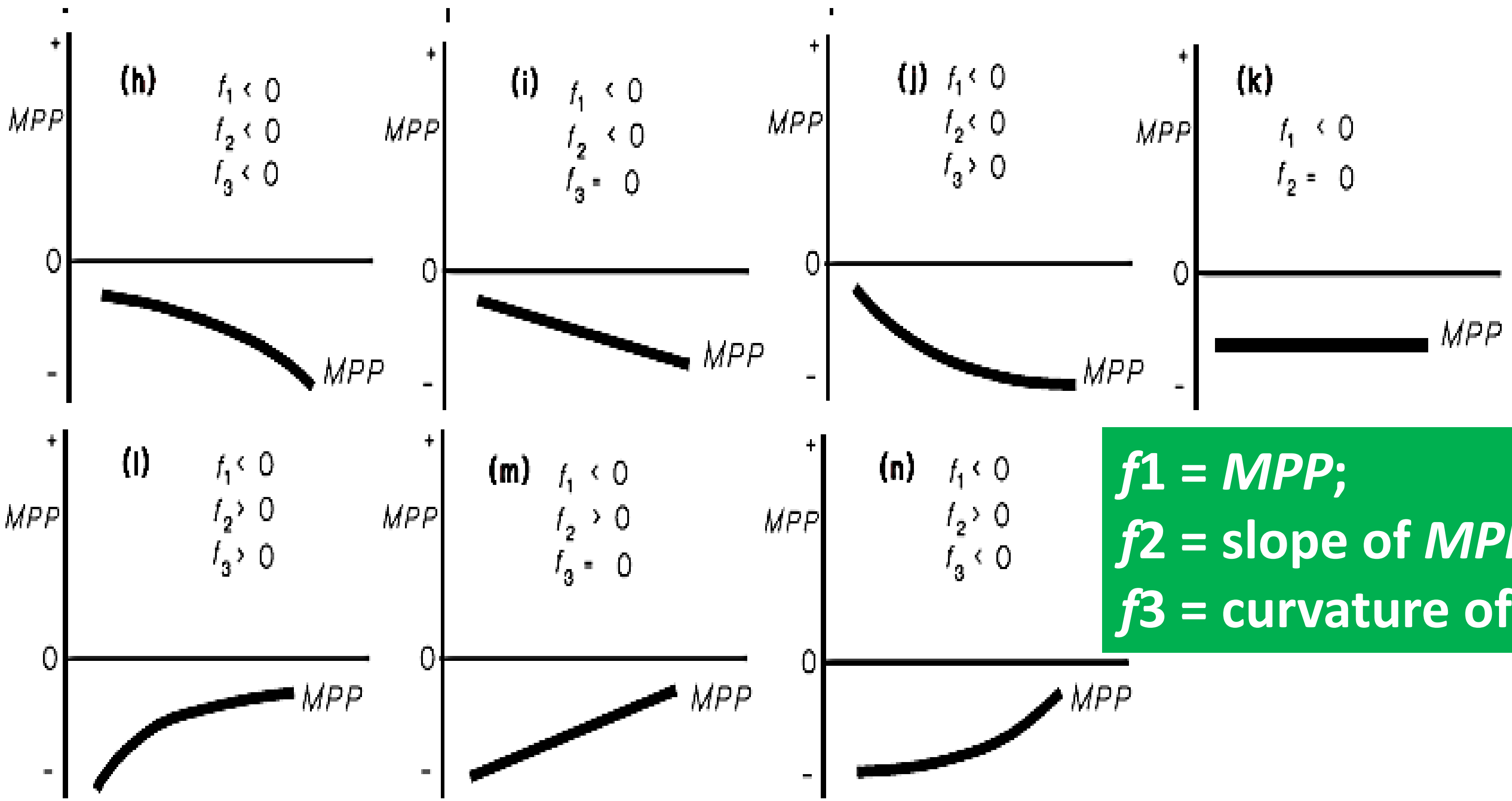
If *MPP* is zero, then $f'(x)$ is also zero, and the production function is likely either **constant** or at its **maximum**.

Figure illustrates seven instances where the first derivative of the *TPP* function (MPP) is positive.



***f*1 = *MPP*;**
***f*2 = slope of *MPP*;**
f*3 = curvature of *MPP

and seven instances where the first derivative is negative .



*f*1 = *MPP*;
*f*2 = slope of *MPP*;
*f*3 = curvature of *MPP*

The first derivative of the *TPP* function could also be zero at the point where the *TPP* function is minimum.

The sign on the second derivative of the *TPP* function is used to determine if the *TPP* function is at a maximum or a minimum.

If the first derivative of the *TPP* function is zero and the second derivative is negative, the production function is at its minimum point.

If the first derivative of the *TPP* function is zero, and the second derivative is positive, the production function is at its minimum point.

If both the first and second derivatives are zero, the function is at an inflection point, or changing from convex to the horizontal axis to concave to the horizontal axis.

However, all inflection points do not have first derivatives of zero.

Finally, if the first derivative is zero and the second derivative does not exist, the production function is constant.

The second derivative of TPP

The second derivative of the production function is the first derivative of the *MPP*, or slope of the *MPP* function.

The second derivative (d^2y/dx^2 or $f''(x)$ or f_2) is obtained by **again differentiating** the production function.

$$d^2y/dx^2 = f''(x) = f_2 = dMPP/dx$$

If equation

$$d^2y/dx^2 = f''(x) = f_2 = dMPP/dx$$

is positive for a particular value of *x*, then *MPP* is increasing at that particular point.

A negative sign indicates that *MPP* is decreasing at that particular point.

If (f_2) is zero, *MPP* is likely at a maximum at that point.

In previous figure, the first derivative of the *MPP* function (f_2) is

positive in (a), (b), and (c), (l), (m),and (n);

negative in (e), (f), (g), (h), (i), and (j), and **zero** in (d) and (k).

The second derivative of the *MPP*

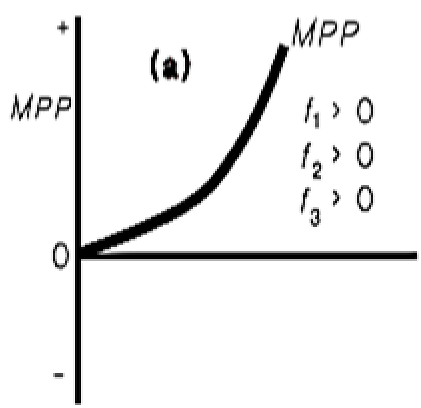
The second derivative of the *MPP* function (f_3) represents the curvature of *MPP* and is the derivative of the original production (or *TPP*) function.

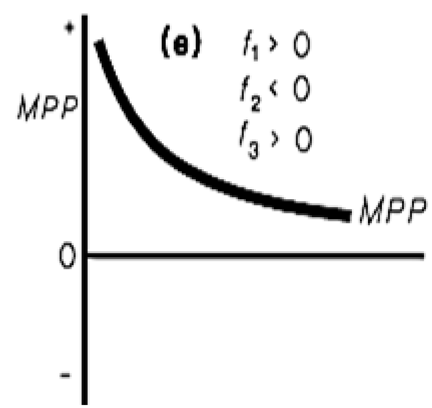
It is obtained by again differentiating the original production function

$$d^3y/dx^3 = f'''(x) = f_3 = d^2MPP/dx^2$$

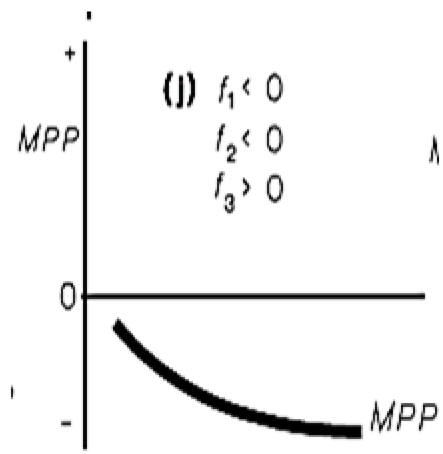
The sign of f_3 for a particular value of x indicates the rate of change in *MPP* at that particular point.

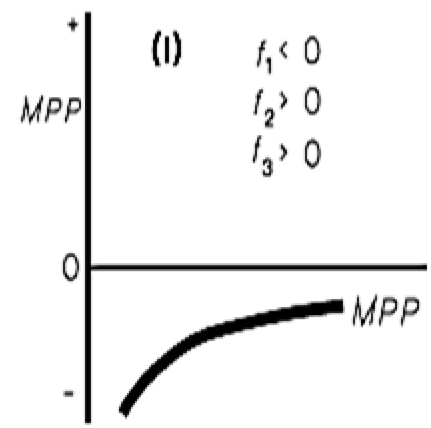
If *MPP* is in the positive quadrant and (f_3) is positive, *MPP* is increasing at an increasing rate as in Figure (a), or decreasing at a decreasing rate as in Figure (e).



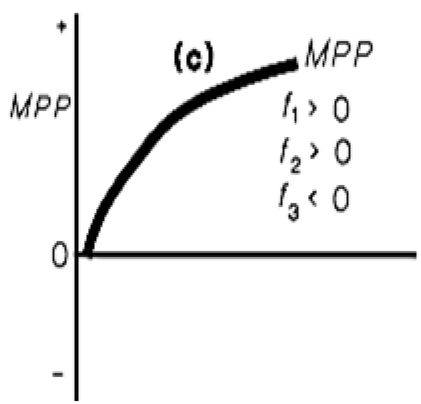


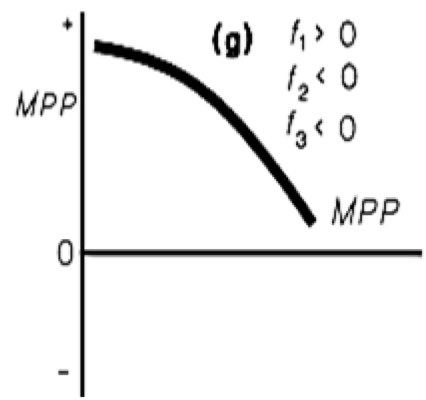
If *MPP* is in the negative quadrant, a positive (f_3) indicates that MPP is either decreasing at a decreasing rate (j), or increasing at a decreasing rate (l).



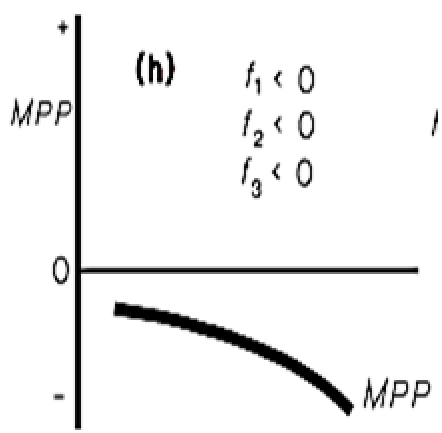


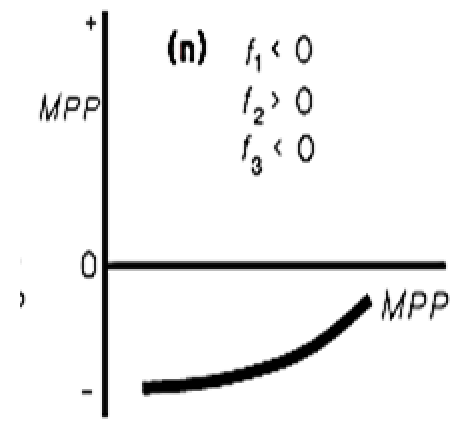
When *MPP* is in the positive quadrant, a negative sign on (f_3) indicates that *MPP* is either increasing at a decreasing rate (c), or decreasing at an increasing rate (g).



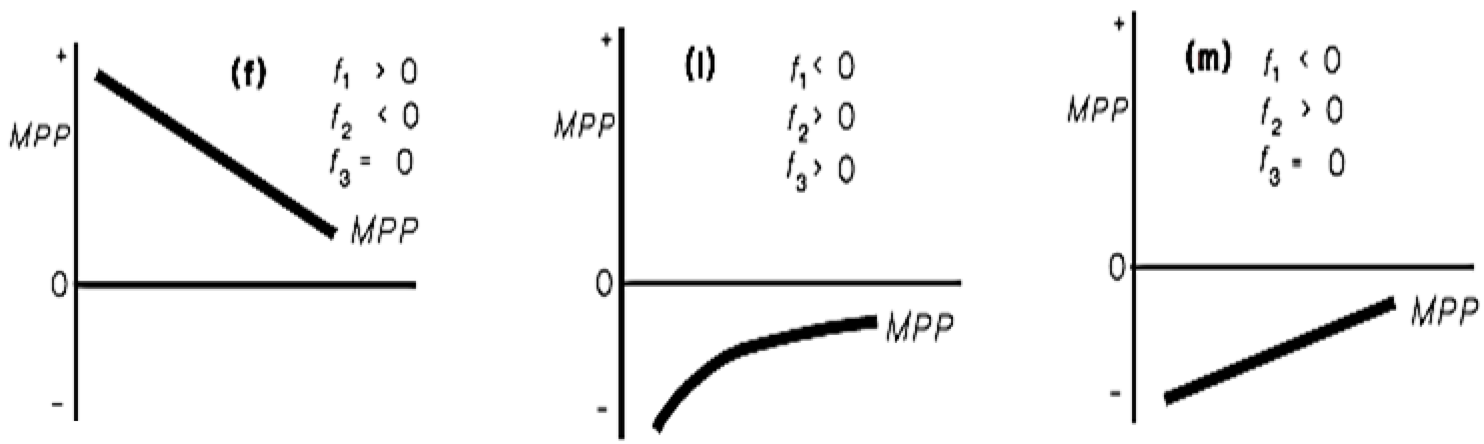


When *MPP* is in the negative quadrant, a negative sign on (f_3) indicates that *MPP* is decreasing at an increasing rate (h) or increasing at an increasing rate (n).

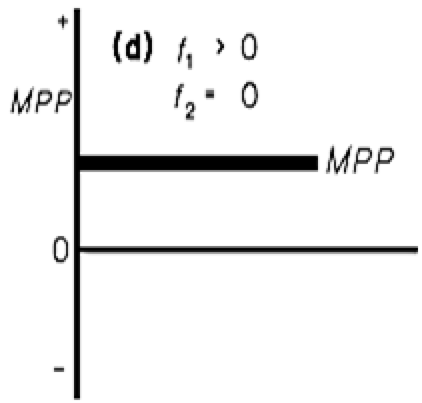


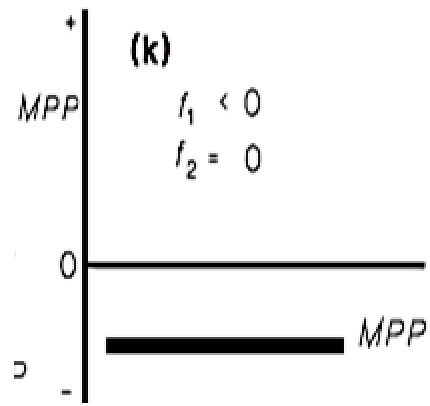


If (f_3) is zero, *MPP* has a constant slope with no curvature as is the case in (f), (l), and (m).



If *MPP* is constant, (f_3) does not exist.





A similar approach might be used for *APP*.

APP equals y/x, and if y and x are positive, then *APP* must also be positive.

As indicated earlier, the slope of *APP* is
$d(y/x)/dx = f\,'(y/x) = d\,APP/dx$

For a particular value of x, a positive sign indicates a positive slope and a negative sign a negative slope.

The curvature of *APP* can be represented by

$d^2(y/x)\mathrm{d}x^2 = f\,''(\mathrm{y}/x) = f_2 = d^2APP/dx^2$

For a particular value of x, a positive sign indicates that *APP* is increasing at an increasing rate, or decreasing at a decreasing rate.

A negative sign on equation ..

$$d^2(y/x)dx^2 = f''(y/x) = f_2 = d^2APP/dx^2$$

indicates that *APP* is increasing at a decreasing rate, or decreasing at an increasing rate.

A zero indicates an *APP* of constant slope.

The third derivative of *APP* would represent the rate of change in the curvature of *APP*.

Here are some examples of how these rules can be applied to a specific production function representing corn yield response to nitrogen fertilizer. Suppose the production function

$$y = 50 + 5.93x^{0.5}$$

where

y = corn yield in bushels per acre

x = pounds of nitrogen applied per acre

$$\mathrm{M}PP = 2.965x^{-0.5} > 0$$

For this equation, *MPP* is always positive for any positive level of input use, as indicated by the sign on equation *MPP*.

If additional nitrogen is applied, some additional response in terms of increased yield will always result.

If x is positive, *MPP* is positive, then the production function has not reached a maximum.

If this equation is negative, *MPP* is slopes downward. Each additional pound of nitrogen that is applied will produce less and less additional corn yield. Thus the law of diminishing (MARGINAL) returns holds for this production function throughout its range.

$$dMPP/dx = f''(x) = -1.48x^{-1.5} < 0$$

If this equation holds, the *MPP* function is decreasing at a decreasing rate, coming closer and closer to the horizontal axis but never reaching or intersecting it. given that incremental pounds of nitrogen always produce a positive response in terms of additional corn.

$$d^2 MPP/dx^2 = f'''(x) = -2.22x^{-2.5} > 0$$

If *x* is positive, *APP* is positive. Corn produced per pound of nitrogen fertilizer is always positive.

$$APP = y/x = 50/x + 5.93x^{-0.5}$$

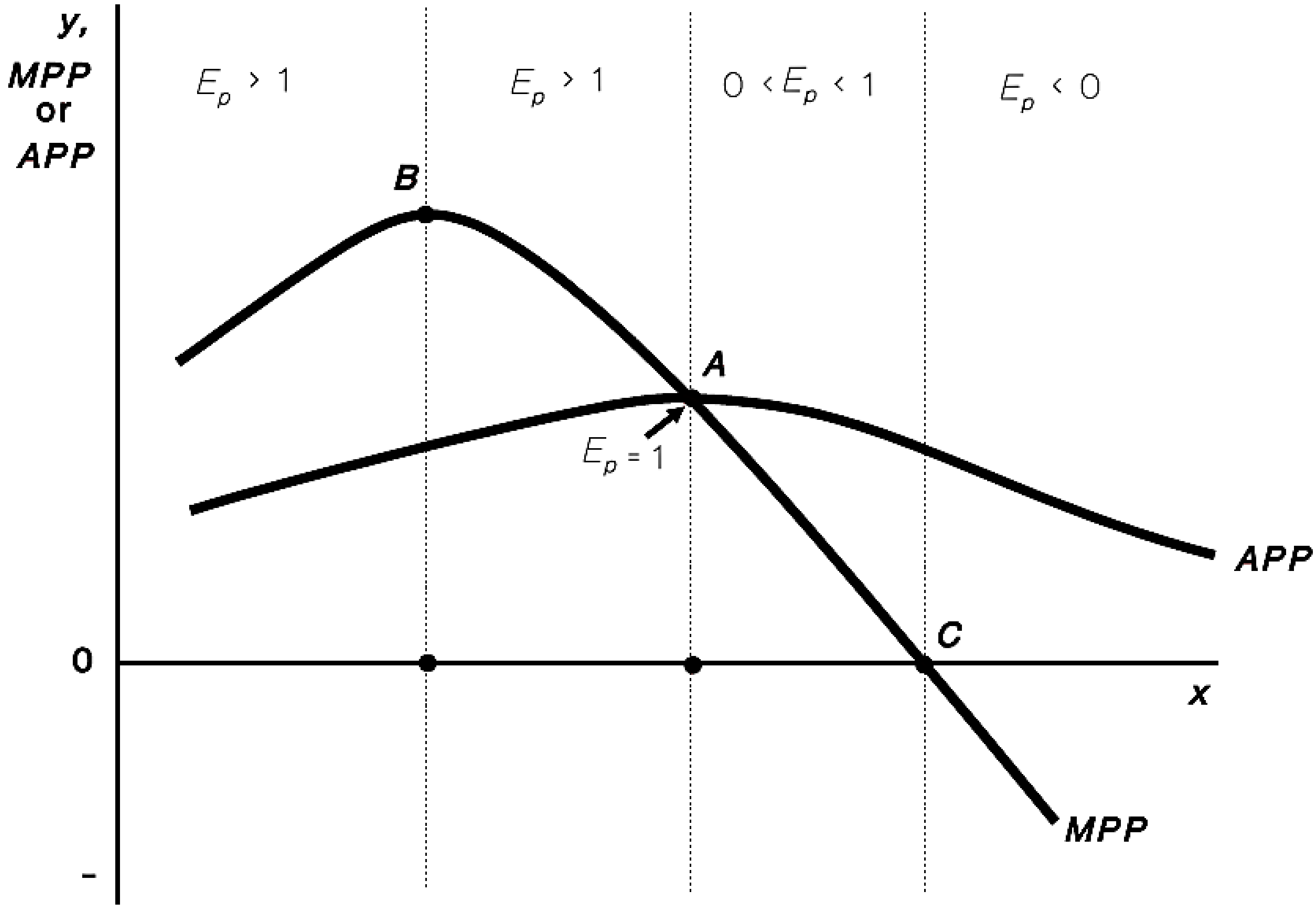
$$= 50x^{-1} + 5.93x^{-0.5} \succ 0$$

If *x* is positive, *APP* is sloped downward. As the use of nitrogen increases, the average product per unit of nitrogen declines

$$dAP\,P/d\,x = d(y/x)xd = -50x^{-2} - 2.97x^{-1.5} \prec 0$$

If *x* is positive, *APP* is also decreasing at a decreasing rate. As the use of nitrogen increases, the average product per unit of nitrogen decreases but at a decreasing rate

$$d^2 APP/dx^2 = d^2(y/x)dx^2 = 100x^{-3} + 4.45x^{-2.5} \succ 0$$


y,
MPP
or
APP
$E_p > 1$
$E_p > 1$
$0 < E_p < 1$
$E_p < 0$
B
A
$E_p = 1$
APP
C
0
x
MPP
-

1. The elasticity of production is greater than 1 until the point is reached where *MPP* = *APP* (point *A*).

2. The elasticity of production is greatest when the ratio of *MPP* to *APP* is greatest. For the neoclassical production function, this normally occurs when *MPP* reaches its maximum at the inflection point of the production function (point *B*).

3. The elasticity of production is less than 1 beyond the point where *MPP* = *APP* (point *A*).

4. The elasticity of production is zero when *MPP* is zero. Note that *APP* must always be positive (point *C*).

5. The elasticity of production is negative when *MPP* is negative and, of course, output is declining (beyond point *C*). If the production function is decreasing, *MPP* and the elasticity of production are negative. Again, *APP* must always be positive.

6. A unique characteristic of the neoclassical production function is that as the level of input use is increased, the relationship between *MPP* and *APP* is continually changing, and therefore the ratio of *MPP* to *APP* must also vary.

Further Topics on the Elasticity of Production

The expression $\Delta x/\Delta y$ is only an approximation of the true *MPP* of the production function for a specific amount of the input *x*. The actual *MPP* at a specific point is better represented by inserting the value of *x* into the marginal product function *dy*/*dx*.

The elasticity of production for a specific level of *x* might be obtained by determining the value for *dy*/*dx* for that level of *x* and then obtaining the elasticity of production from the expression

$$E_p = (dy/dx)*x/y$$

Now suppose that instead of the neoclassical production function, a simple linear relationship exists between y and x.

Thus .. $T\,PP = y = bx$

where b is some positive number.

Then .. $dy/dx = b$, but note also that since $y = bx$, then $y/x = bx/x = b$. Thus MPP (dy/dx) = APP (y/x) = b. Hence, $MPP/APP = b/b = 1$.

The elasticity of production for any such function is 1. This means that a given percentage increase in the use of the input *x* will result in exactly the same percentage increase in the output *y*.

Moreover, any production function in which the returns to the variable input are equal to some constant number will have an elasticity of production equal to 1.

Now suppose a slightly different production function

$$y = a\sqrt{x}$$

Another way of writing equation is $y = ax^{0.5}$

In this case $dy/dx = 0.5x^{-0.5}$

And $y/x = ax^{-0.5}$

Thus, $(dy/dx)/(y/x) = 0.5$

Hence the elasticity of production is 0.5. This means that for any level of input use *MPP* will be precisely one half of *APP*. In general, the elasticity of production will be *b* for any production function of the form

$$y = ax^b$$

where *a* and *b* are any numbers. Notice that

$$dy/dx = abx^{b-1}$$

and that

$$y/x = ax^b/x = ax^b x^{-1}$$

Thus the ratio of *MPP* to *APP*-the elasticity of production- for such a function is always equal to the constant *b*. This is not the same as the relationship that exists between *MPP* and *APP* for the neoclassical production function in which the ratio is not constant but continually changing as the use of *x* increases

Problems and Exercises

1. Suppose the following production function data. Fill in the blanks.

x (Input)	y (Output)	*MPP*	*APP*
0	0		——
		——	
10	50		——
		——	
25	75		——
		——	
40	80		——
		——	
50	85		——

2. For the following production functions, does the law of diminishing returns hold?

a. $y = x^{0.2}$
b. $y = 3x$
c. $y = x^3$
d. $y = 6x - 0.10x^2$

3. Find the corresponding *MPP* and *APP* functions for the production functions given in problem number 2.

4. Assume a general multiplicative production function of the form

$$y = 2x^b$$

Derive the corresponding *MPP* and *APP* functions, and draw on a sheet of graph paper *TPP*, *APP* and *MPP* when the value of b is

a. 5
b. 3
c. 2
d. 1.5
e. 1.0
f. 0.7
g. 0.3
h. 0
i. -0.5
j. -1.0

Be sure to show the sign, slope and curvature of *MPP* and *APP*. What is the value for the elasticity of production in each case? Notice that the curves remain at fixed proportion from each other.

5. Graph the production function

$$y = 0.4x + 0.09x^2 - 0.003x^3$$

for values of x between 0 and 20. Derive and graph the corresponding *MPP* and *APP*. What is the algebraic expression for the elasticity of production in this case? Is the elasticity of production constant or variable for this function? Explain.

6. Suppose that the coefficients or parameters of a production function of the polynomial form are to be found. The production function is

$$y = ax + bx^2 + cx^3$$

where y = corn yield in bushels per acre

x = nitrogen application in pounds per acre

a, b and c are coefficients or unknown parameters

The production function should produce a corn yield of 150 bushels per acre when 200 pounds of nitrogen is applied to an acre. This should be the maximum corn yield ($MPP = 0$). The maximum APP should occur at a nitrogen application rate of 125 pounds per acre. Find the parameters a, b and c for a production function meeting these restrictions. *Hint:* First find the equation for APP and MPP, and the equations representing maximum APP and zero MPP. Then insert the correct nitrogen application levels in the three equations representing TPP, maximum APP and zero MPP. There are three equations in three unknowns (a, b, and c). Solve this system for a, b, and c.

NEXT

Profit Maximization with One Input and One Output