1243

FLUID MECHANICS

COURSE CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Nature of fluids

A fluid is a material that takes the shape of its container. These are divided into incompressible fluids (Liquids) and compressible fluids (Gases). The former types do not show appreciable change in volume upon application of an external pressure, while the latter show a decrease in volume under pressure.

On the other hand, the main difference between solids and fluids is the ability of most solids to resist shear stresses. In this respect, liquids and gases deform under the application of shear stresses.

1.2 Dimensions and Units

1.2.1 Dimensions

A dimension is the measure by which a physical variable is expressed quantitatively while a unit is a particular way of attaching a number to the quantitative dimension. The main dimensions to be dealt with are as follows:

- **Dimension of length**: This is represented by [L] and is used to express lengths, distances, width, height, etc.
- **Dimension of time**: This is represented by [T].
- **Dimension of mass**: this is represented by [M].

Out of these main dimensions, are derived other entities the dimensions of which can be expressed by the above three main dimensions. Their dimensions are simply obtained from their defining expressions. Table (1.1) shows the main variables with their corresponding dimensions. This table is not complete, as some other physical properties will be introduced in due course.

Any equation must be dimensionally correct, that is, the dimensions of both sides should be the same. For example, during the course, we will face the following expression expressing the pressure of a fluid:

$$p = \frac{1}{2}\rho v^2 + \rho g h$$

From table (1.1), the LHS has the dimension $[M].[L]^{-1}.[T]^{-2}$

The term $\frac{1}{2}\rho v^2$ has the dimension: $[M].[L]^{-3}[L]^{2}.[T]^{-2} = [M].[L]^{-1}.[T]^{-2}$

The term $\rho g h$ has the dimension: $[M].[L]^{-3}[L].[T]^{-2}.[L] = [M].[L]^{-1}.[T]^{-2}$

Table 1.1: Dimensions of some entities

Entity	Defining expression	Dimension
Area	$A = x^2$	$[L]^2$
Volume	$V = x^3$	$[L]^3$
Velocity	$v = \frac{dx}{dt}$	$[L].[T]^{-1}$
Acceleration	$a = \frac{d^2x}{dt^2}$	$[L].[T]^{-2}$
Force	$F = ma$	$[M].[L].[T]^{-2}$
Pressure or stress	$p = \frac{F}{A}$	$[M].[L]^{-1}.[T]^{-2}$
Density	$\rho = \frac{m}{V}$	$[M].[L]^{-3}$
Work or energy	$W = \int F.dx$	$[M].[L]^2.[T]^{-2}$
Power $\mathscr{P}$	$P = \frac{dW}{dt}$	$[M].[L]^2.[T]^{-3}$

1.2.2 Units

There are two main systems of units: The SI units (Systême International) and the British units. In the last two decades, the first system has taken the lead except in some cases where the British system still prevails. (Like the thickness of steel sheets or diameters of pipes).

The SI unit expresses the [L] dimensions in meter (m) or (mm) and less commonly in kilometer (km), centimeter (cm) and micrometer (μm). Lately, the nanometer (nm) has also been widely used. The following shows the conversions from one length unit to the other:

1 km = 10^3 m – 1 cm = 10^{-2} m – 1 mm = 10^{-3} m – 1 μm = 10^{-6} m – 1 nm = 10^{-9} m.

The SI unit expresses the [T] dimension in second (s) with two derived units: 1 hour (h) = 3600 s and 1 minute (min) = 60 s.

In this system, the [M] dimension main unit is the kilogram (kg) with the following derived units: 1 ton = 10^3 kg – 1 gram (g) = 10^{-3} kg and 1 milligram (mg) = 10^{-6} kg.

On the other hand, the less commonly used **British system** expresses the length in foot (ft) or inch ("), where 1 ft = 12".

To convert from ft to m: 1 ft = 0.305 m and 1" = 0.0254 m.

The mass unit is the pound (lb) where 1 lb = 0.454 kg.

Physical entities have corresponding units derived from their dimensions, although some of them bear special names after some eminent scientists. (Table 1.2).

Table 1.2: Units of some entities

Entity	SI Unit	British unit
Area	m^2	ft^2
Volume	m^3	ft^3
Velocity	$m.s^{-1}$	$ft.s^{-1}$
Acceleration	$m.s^{-2}$	$ft.s^{-2}$
Force	Newton N = $kg.m.s^{-2}$	$lb.ft.s^{-2}$ or lb_f
Pressure or stress	Pascal Pa = $N.m^{-2}$	$lb.ft^{-2}$ or psi
Density	$kg.m^{-3}$	$lb.ft^{-3}$
Work or energy	Joule (J) = N.m	$lb.ft^2.s^{-2}$ or $lb_f.ft$
Power $\mathscr{P}$	Watt (W) = $N.m.s^{-1}$	$lb.ft^2.s^{-3}$ or $lb_f.ft.s^{-1}$

In particular, the force unit in the SI system is sometimes expressed in kilogram force (kg_f) which is equal to 9.81 N, while in the British system it is usually expressed in pound force (lb_f) which equals 32.2 $lb.ft.s^{-2}$.

1.2.3 Units of volume

The liter (L) is sometimes used as measuring unit of volume: 1 m^3 = 10^3 L. The milliliter = 10^{-3} L = 10^{-6} m^3.

The British units sometimes use the Gallon (Gal) as measure of volume: 1 Gal = 3.785 L.

1.2.3 The units of pressure

The units of pressure are particularly important as this variable plays an essential role in fluid mechanics.

As mentioned before, the SI unit is the Pascal (Pa). Other important derived units are:

- The Megapascal (MPa) = 10^6 Pa
- The atmosphere: 1 atm = 1.0132×10^5 Pa.
- The bar: 1 bar = 10^5 Pa = 0.1 MPa
- 1 mm Hg = $\frac{1}{760}$ atm = 133.3 Pa
- 1pound per square inch (psi) = $\frac{1}{14.7}$ atm = 6892.5 Pa

1.3 Viscosity

1.3.1 Definition

This is by far one of the most important physical properties of fluids. It is defined as follows:

Consider a fluid placed between two horizontal plates (Figure 1.1).

Fig 1.1: Definition of viscosity

The lower plate is stationary while the upper one is moved at a velocity v. The relative velocity between the two plates = $\Delta v = v - 0 = v$.

The shear strain is defined as the ratio between the horizontal distance and the distance between the two plates = $\Delta x/\Delta y$

The shear rate ($\dot{\gamma}$) is defined is the shear strain per unit time = $(\Delta x/\Delta t)/\Delta y$

$\dot{\gamma} = \Delta v/\Delta y$ s^{-1} (1.1)

The shear stress (τ) is the force per unit area $\tau = F/A$ (Pa) (See Table 1.2). The shar stress is proportional to the shear rate so that:

$$\tau = \mu.\dot{\gamma} \quad (1.2)$$

1.3.2 Effect of temperature

Viscosity is the constant of proportionality in the previous equation. It has the units Pa.s, although another important unit is sometimes used: The Centipoise (cP) which is equal to 10^{-3} Pa.s. This unit has been chosen to fix a reference value for the viscosity by setting the viscosity of water as 1 cP.

The viscosity of liquids decreases with increased temperature since the liquid tends to display a higher shear rate for a given stress. The following expression best relates viscosity of liquids to temperature (K):

$$\mu = Ae^{\frac{E}{RT}} \quad (1.3)$$

In this expression, A is a pre-exponential factor having the same dimensions as viscosity.
R is the general gas constant (= 8.314 $J.mol.K^{-1}$)
E is called the activation energy for viscosity ($J.mol^{-1}$) and denotes the sensitivity of viscosity to changes in temperature.

The viscosity of gases, as opposed to liquids, increases with temperature, following the empirical rule:

$$\mu = C.T^n \quad (1.4)$$

Where C is a constant depending on the nature of the gas and n ranging from 0.5 to 1.1.

1.3.3 Kinematic viscosity

Sometimes, liquids are rather characterized by their **kinematic viscosity**

$$\nu = \mu/\rho \tag{1.5}$$

This is obtained by dividing the viscosity by the density. Its SI unit is $m^2.s^{-1}$ although the practical unit is the centistoke (cSt): 1 cSt = 1 $mm^2.s^{-1}$ = 10^{-6} $m^2.s^{-1}$. The use of kinematic viscosity to characterize liquids is widespread in the oil industry.

1.4 Other fluid properties

1.4.1 Density

Density, as defined by mass per unit volume is practically temperature independent for liquids. For ideal gases, it is calculated from the following expression:

$$\rho = \frac{pM}{10^3 RT} \tag{1.6}$$

To apply this equation in proper units, the molecular weight of the gas M must be expressed in kg per mol. Since its units are g per mol, it must be divided by 10^3.

For example, air at 25°C and 1 atm pressure will have a density of:

$$\rho = \frac{1.0132\times10^5\times29}{10^3\times8.314\times(25+273)} = 1.186 \text{ kg.m}^{-3}.$$

On the other hand, density is often expressed in the form of **specific gravity.** This is the ratio of the density of the fluid and that of water. Since this latter = 1000 $kg.m^{-3}$, the density of the fluid can be readily calculated by multiplying its specific gravity by 1000.

1.4.2 Surface tension

Consider a liquid wetting a solid surface. Due to the cohesive forces, a molecule located away from the surface is pulled equally in every direction by neighboring liquid molecules, resulting in a net force of zero. The molecules at the surface do not have the same molecules on all sides of them and therefore are pulled inward. This creates some internal pressure and forces liquid surfaces to contract. (Figure 1.2).

Surface tension is responsible for the spherical shape of liquid droplets as the cohesive forces between the liquid molecules tend to reach an equilibrium state where the droplet is at a minimum energy level. This was proved to take place when the droplets assume a spherical shape.

Surface tension is expressed as the force acting on the liquid surface per unit length. That is why its SI units are $N.m^{-1}$. This is equivalent to $N.m.m^{-2}$, that is $J.m^{-2}$, which represents the surface energy per unit area of liquid.

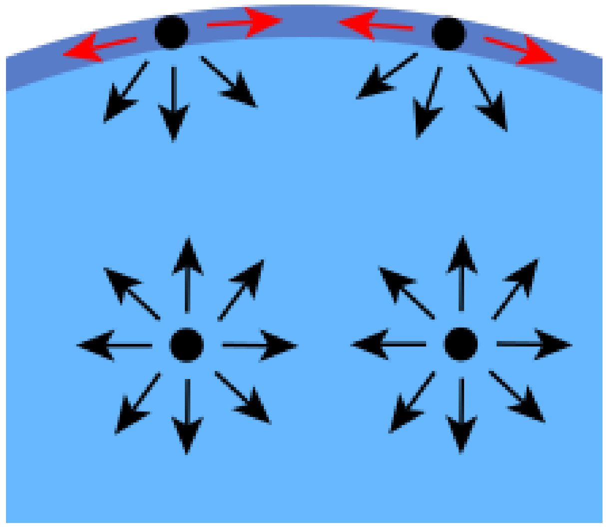


Fig 1.2: Origin of surface tension

1.4.3 Compressibility

The compressibility coefficient for a material is defined by:

$$\beta = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \qquad (1.7)$$

Liquids possess low compressibility: For example, water at 25°C possesses a compressibility coefficient of 0.00045 MPa^{-1}.

Gases, on the other hand, are highly compressible. For ideal gases: $V = \frac{nRT}{p}$ so that:

$$-\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = \frac{nRT}{V}.p^{-2} = p.p^{-2} = 1/p$$

Hence, at pressure of 1 bar = 0.1 MPa, the compressibility of a gas = 10 MPa^{-1}.

1.4.4 Vapor pressure

At any temperature higher than 0 K, some of the liquid molecules can overcome the cohesion forces, due to an increase in their kinetic energy, to move into the vapor phase. The pressure exerted by the vapor at any temperature is termed **the vapor pressure** of the liquid. It is obvious that this pressure will increase its temperature. As the temperature of the liquid reaches the ambient temperature, the liquid starts to boil. This occurs at the boiling temperature of the liquid. Figure (1.3) displays the increase in the vapor pressure of water with temperature.

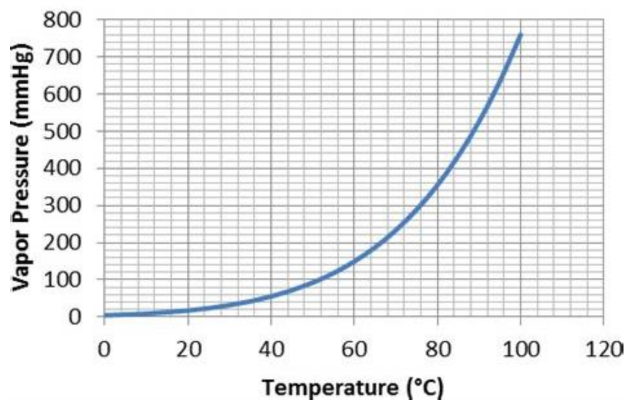


Fig 1.3: Effect of temperature on the vapor pressure of water

Example 1.1:
The activation energy of a lubricating oil = 26000 J.mol^{-1}. If its viscosity at 25°C is 0.45 Pa.s. Evaluate its viscosity at a working temperature of 50°C. At what temperature will the viscosity of the oil is equal to 0.8 Pa.s?

Solution:

$\mu = Ae^{\frac{E}{RT}} \qquad 0.45 = Ae^{\frac{26000}{298\times8.314}} \quad A = 1.246\times10^{-5}$

At 50°C (323K):

$\mu = 1.246\times10^{-5}e^{\frac{26000}{323\times8.314}} = \mathbf{0.2\ Pa.s}$

$0.8 = 1.246\times10^{-5}e^{\frac{26000}{8.314T}}$

$T = 282.5\text{K} \equiv \mathbf{9.5^oC}$

Example 1.2:
Under atmospheric pressure, the density of a liquid = 880 kg.m^{-3}. Under a pressure of 300 bar, its density increases to 889.6 kg.m^{-3}. What is the compressibility coefficient of that liquid in Pa^{-1}?

Solution:

$\beta = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$

Since the mass of the liquid remains constant, and $V = m/\rho$, then the previous definition can be written in the form:

$\beta = -\frac{\rho}{m}\left(\frac{\partial\frac{m}{\rho}}{\partial p}\right)_T = \frac{\rho}{\rho^2}.\left(\frac{\partial\rho}{\partial p}\right)_T \approx \frac{1}{\rho}.\frac{\Delta\rho}{\Delta p}$

$\beta = \frac{1}{880}.\frac{9.6}{300-0.987} = 3.65\times10^{-5}\text{bar}^{-1} \equiv \mathbf{3.65\times10^{-10}Pa^{-1}}$

1.5 An introduction to dimensional analysis

Sometimes, a certain process parameter is known to depend on some independent variables. It is then possible to deduce the probable relation between the parameter and these variables using a tool known as dimensional analysis. The following example explains the use of that tool.

Example 1.3

It is known that the pressure drop per unit length of a fluid flowing through a pipe, under some conditions, depends on the pipe diameter D, the fluid viscosity μ, and the fluid velocity v. What would be the form of the dependence function?

Solution:

$$\frac{\Delta p}{L} = f(D, \mu, V)$$

We assume that this function takes the form: $\frac{\Delta p}{L} = k.D^a.\mu^b.v^c$

The dimension of the LHS is: $[F].[L]^{-2}.[L]^{-1} \rightarrow [M].[L].[T]^{-2}. L]^{-2}.[L]^{-1} \rightarrow [M].[L]^{-2}.[T]^{-2}$

The constant k being dimensionless, the dimensions of the RHS will show as:

$[L]^a.([M].[L]^{-1}[T]^{-1})^b.([L].[T]^{-1})^c \rightarrow [M]^b.[L]^{a-b+c}.[T]^{-b-c}$

Equating the exponents of the two sides, we get the following set of linear equations:

$b = 1$ (i)

$a - b + c = -2$ (ii)

$-b - c = -2$ (iii)

Solving, we get: $a = -2, b = 1, c = 1$

Hence: $\frac{\Delta p}{L} = k.D^{-2}.\mu^1.v^1$

The dependence of $\frac{\Delta p}{L}$ will therefore take the form:

$$\frac{\Delta p}{L} = k.\frac{\mu.v}{D^2}$$

This expression will be proved in Chapter 4, where it will be shown that $k = 32$.

CHAPTER 2

STATICS OF FLUIDS

2.1 Pressure exerted by a fluid

If a liquid of density ρ is placed in a vessel of cross-sectional area A, the pressure exerted at its bottom is due to the weight of the liquid + atmospheric pressure. This is called **absolute pressure,** and its SI unit is often written as Pa abs. The pressure of the liquid in excess of the atmospheric pressure is the effective pressure applied on the bottom since the other side of the bottom is also under atmospheric pressure. This is called **gauge (or gage) pressure**. The weight of the liquid = $\rho V g$ N, and the volume $V = A.h$, so that the gauge pressure obtained by dividing the weight by the area is:

$$p = \rho g h \qquad (2.1)$$

In the case of a liquid, the pressure at any level will increase linearly with the depth and reaches its maximum value at the bottom. (Figure 2.1). This pressure is the same in all radial directions at any specific height.

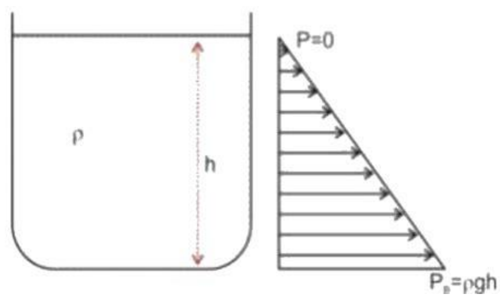


Fig 2.1: Pressure gradient along the depth of a liquid

In case of gases filling an enclosed space, the pressure is the same on all the internal surface of the enclosure.

Example 2.1

65 L of oil of specific gravity 0.87 float over 120 L of water in a cylindrical container with a diameter of 0.6 m. Calculate the maximum pressure exerted on the walls of the container.

Solution:

Area of base = $\frac{1}{4}\pi \times 0.6^2 = 0.2827$ m^2

Weight of oil + water = $(0.065 \times 870 + 0.12 \times 1000) \times 9.81 = 1732$ N

Hence, the maximum applied pressure (on bottom) = $\frac{1732}{0.2827}$ = **6126.5 Pa**

2.2 **The U – tube manometer**

This consists of two vertical tubes of open sides, joined at their bottom be a third tube. As the tube is filled with a liquid, the pressures exerted on each side being equal, the liquid in the two branches will be at the same level. If a pressure is applied on one of its branches, then the level of the liquid in that branch will drop with respect to the level in the other by an amount corresponding to the applied pressure, which can be calculated from Equation (2.1). (Figure 2.2)

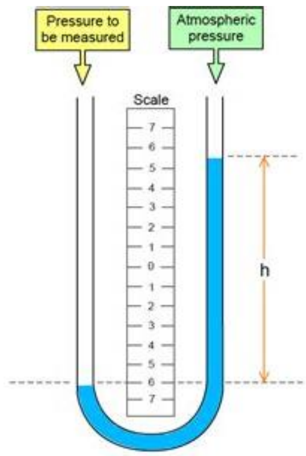
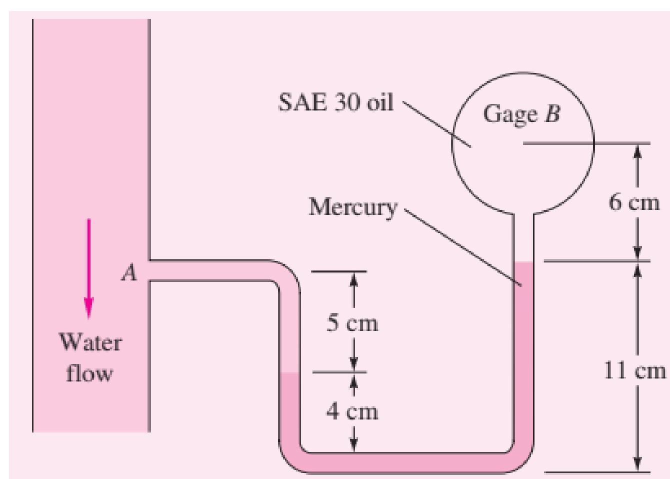


Fig 2.2: Measuring pressure by means of a U-tube manometer

Example 2.2

In the figure, water flows through a pipe, and a manometer is used to estimate its pressure. The manometer fluid is mercury (Specific gravity = 13.6) over which rest 6 cm of oil of specific gravity = 0.689 subjected to a pressure of 87 kPa. Figure (2.3) displays the arrangement. Estimate the water pressure in Pa.



Solution:

At the mercury – water interface the pressures are equal:

At the right side, the total pressure is:

$p_{water} + 1000 \times 9.81 \times 0.05 = p_{water} + 490.5$

On the right side, the total pressure is:

$$87000 + 689 \times 9.81 \times 0.06 + 13600 \times 9.81 \times (0.11 - 0.04) = 96744.66 \text{ Pa}$$

$$p_{water} + 490.5 = 96744.66 \rightarrow p_{water} = 96254.16 \text{ Pa} \approx \mathbf{96.24\ kPa}$$

Example 2.3

18.7 kg of natural gas of molecular weight = 19 is stored in a cylindrical reservoir of 2.2 m diameter and 3.2 m height at 25ºC. Calculate in bar, the pressure exerted on the walls of the reservoir.

Solution:

$p = \frac{nRT}{V}$

$n =$ The number of mols of gas = $\frac{18.7 \times 1000}{19}$ = 984.2 mol.

The radius of the spherical container = 1.6 m.

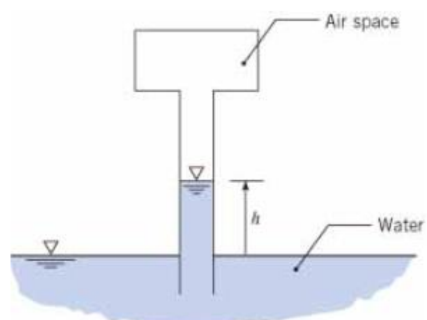
$V = \frac{\pi D^2 h}{4} = \frac{\pi \times 2.2^2 \times 3.2}{4}$ = 12.164 m^3

$p = \frac{984.2 \times 8.314 \times (25+273)}{12.164}$ = 200462 Pa ≡ **2 bar**

Example 2.4

As shown, an air space above a long tube is pressurized to 50 kPa. Water (15°C) from a reservoir fills the tube to a height h. If the pressure in the air space is changed to 75 kPa, will h increase or decrease and by how much? Assume atmospheric pressure is 100 kPa.

Solution:



When the pressure is 50000 Pa, it differs from the pressure on the free liquid surface by 100000 – 50000 = 50000 Pa

$50000 = 1000 \times 9.81h \rightarrow h = 5.097$ m

When the pressure is 75000 Pa, the difference with the level at free surface = 100000 – 75000 = 25000 Pa

$25000 = 1000 \times 9.81h' \rightarrow h' = 2.548$ m

The level **decreases** by $5.097 - 2.548 = \mathbf{2.548}$ **m**

CHAPTER 3

BASIC EQUATIONS OF FLUID FLOW

3.1 The continuity equation

In Figure (3.1), a fluid flows through the given hollow volume. This volume is called the **control volume of flow.** The mass flow rate of the fluid $\dot{m}$ kg.s^{-1} is obviously constant since the flow is continuous. Consider an infinitesimal thickness dx along the surface. If the cross-sectional area at that section = A, then the volume of this element = $A.dx$ and the elemental mass contained in that volume = $\rho.A.dx$

The mass rate of flow $\dot{m} = \frac{dm}{dt} = \frac{\rho.A.dx}{dt} = \rho.A.v = \text{Constant.}$

Referring to the same figure, if the inlet velocity to the control volume = v_1 and the exit velocity = v_2, then:

$$\rho_1.A_1.v_1 = \rho_2.A_2.v_2 \qquad (3.1)$$

this equation is known as the continuity equation of flow.

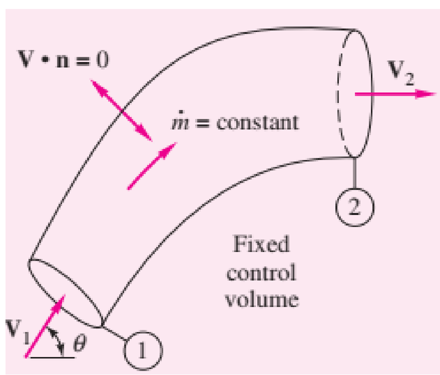


Fig 3.1: continuity of flow in control volume

In the case of liquids, the density is practically unaffected by changes in pressure or temperature, so that this equation can be simplified to:

$$A_1.v_1 = A_2.v_2 \qquad (3.2)$$

The product $A.v$ is of major importance in fluid flow and is termed the **volumetric flow rate** or simply, the flow rate (m^3.s^{-1}), of symbol Q.

Example 3.1

Figure (3.2) shows oil of specific gravity 0.87 flowing from a large pipe section of diameter 4” to a narrower section of diameter 2”. If the velocity of the liquid through the first section = 0.3 m.s^{-1}, calculate:

(a) The mass flow rate in $kg.h^{-1}$
(b) The volumetric flow rate in $m^3.h^{-1}$
(c) The velocity of oil at the narrow section.

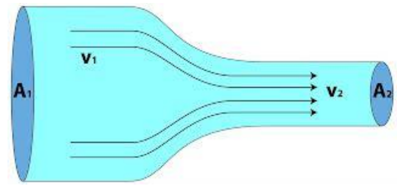


Fig 3.2

Solution:

$A=\frac{1}{4}\pi D^2 = 0.25\pi \times (4 \times 0.0254)^2 = 0.0081\ m^2$

$\frac{dm}{dt} = \rho.A.v = 870 \times 0.0081 \times .3 \times 3600 = \mathbf{7617.65\ kg.h^{-1}}$

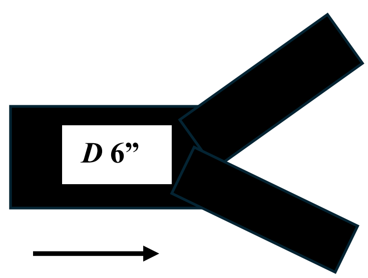
$Q = A_1.v_1 = 0.0081 \times .3 \times 3600 = \mathbf{8.755\ m^3.h^{-1}}$

$A=\frac{1}{4}\pi D^2 = 0.25\pi \times (2 \times 0.0254)^2 = 0.00203\ m^2$

$A_1.v_1 = A_2.v_2 \rightarrow 0.0081 \times 0.3 = 0.00203\ v_2 \rightarrow \boldsymbol{v_2} = \mathbf{1.2\ m.s^{-1}}$

Example 3.2

In Figure (3.3), the diameters of the pipes are in inches and the velocity of oil in the main pipe = 2.6 $m.s^{-1}$. If the main pipe forks into two pipes, as shown, calculate the flow rate in each pipe and the velocity of oil in the 4" pipe, if the velocity in the 3" pipe is 4 $m.s^{-1}$.



Solution:

Flow rate in main pipe = $Q = \frac{1}{4}\pi D^2 v = \frac{1}{4}\pi(6 \times 0.0254)^2 \times 2.6 = \mathbf{0.0474\ m^3.s^{-1}}$

The continuity of flow requires that the flow rate in the main pipe equals the sum of the flow rates in the two smaller pipes:

$A.v = A_1.v_1 + A_2.v_2$ $\qquad$ $\frac{1}{4}\pi D^2 v = \frac{1}{4}\pi D_1^2 v_1 + \frac{1}{4}\pi D_2^2 v_2$

$(0.0254 \times 6)^2 \times 2.6 = (0.0254 \times 4)^2 \times v_1 + (0.0254 \times 3)^2 \times 4 \rightarrow \boldsymbol{v_1} = \mathbf{3.6\ m.s^{-1}}$

$Q_1 = \frac{1}{4}\pi \times (4 \times 0.0254)^2 \times 3.6 = \mathbf{0.0292\ m^3.s^{-1}}$

$Q_2 = \frac{1}{4}\pi \times (3 \times 0.0254)^2 \times 4 = \mathbf{0.0182\ m^3.s^{-1}}$

Check: $0.0292 + 0.0182 = 0.0474$.

Example 3.3

Compressed warm air at 1.2 bar flows in a 100 mm pipe at 36 m.s^{-1} and 50^{o}C. Due to heat losses, its temperature reaches 30^{o}C as it is discharged to ambient atmosphere. Calculate the velocity at the discharge end.

Solution:

At entrance: $\rho = \frac{pM}{10^3 RT} = \frac{1.2\times10^5\times29}{10^3\times8.314\times323}$ = 1.296 kg.m^{-3}.

At outlet: $\rho = \frac{1.0132\times10^5\times29}{10^3\times8.314\times303}$ = 1.166 kg.m^{-3}.

Since the pipe is the same, then $A_1 = A_2$ in equation (3.1):

$\rho_1.v_1 = \rho_2.v_2 \rightarrow 1.296 \times 36 = 1.166v_2 \rightarrow \boldsymbol{v_2} = \mathbf{40}$ **m.s^{-1}**

3.2 Bernoulli equation

When a solid body moves in a plane, the sum of its kinetic and potential energies remains constant, as long as there are no friction losses. For liquids, the situation is similar except that extra energy arises due to the pressure exerted by the liquid on the duct walls.

Consider a small portion of the duct of length dx and cross-sectional area A. the mass of the portion is $dm = \rho.A.dx$.

The kinetic energy of the fluid in that portion = $\frac{1}{2}dm.v^2 = \frac{1}{2}\rho.A.v^2.dx$ and the potential energy = $\rho.A.g.h.dx$.

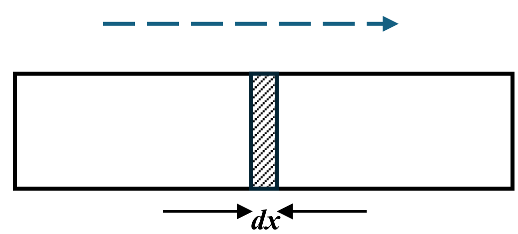


Fig 3.3: Flow through an elemental volume

The pressure developed = p and the net force = $p.A$ The work necessary to produce that pressure difference = $p.A.dx$.

Following the principle of conservation of energy, the total energy throughout the control volume is constant:

$$\frac{1}{2}\rho.A.v^2.dx + p.A.dx + \rho.A.g.h.dx = \text{Const.}$$

The LHS is then divided by $\rho.A.g.dx$ to obtain:

$$\frac{v^2}{2g} + \frac{p}{\rho g} + h = \text{Const}$$

The three terms of the LHS have dimensions of length [L]. They are called **heads**.

$\frac{v^2}{2g}$ is the **velocity head**.

$\frac{p}{\rho g}$ is the **pressure head**.

h is the **elevation (or potential) head.**

This means that between any two positions on the flow path, one may write:

$$\frac{v_1^2}{2g} + \frac{p_1}{\rho_1 g} + h_1 = \frac{v_2^2}{2g} + \frac{p_2}{\rho_2 g} + h_2 \qquad (3.3)$$

This equation, known as the **Bernoulli equation**, states that the total head of any fluid is constant. This equation was first suggested by Daniel Bernoulli in 1738 and written in the previous form by Euler in 1752.

The following form, in terms of total pressures can also be used:

$$\frac{1}{2}\rho v_1^2 + p_1 + \rho_1 g h_1 = \frac{1}{2}\rho v_2^2 + p_2 + \rho_2 g h_2 \qquad (3.4)$$

The term $\frac{1}{2}\rho v^2$ is termed the **dynamic pressure of the fluid.** This is the kinetic energy of the fluid per unit volume.

Example 3.4

Oil of specific gravity 0.85 flows through a circular pipe of diameter 100 mm at a velocity = 5.3 m.s^{-1} and at ground level. A gauge records its pressure as 2.3 bar. Calculate its velocity as it enters a larger duct of diameter 150 mm, situated 3 m over ground level, assuming no energy losses, what would be the oil pressure at that larger section?

Solution:

$A_1.v_1 = A_2.v_2 \rightarrow D_1^2 v_1 = D_2^2 v_2 \rightarrow 0.1^2 \times 5.3 = 0.15^2 \times v_2 \rightarrow \boldsymbol{v_2 = 2.36}$ **m.s^{-1}.**

$$\frac{v_1^2}{2g} + \frac{p_1}{\rho_1 g} + h_1 = \frac{v_2^2}{2g} + \frac{p_2}{\rho_2 g} + h_2 \rightarrow \frac{5.3^2}{2\times 9.81} + \frac{2.3\times 10^5}{850\times 9.81} = \frac{2.36^2}{2\times 9.81} + \frac{p_2}{850\times 981} + 3$$

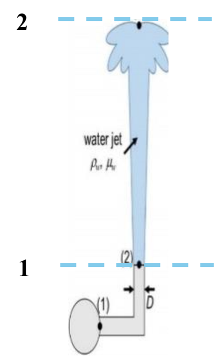
This yields: $\boldsymbol{p_2 = 215555.7}$ **Pa** $\equiv$ **2.146 bar**.

Example 3.5

Water flows upwards in a vertical pipe of 5 cm diameter open upwards at the rate of 50 m^3.h^{-1} and a pressure of 50 kPa. As it leaves the pipe, what is the highest level it can reach above the pipe outlet?

Solution:

$$\frac{v_1^2}{2g} + \frac{p_1}{\rho_1 g} + h_1 = \frac{v_2^2}{2g} + \frac{p_2}{\rho_2 g} + h_2$$



$A_1 = 0.25\pi \times 0.05^2 = 0.001964$ m^2, $v_1 = \frac{50}{3600\times 0.001964} = 7.07$ m.s^{-1}.

$$\frac{7.07^2}{2\times 9.81} + \frac{50000}{1000\times 9.81} + 0 = 0 + 0 + h \quad \rightarrow \boldsymbol{h = 7.64}\ \mathbf{m}$$

3.3 Applications

3.3.1 Liquid flowing out of an orifice

Consider Figure (3.4) showing a tank filled with a liquid with an orifice at the side bottom.

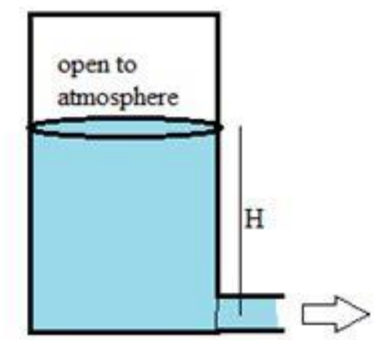


Fig 3.4: Tank with side orifice

Let the original height of liquid in the tank = H, the tank diameter D and the orifice diameter d. It is required to determine h, the height of the liquid in the tank at any time t, as function of that time. On applying Bernoulli equation between the free surface of the liquid in the tank and the outlet orifice, we note that both points are at atmospheric pressure.

$$\frac{v_1^2}{2g} + h = \frac{v_2^2}{2g} + 0 \qquad (3.5)$$

And, from continuity: $D^2 v_1 = d^2 v_2 \rightarrow v_2 = v_1 . \left(\frac{D}{d}\right)^2$

Substituting in Bernoulli equation:

$$\frac{v_1^2}{2g} + h = \frac{v_1^2}{2g}\left(\frac{D}{d}\right)^4 \qquad (3.6)$$

$$v_1 = \sqrt{h}\sqrt{\frac{2g}{\left(\frac{D}{d}\right)^4 - 1}}$$

The velocity of the liquid in the tank is the rate at which the level varies $\frac{dh}{dt}$. Hence, the previous equation can be written as:

$$h^{-1/2}\frac{dh}{dt} = \sqrt{\frac{2g}{\left(\frac{D}{d}\right)^4 - 1}}$$

Integrating between the original height of the liquid in the tank H and any height h:

$$\int_H^h h^{-1/2}.dh = \sqrt{\frac{2g}{\left(\frac{D}{d}\right)^4 - 1}}\int_0^t dt$$

Hence:

$$\sqrt{H} - \sqrt{h} = \frac{1}{2}\sqrt{\frac{2g}{\left(\frac{D}{d}\right)^4 - 1}}.t$$

The denominator under the root can be approximated since $d \ll D$, as follows:

$$\left(\frac{D}{d}\right)^4 - 1 = \frac{D^4 - d^4}{d^4} \approx \frac{D^4}{d^4} = \left(\frac{D}{d}\right)^4$$

The above equation then becomes:

$$\sqrt{H} - \sqrt{h} = \frac{1}{2}\left(\frac{d}{D}\right)^2\sqrt{2g}.t$$

This equation assumes no friction losses at the orifice, which is not practical, because of its narrow opening. That is why, a factor is introduced in that equation to account for that loss: **The discharge coefficient** C_d. This factor usually ranges from 0.65 to 0.9. This way, the previous equation becomes:

$$\sqrt{H} - \sqrt{h} = \frac{C_d}{2}\left(\frac{d}{D}\right)^2\sqrt{2g}.t \qquad (3.7)$$

An interesting result can be obtained by setting $h = 0$ in equation (3.6), which would give the time required to empty the tank, if the orifice is at its lowest level.

On the other hand, in Equation (3.5), if the rate at which the liquid level in the tank is neglected with respect to the velocity of discharge from the orifice, we get:

$$h \approx \frac{v_2^2}{2g}$$

From this equation, one can determine the rate of discharge from the orifice. The discharge coefficient is included in the following equation, where a represents the area of the orifice.

$$Q = a.C_d.\sqrt{2gh} \tag{3.8}$$

Example 3.6

A cylindrical tank 1.6 m in diameter contains 4 m^3 of water. An orifice of diameter 20 mm is opened at the bottom side of the tank. Calculate the rate of discharge from the orifice when the liquid reaches half its original level. Find the time required to empty the tank. (Take $C_d = 0.7$)

Solution:

$4 = \frac{\pi}{4} \times 1.6^2 H \rightarrow H = 1.99 \text{ m} \approx 2 \text{ m}$ At one-half level: $h = 1$ m

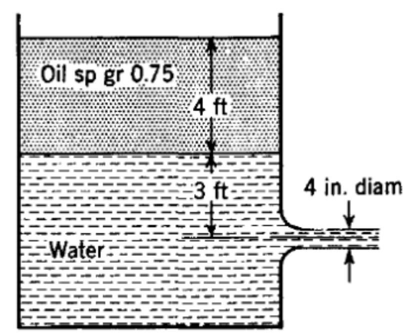
$a = \frac{\pi}{4} \times 0.02^2 = 0.000314 \text{ m}^2$

$Q = a.C_d.\sqrt{2gh} \rightarrow Q = 0.000314 \times 0.7 \times \sqrt{2 \times 9.81 \times 1} = \mathbf{9.736 \times 10^{-4}}$ **m^3.s^{-1}**

$$\sqrt{H} - \sqrt{h} = \frac{C_d}{2}\left(\frac{d}{D}\right)^2 \sqrt{2g}.t \rightarrow \sqrt{2} = \frac{0.7}{2}\left(\frac{0.02}{1.6}\right)^2 \sqrt{2 \times 9.81}t \rightarrow \boldsymbol{t = 5838\text{ s}}$$

Example 3.7

In the figure, the discharge coefficient = 0.8. Calculate the rate of discharge from the side opening.



Solution:

First, the linear dimensions are converted to m.

$4 \text{ ft} \equiv 4 \times 0.305 = 1.22 \text{ m}$

$3 \text{ ft} \equiv 3 \times 0.305 = 0.915 \text{ m}$

$4'' \equiv 4 \times 0.0254 = 0.1016 \text{ m}$

Applying Bernoulli equation in the form (3.8) between free liquid surface and opening (neglect the velocity of liquids in the tank):

$$0 + 750 \times 9.81 \times 1.22 + 1000 \times 9.81 \times 0.915 = \frac{1}{2} \times 1000 v_2^2 + 0 + 0$$

Hence: $v_2 \approx 6$ m.s^{-1}

$Q = a.v_2 = \frac{\pi}{4} \times 0.1016^2 \times 6 = \mathbf{0.0486}$ **m^3.s^{-1}**

3.3.2 The venturi-meter

The Bernoulli equation can be used to measure the flow rate of a liquid in a duct by installing a constriction in its path. This can be a simple circular orifice, or more frequently a type of throat known as **venturi-meter.** Its principle relies on relating the flow rate to the pressure drop occurring when a fluid is forced to pass through a restriction in its path. It was named after the Italian physicist Giovanni venturi who discovered this effect in 1797. This device is shown in Figure (3.5). It consists of two conical pipes joined by a throat. It is installed in the path of the liquid duct as shown in Figure (3.5). A U-tube manometer containing a liquid of density ρ_l. Let the diameter of the pipe = D and that of the venturi throat d. Let the difference between the liquid in manometer = h.

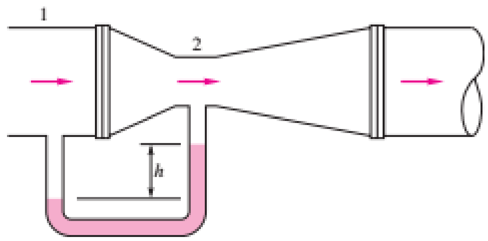


Fig 3.5: The venturi meter

Applying Bernoulli equation between a point on the pipe and the throat:

$$\frac{v_1^2}{2g} + \frac{p_1}{\rho_1 g} + 0 = \frac{v_2^2}{2g} + \frac{p_2}{\rho_2 g} + 0 \qquad \text{And since } \rho_1 = \rho_2 = \rho$$

This equation can be written in the form:

$$v_1^2 = v_2^2 - \frac{2\Delta p}{\rho} \tag{3.9}$$

To obtain the value of pressure drop, consider the lower horizontal level of the liquid in manometer (to the left):

Δp is the pressure drop between the two sides = $\rho g h - \rho_l g h = gh(\rho - \rho_l)$.

The positive numerical value is $\Delta p = gh(\rho_l - \rho)$

Substituting in (3.9):

Hence: $v_1^2 = v_2^2 - \frac{2gh(\rho_l - \rho)}{\rho}$

$$D^2 v_1 = d^2 v_2 \rightarrow v_1 = v_2.\left(\frac{d}{D}\right)^2 \rightarrow v_2^2.\left(\frac{d}{D}\right)^4 = v_1^2 = v_2^2 - \frac{2(\rho_l - \rho)}{\rho} hg$$

$$v_2^2\left(1 - \frac{d^4}{D^4}\right) = \frac{2(\rho_l - \rho)}{\rho} hg \rightarrow v_2 = \sqrt{\frac{\frac{2(\rho_l - \rho)}{\rho} hg}{\left(1 - \frac{d^4}{D^4}\right)}}$$

Since $Q = \frac{\pi}{4} d^2 v_2$, the latter equation can be written in the final form, assuming $C_d \neq 1$:

$$Q = C_d \frac{\pi}{4} d^2 \sqrt{\frac{\frac{2(\rho_l-\rho)}{\rho}hg}{\left(1-\frac{d^4}{D^4}\right)}} \tag{3.10}$$

If the venturi is used to meter the flow rate of a gas, then it is common to neglect the density of the gas with respect to the liquid in the manometer, so that the previous equation reads:

$$Q = C_d \frac{\pi}{4} d^2 \sqrt{\frac{\frac{2\rho_l}{\rho}hg}{\left(1-\frac{d^4}{D^4}\right)}} \tag{3.11}$$

Example 3.8

A venturi meter is installed in a tube of 4" diameter with a 2" orifice to measure the flow rate of water. The manometric fluid is mercury and indicates a pressure drop of 45 mmHg. Calculate the flow rate in $m^3.h^{-1}$ if the discharge coefficient = 0.97.

Solution:

$$\frac{d}{D} = \frac{1}{2} \rightarrow \left(\frac{d}{D}\right)^4 = \frac{1}{16}$$

$$\frac{2(\rho_l - \rho)}{\rho} gh = \frac{2 \times (13600 - 1000)}{1000} \times 9.81 \times 0.045 = 11.124$$

$$Q = 0.97 \times \frac{\pi}{4} (2 \times 0.0254)^2 \sqrt{\frac{11.124}{\left(1-\frac{1}{16}\right)}} = 6.77 \times 10^{-3} \text{ m}^3.\text{s}^{-1} \equiv \mathbf{24.37\ m^3h^{-1}}$$

Example 3.9

A venturi meter is installed in a pipe of 200 mm in diameter where pressurized air at 1.8 bar (abs.) and 15°C flows. The orifice of the venturi meter is 80 mm in diameter and the mercury reading = 230 mm. Estimate the flow rate of air assuming a discharge coefficient of 0.98.

Solution:

$$\rho = \frac{pM}{10^3RT} = \frac{1.8\times10^5\times29}{8314\times(15+273)} = 2.18 \text{ kg.m}^{-3}$$

Substituting in Equation (3.11):

$$Q = 0.98 \times \frac{\pi}{4} 0.08^2 \sqrt{\frac{\frac{2\times13600}{2.18}\times0.23\times9.81}{\left(1-\frac{80^4}{200^4}\right)}} = \mathbf{0.837\ m^3.s^{-1}}$$

FLUID FLOW IN PIPES

4.1 Pattern of flow in ducts

The flow of fluids in ducts can be streamlined, as if the liquid is assumed to be composed of parallel layers, these flow parallel to the axis of the duct. This is called **laminar** (or **streamline flow**).

If, on the contrary, the liquid particles move in their general direction in an erratic way, this is called **turbulent flow**.

The criterion deciding about the type of flow in a circular duct is the Reynolds number $\boldsymbol{Re}$. This is a dimensionless entity defined by:

$$Re = \frac{\rho v D}{\mu} \tag{4.1}$$

The value of this number decides about the pattern of flow in the following way:

- If $Re < 2000$, the flow is laminar.
- If $Re > 4000$, the flow is turbulent.
- For values of Re such that $2000 < Re < 4000$, the flow passes through an intermediate transition zone, gradually shifting from laminar to turbulent.

Example 4.1

Crude oil flows in a pipe 12" in diameter at the rate of 600 $m^3.h^{-1}$ at ^{25oC}. At that temperature, its density = 850 kg,m^{-3} and its viscosity = 11 cP. State whether the flow is laminar or not.

Solution:

$$A = \frac{\pi}{4} \times (12 \times 0.0254)^2 = 0.073 \text{ m}^2$$

$$\frac{600}{3600} = 0.073v \rightarrow v = 2.28 \text{ m.s}^{-1}$$

$$Re = \frac{850 \times 2.28 \times 12 \times 0.0254}{11 \times 10^{-3}} = 53700 > 4000 \rightarrow \textbf{Turbulent flow}$$

4.2 Pressure drop in flow in circular ducts

4.2.1 The Darcy equation

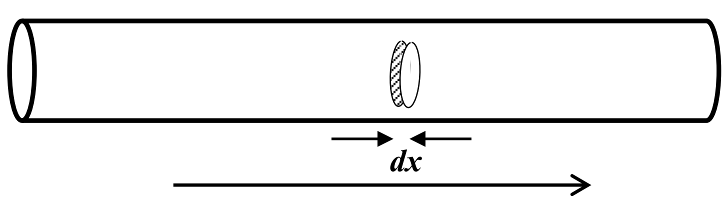


Fig 4.1: Elemental section in circular duct

Consider a fluid flowing through a circular duct of inner radius R and let an infinitesimal portion of length dx with radius r be considered as a control volume of the fluid. The pressure just before this volume is p and decreases to $p + dp$ at the end of the considered portion $(dp < 0)$. Therefore the net force acting on the fluid is $-\pi r^2 . dp$. Also, the shear force due to movement along the walls is $-2\pi r . \tau dx$, where τ is the shear stress developed along a peripheral area = $-2\pi r . dx$. If the velocity of the fluid is considered uniform, then the net force = 0. We get:

$$-\pi r^2 . dp - 2\pi r . dx = 0$$

$$\frac{dp}{dx} = -\frac{2\tau}{r} \qquad (4.2)$$

A similar equation can be deduced for a control volume of radius R and length dx, where the shear stress will be developed at walls τ_w.

$$\frac{dp}{dx} = -\frac{2\tau_w}{R} \qquad (4.3)$$

A **friction factor f** is defined as being the ratio between the shear stress at pipe walls τ_w and the kinetic energy per unit volume of fluid $\left(\frac{1}{2}\rho v^2\right)$. From Equation (4.3):

$$\tau_w = -\frac{R}{2} . \frac{dp}{dx} \qquad (4.4)$$

$$f = \frac{-\frac{R}{2} . \frac{dp}{dx}}{\frac{1}{2}\rho v^2} \qquad (4.5)$$

The term $\frac{dp}{dx}$ can be approximated by $-\frac{\Delta p}{L}$, where L is the length of the duct. This makes use of the following assumption: **The pressure drop per unit length is constant along the pipe**. The following equation, known as the Darcy equation, is obtained, bearing in mind that $D = 2R$:

$$\frac{\Delta p}{L} = \frac{2 f \rho v^2}{D} \qquad (4.6)$$

Example 4.2

Water flows in a 50 mm pipe 120 m long at the rate of 23.9 $m^3.h^{-1}$. If the friction factor is assumed to equal 0.001, calculate the pressure drop due to walls friction.

Solution:

$$A = \frac{\pi}{4} \times 0.05^2 = 1.9635 \times 10^{-3} \mathrm{m}^2 \qquad \frac{23.9}{3600} = 1.9635 \times 10^{-3} v \rightarrow v = 3.381 \mathrm{m.s}^{-1}$$

Hence:

$$\frac{\Delta p}{120} = \frac{2 \times 0.001 \times 1000 \times 3.381^2}{0.05}$$

Hence: $\boldsymbol{\Delta p = 54875\ \mathrm{Pa}}$

4.2.2 Velocity profile in circular duct at laminar flow

When the flow is laminar, two assumptions can be made:

- First, the velocity at walls is negligible with respect to that at axis.
- Second, the velocity is a function of r only, and not of x.

From Equations (4.2) and (4.3), we get:

$$\tau = \frac{\tau_w . r}{R} \tag{4.7}$$

Following the definition of viscosity:

$$\tau = \mu \frac{dv}{dr} \tag{4.8}$$

Equating (2.7) and (2.8), one gets:

$$\frac{\tau_w . r}{R} = \mu \frac{dv}{dr}$$

$$v = \frac{\tau_w}{\mu . R} \int_r^R r . dr$$

$$v = \frac{\tau_w}{2\mu . R} (R^2 - r^2) \tag{4.9}$$

This equation reveals that the distribution of velocity in laminar flow is parabolic as seen in Figure (4.2).

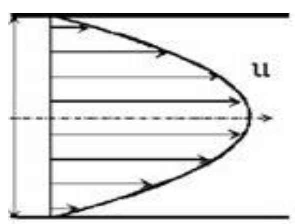


Fig 4.2: Radial velocity profile in laminar flow

The maximum velocity v_{max} is seen to exist at the center line, as $r = 0$. Substituting in the above equation, we get:

$$v_{max} = \frac{\tau_w . R}{2\mu} \tag{4.10}$$

From Equations (4.9) and (4.10), we get:

$$v = v_{max} \left(1 - \left(\frac{r}{R}\right)^2\right) \tag{4.11}$$

The average velocity can be proved to equal ½ the maximum velocity. From equation (4.10), it equals:

$$\bar{v} = \frac{\tau_w . R}{4\mu} \tag{4.12}$$

4.2.3 Pressure drop for laminar flow in circular ducts

In the next sections, we will only deal with the average velocity of the fluid, so that it will be more convenient to simply write it as v instead of $\bar{v}$.

From Equations (4.12) and (4.4):

$v = \frac{\tau_w . R}{4\mu}$ and $\tau_w = -\frac{R}{2} . \frac{dp}{dx}$

Making use of the assumption that $-\frac{dp}{dx} = \frac{\Delta p}{L}$, we can write: $v = \frac{\frac{D}{4} . \frac{\Delta p}{L} . \frac{D}{2}}{4\mu}$

$$\frac{\Delta p}{L} = \frac{32\mu v}{D^2} \qquad (4.13)$$

This equation, known as the **Hagen – Poiseuille equation**, is used to predict the pressure drop in laminar flow.

It is worth noticing that the shear rate at walls can be related to the average fluid velocity from Equation (4.12) since $\tau_w = \mu\dot{\gamma}_w$. This yields:

$$v = \frac{\dot{\gamma}_w . R}{4} = \frac{\dot{\gamma}_w . D}{8}$$

$$\dot{\gamma}_w = \frac{8v}{D} \qquad (4.14)$$

4.2.4 Determination of the friction factor *f*

Comparing Equation (4.13) with (4.6):

$$\frac{2f\rho v^2}{D} = \frac{32\mu v}{D^2} \qquad f = \frac{16\mu}{\rho v D}$$

Hence, **for laminar flow**:

$$f = \frac{16}{Re} \qquad (4.15).$$

For **transition and turbulent flow**, the value of f not only depends on the Reynolds number, but also on the roughness of the internal surface of the pipe. Because of manufacturing defects and eventual erosion and corrosion, asperities develop along the internal surface. The absolute roughness ϵ is the average height or depth of these defects. The **relative roughness** is obtained by dividing the average absolute roughness by the internal pipe diameter = ϵ/D.

The value of f is obtained from generalized charts known as **Moody charts**. These are curves obtained experimentally and shown as a log – log plot between f and Re for different values of relative roughness.

For laminar flow, if the logarithms of both sides of Equation (4.15) are taken we get:

$\ln f = \ln 16 - \ln Re$

On a $\ln f - \ln Re$ plot, this relation will show up as a straight line with slope = -1., that is sloping 135° to the horizontal.

Moody charts are shown in Figure (4.3). The line corresponding to laminar flow does not look like sloping 135°, as predicted. The reason is that the horizontal axis on the chart starts at number 10^2.

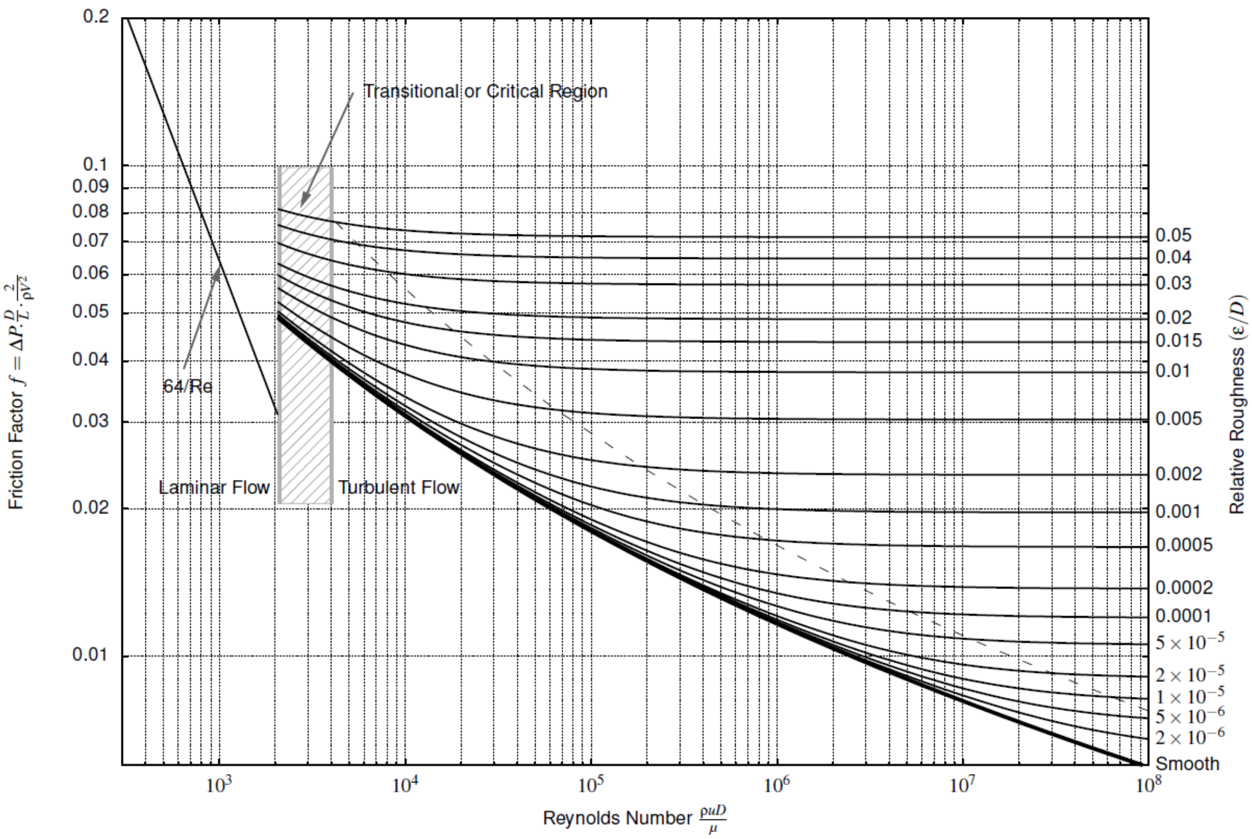


Fig 4.3: Moody charts

Example 4.3

Crude oil (Specific gravity = 0.86) flows inside a stainless-steel pipe of 18" diameter at 1800 $m^3.h^{-1}$, with relative roughness = 2×10^{-5}. The pipe is 200 m long and the kinematic viscosity of the crude = 11 cSt. Determine the pressure drop in bar.

Solution:

$\rho = 860$ kg.m^{-3} and $\nu = 11 \times 10^{-6}$ m^2.s-1

Hence, $\mu = 860 \times 11 \times 10^{-6} = 0.00946$ Pa.s

$A = \frac{\pi}{4} \times (18 \times 0.0254)^2 = 0.1642$ m^2

$v = \frac{\frac{1800}{3600}}{0.1642} = 3.0456$ m.s^{-1}

$Re = \frac{860 \times 3.0456 \times 18 \times 0.0254}{0.0946} = 126584 > 4000$ Turbulent flow

From Figure (4.3), at the given relative roughness and $Re \approx 125000$, $f = 0.016$

From equation (4.6):

$$\frac{\Delta p}{200} = \frac{2 \times 0.016 \times 860 \times 3.0456^2}{18 \times 0.0254} = 111665 \text{ Pa} \equiv \mathbf{1.117\ bar}$$

Example 4.4

Molasse is being transported in a pipe 30 m long and 60 mm in diameter, at the rate of 45 $L.min^{-1}$, The density and the viscosity of molasse at the prevailing temperature are 1500 $kg.m^{-3}$ and 8100 cP respectively. Evaluate the pressure drop in kPa.

Solution:

$$A = \frac{\pi}{4} \times 0.06^2 = 2.2827 \times 10^{-3} \text{m}^2$$

$$v = \frac{\frac{0.045}{60}}{2.2827 \times 10^{-3}} = 0.265 \text{ m.s}^{-1}$$

$$Re = \frac{1500 \times 0.265 \times 0.06}{8.1} = 2.947 < 2000 \quad \text{Laminar flow}$$

$$\frac{\Delta p}{30} = \frac{32 \times 8.1 \times 0.265}{0.06^2} \rightarrow \Delta p = 572957 \text{ Pa} \equiv \mathbf{573\ kPa}$$

4.2.5 Head losses due to interruptions in the fluid path

When a fluid flows in a pipe, it often encounters obstacles that interrupt its flow, causing the fluid pressure to drop. The main interruptions to flow are:

- Sudden enlargement
- Sudden contraction
- Gradual enlargement
- Gradual contraction
- 90° threaded elbows
- 90° flanged elbows
- Valves (Globe)

The general form of head losses due to interruptions is:

$$h_{loss} = K.\frac{v^2}{2g} \tag{4.16}$$

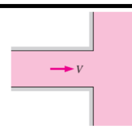
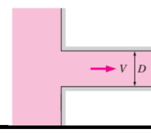
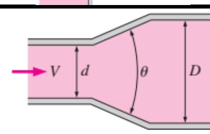
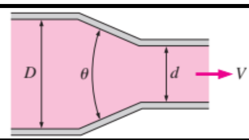
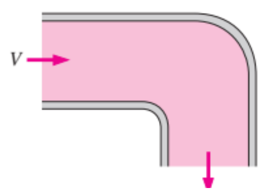
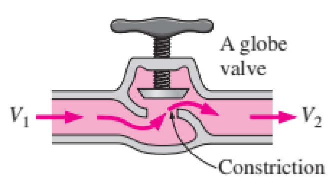
The values of the constant K for different types of interruptions, together with diagrammatic representations of these cases are illustrated in Table (4.1).

Therefore, in flow of fluid through circular ducts, the head losses due to friction and interruptions have both to be taken into consideration. This results in rewriting Bernoulli equation in the following generalized form:

$$\frac{v_1^2}{2g} + \frac{p_1}{\rho_1 g} + h_1 = \frac{v_2^2}{2g} + \frac{p_2}{\rho_2 g} + h_2 + \frac{2fLv^2}{g.D} + \sum K_{Li} . \frac{v^2}{2g} \qquad (4.17)$$

The friction head loss $\frac{2fLv^2}{g.D}$ is deduced from Equation (4.6) by dividing by ρg.

Table 4.1: Values of *K* for flow interruptions

Type	**Diagrammatic Representation**	***K***
Sudden enlargement		$(\mathbf{1} - (\boldsymbol{d}/\boldsymbol{D})^2)^2$
Sudden contraction		**0.2 – 0.5**
Gradual enlargement		**0.02 – 0.05**
Gradual contraction		**0.2 – 0.6**
90º threaded elbow		**0.9**
90º flanged elbow		**0.3**
Open globe valve		**10**

4.2.6 Pumping power

Pumps are used to overcome the different lead losses besides imparting higher velocities and pressures to the pumped liquid.

The pumping power (W) is calculated by knowing the total pressure difference (Pa) and the volumetric flow rate ($m^3.s^{-1}$) by use of the following equation:

$$\mathscr{P} = \frac{Q.\Delta P}{\eta} \tag{4.18}$$

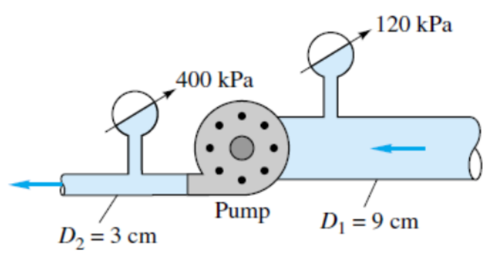
Here, η is the pump efficiency, which normally varies from 0.6 to 0.9.

It is also possible to express the power in horsepower (hp) by dividing the power in Watt by 735. This is known as the **Break Horsepower** of the pump (BHP).

The pressure difference is obtained as follows: A pump must in general supply the three types of heads besides overcoming head losses. The pressure difference that must be overcome by the pump Δp is calculated from the Bernoulli equation as follows:
Let the initial pressure be p_1, the velocity v_1 and the vertical height h_1 and let the corresponding conditions following the pump be p_2, v_2 and h_2. Then:

$$\Delta p = \frac{1}{2}\rho_2 v_2^2 + \rho_2 g h_2 + p_2 - \left(\frac{1}{2}\rho_1 v_1^2 + \rho_1 g h_1 + p_1\right) + \Delta p_{losses} \tag{4.19}$$

Example 4.5
The pump shown discharges water at 57 $m^3.h^{-1}$. Estimate the pump horsepower in kW, assuming 70% efficiency.



Solution:

$A_1 = \frac{\pi}{4} \times 0.09^2 = 6.636 \times 10^{-3} \text{m}^2$ $\qquad v_1 = \frac{\frac{57}{3600}}{6.636\times10^{-3}} = 2.488 \text{ m.s}^{-1}$

$A_2 = \frac{\pi}{4} \times 0.03^2 = 7.069 \times 10^{-4} \text{m}^2$ $\qquad v_2 = \frac{\frac{57}{3600}}{7.069\times10^{-4}} = 22.4 \text{ m.s}^{-1}$

$p_1 = 120000$ Pa

$p_2 = 400000$ Pa

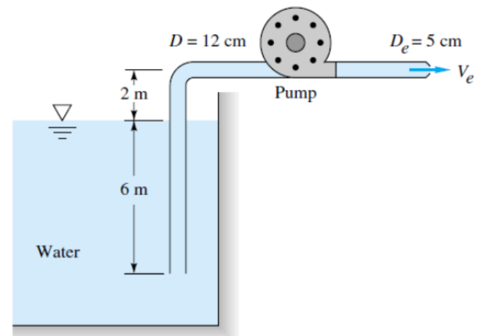
Substituting in Equation (4.19), assuming no losses:

$$\Delta p = \frac{1}{2} \times 1000 \times 22.4^2 + 400000 - \left(\frac{1}{2} \times 1000 \times 2.488^2 + 120000\right) = 527770 \text{ Pa}$$

$$\mathscr{P} = \frac{\frac{57}{3600}\times527770}{0.7\times1000} = \mathbf{11.94\ kW}$$

Example 4.6

When the pump shown in figure draws water from the reservoir at the rate of 220 $m^3.h^{-1}$, the head losses = 5 m. Estimate the pump theoretical power in kW.



Solution:

$$A_1 = \frac{\pi}{4} \times 0.12^2 = 0.01131 \text{m}^2$$

$$v_1 = \frac{\frac{220}{3600}}{0.01131} = 5.403 \text{ m.s}^{-1}$$

$$A_2 = \frac{\pi}{4} \times 0.05^2 = 1.963 \times 10^{-3} \text{m}^2$$

$$v_2 = \frac{\frac{220}{3600}}{1.963\times10^{-3}} = 31.131 \text{ m.s}^{-1}$$

$$\Delta p_{losses} = 5 \times 1000 \times 9.81 = 49050 \text{ Pa}$$

Applying Equation (4.19):

$$\Delta p = \frac{1}{2} \times 1000 \times 31.131^2 + 2 \times 1000 \times 9.81 + 49050 - \left(\frac{1}{2} \times 1000 \times 5.403^2\right) = 538643 \text{ Pa}$$

$$\mathscr{P} = \frac{\frac{220}{3600}\times538643}{1000} = \mathbf{32.9 \text{ kW}}$$

Example 4.7

In the figure, water flows at 54 $m^3.h^{-1}$ in a 60 mm pipe in position (1). The head friction losses amount to 5.2 m and two 90° flanged elbows are placed on the line. A pump is to raise water at a level of 3 m and delivers it to a nozzle 30 mm in diameter at position (2). Assuming 70% efficiency, calculate the BHP of the pump.

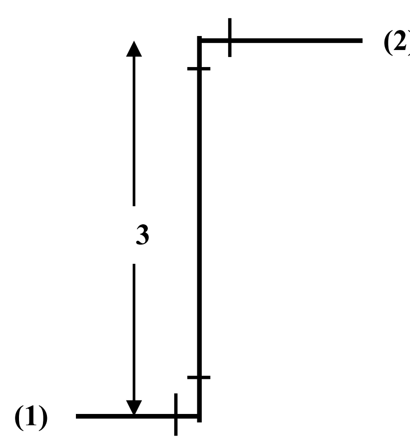
Solution:

$$A_1 = \frac{\pi}{4} \times 0.06^2 = 2.827 \times 10^{-3} \text{m}^2 \qquad v_1 = \frac{\frac{54}{3600}}{2.827\times10^{-3}}$$

= 5.306 $m.s^{-1}$

$$A_2 = \frac{\pi}{4} \times 0.03^2 = 7.069 \times 10^{-4} \text{m}^2 \qquad v_2 = \frac{\frac{54}{3600}}{7.069\times10^{-4}}$$

= 21.22 $m.s^{-1}$



$$\Delta p_{loss} = 5.2 \times 1000 \times 9.81 = 51012 \text{ Pa}$$

$$\Delta p_{elbow1} = 1000 \times 0.3 \times \frac{5.306^2}{2} = 4223 \text{ Pa}$$

$$\Delta p_{elbow2} = 1000 \times 0.3 \times \frac{21.22^2}{2} = 67543 \text{ Pa}$$

$$\Delta p = \left(\frac{1}{2} \times 1000 \times 21.22^2 + 3 \times 9.81 \times 1000 + 4223 + 67543 + 51012\right) - \left(\frac{1}{2} \times 1000 \times 5.306^2\right) = 363275 \text{ Pa}$$

$$\mathscr{P} = \frac{\frac{54}{3600} \times 363275}{0.75 \times 735} = 9.88 \rightarrow \mathbf{10\ hp}$$

4.3 Pressure drop of compressible fluids

To evaluate the pressure drop in the flow of compressible fluids, care must be taken that gases suffer from changes in their densities as pressure is varied. Also, A variation in temperature will likewise cause a change in density. The continuity equation must be applied therefore in its general form:

$$\rho_1 . A_1 . v_1 = \rho_2 . A_2 . v_2 \qquad (3.1)$$

The density is determined from the expression:

$$\rho = \frac{pM}{10^3 RT} \qquad (1.6)$$

Example 4.9

Warm air enters a circular duct of 12" diameter at a pressure of 1.08 bar and at 35ºC. Along the path cools down and at the end of the pipe, it is discharged to the atmosphere at 25ºC. If the velocity of warm air = 6 m.s^{-1}, at what velocity is it discharged at the end of the duct? (Neglect pressure losses).

Solution:

Since there is no variation in pipe diameter, then: $\rho_1 . v_1 = \rho_2 . v_2$

Equation (1.6) reveals that density is inversely proportional to temperature (K). Hence:

$$\frac{\rho_2}{\rho_1} = \frac{p_2 T_1}{p_1 T_2} = \frac{1}{1.08} \times \frac{35 + 273}{25 + 273} = \frac{308}{321.84} = \frac{v_1}{v_2} = \frac{6}{v_2}$$

Consequently:

$$\boldsymbol{v_2 = 6.27\ \mathrm{m.s^{-1}}}$$

Example 4.10

Natural gas possesses a molecular weight=17 and is assumed to behave ideally. It enters a horizontal circular duct 18" in diameter at a speed of 36 m.s^{-1} and 30ºC at an absolute pressure of 15.2 psi. What would be the pressure at the outlet if it cools down to 20ºC. (Neglect head losses)

Solution:

The Bernoulli equation is applied in the form:

$$\frac{1}{2}\rho_1 v_1^2 + \rho_1 g h_1 + p_1 = \frac{1}{2}\rho_2 v_2^2 + \rho_1 g h_2 + p_2$$

The inlet pressure = 15.2 psi ≡ 1.034 atm. ≡ 1.0476 bar ≡ 1.0476×10^5 Pa

$$\rho_1 = \frac{p_1 M}{10^3 RT} = \frac{1.0476\times10^5\times17}{1000\times8.314\times(30+273)} = 0.707 \text{ kg.m}^{-3}.$$

$$\rho_2 = \frac{p_2 M}{10^3 RT} = \frac{p_2\times17}{1000\times8.314\times(20+273)} = 6.98\times10^{-6} p_2 \text{ kg.m}^{-3}$$

$$\rho_1.v_1 = \rho_2.v_2 \rightarrow 0.707\times36 = 6.98\times10^{-6} p_2\, v_2 \rightarrow v_2 = \frac{3.646\times10^6}{p_2} \quad \text{(i)}$$

$$\frac{1}{2}\rho_1 v_1^2 + p_1 = \frac{1}{2}\rho_2 v_2^2 + p_2$$

$$\frac{1}{2}\times0.707\times36^2 + 1.0476\times10^5 = \frac{1}{2}\times6.98\times10^{-6} p_2 \times\left(\frac{3.646\times10^6}{p_2}\right)^2 + p_2$$

$$105218.14 = 3.49\times10^{-6}\times\frac{13.065\times10^{12}}{p_2} + p_2$$

$$105218.14 p_2 = 4.559\times10^7 + p_2^2$$

Solving the quadratic equation $p_2^2 - 105218.14 p_2 + 4.559\times10^7 = 0$ yields two positive solutions:

$p_2 = 1.0478\times10^5$ Pa and 435 Pa.

The inlet total pressure (Static + dynamic) being 1.274×10^5 Pa, the first solution is the more logical one.

$\boldsymbol{p_2 = 1.0478\times10^5}$ **Pa**

Had the second solution been chosen, the outlet speed would have equaled 8381 m.s^{-1}. That is, more than five times the speed of sound in water (about 1500 m.s^{-1}).

4.4 Choice of pipes

4.4.1 The schedule number

As materials with higher mechanical strength and higher corrosion resistance properties were developed, it became possible to vary the thickness of pipes considerably. To accommodate the many pipe thickness sizes developed, the **schedule (SCH) system** was invented. This system standardizes pipe thickness. Actually, in all previous sections, the symbol *D* referred to the internal pipe diameter whereas pipes are usually defined by their **nominal diameter**.

The nominal diameter is a standard diameter characterizing the pipe, to which may correspond more than one inner and outer diameter.

For example, a pipe with nominal diameter 40 mm, can have several inner diameters corresponding to a fixed outer diameter, depending on its schedule number which determines its thickness, as can be followed in the following table:

Table 4.2: Schedule numbers of pipe with nominal diameter 40 mm

Schedule #	5	10 – 20	30	40	80
D_{out} mm	48.26 mm				
Thickness mm	1.651	2.769	3.175	3.683	5.080
D_{in} mm	44.958	42.722	41.985	40.894	38.10

4.4.2 Choice of proper pipe schedule

The thickness of a pipe depends on the maximum gauge pressure developed inside the pipe and the material of construction of the pipe. In general, for a thickness t mm, the internal pressure is related to the maximum allowed tensile stress of the pipe material by the formula:

$$\sigma_{max} = \frac{p.D_{in}}{2t-C} \tag{4.20}$$

In that equation:

p is the gauge pressure (Pa)

D_{in} is the inner pipe diameter (mm)

C is termed corrosion allowance (mm)

σ_{max} is the maximum allowable stress (Pa)

The value of C depends on the type of fluid and its corrosive or erosive nature. It usually ranges from 0 to 6 mm.

The maximum allowable strength depends on the type of material used for the pipe. Table (4.3) gives it value for some common pipe materials.

Table 4.3: Approximate maximum allowable stress (MPa)

Material	**Low C steel**	**Stainless steel**	**Nickel**	**Polypropylene**
Max All. Stress	124 – 138	138 – 186	275 – 372	1.93

Example 4.11

A 40 mm Schedule 40 polypropylene pipe is used to handle domestic water. Considering a zero-corrosion allowance, what is the maximum gauge pressure this pipe can withstand?

Solution:

$D_{in} = 40.894$ mm

$t = 3.683$ mm

$C = 0$

$1.93 \times 10^6 = \frac{p \times 40.984}{2 \times 3.683}$ Hence, $\boldsymbol{p_{max} = 346876\ \mathrm{Pa} \equiv 3.47\ \mathrm{bar}}$

Example 4.12

Water is pumped from the basement of a building from a water tank to a level 30 m higher at the rate of 10.4 $m^3.h^{-1}$ in a 40 mm Schedule 30 carbon steel pipe to deliver it at atmospheric pressure. The total losses correspond to a head of 6.3 m. Calculate the pressure developed by the pump and find whether the pipe can withstand that pressure. (Assume $C = 0$.and neglect the initial velocity of water)

Solution:

From Table (4.2), for the chosen pipe: $D_{in} = 0.041985$ m

$A_1 = \frac{\pi}{4} \times 0.041985^2 = 1.384 \times 10^{-3} \mathrm{m}^2$ $v_1 = \frac{\frac{10.4}{3600}}{1.384 \times 10^{-3}} = 2.087$ $m.s^{-1}$

$\Delta p = \frac{1}{2} \times 1000 \times 2.087^2 + (6.3 + 30) \times 9.81 \times 1000 = 358281$ Pa

Maximum allowable stress = 1.24×10^8 Pa Lowest strength from Table (4.3)

Thickness = 3.175 mm

$\sigma = \frac{358281 \times 41.985}{2 \times 3.175} = 2.37 \times 10^6 < 1.24 \times 10^8$

Therefore, the pipe will withstand the pressure.

CHAPTER 5

FLOW PAST IMMERSED BODIES

5.1 Flow streamlines

These are hypothetical lines showing the direction of the velocity vector of the fluid in its different regions. These indicate the direction of motion of the different layers of the fluid (Figure 5.1).

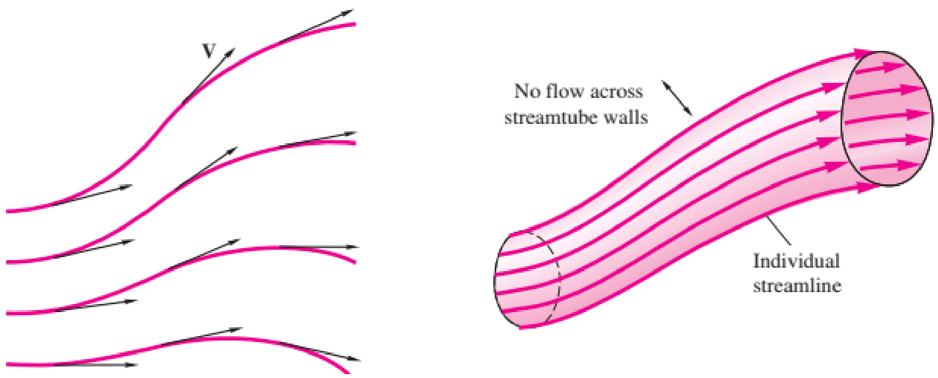


Fig 5.1: Streamlines in flow through a duct

In the case of laminar flow, these lines are almost parallel during the flow of a fluid in a duct, whereas they take more erratic directions in the case of turbulent flow. (Figure 5.2).

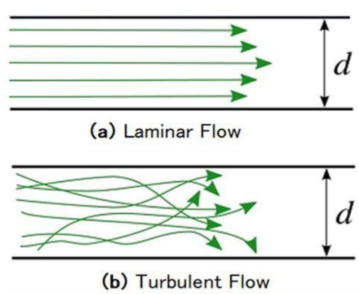


Fig 5.2: Streamlines in laminar and turbulent flow

5.2 Flow past rigid bodies

5.2.1 Patterns of flow

In case of flow past a body, like a cylindrical disk, the idealized pattern would indicate regular streamlines only disturbed by the body, to re-align once more after crossing the body. What actually happens is that strong disturbances occur as a result of impingement against the body, strongly disrupting the regular flow pattern. (Figure 5.3)

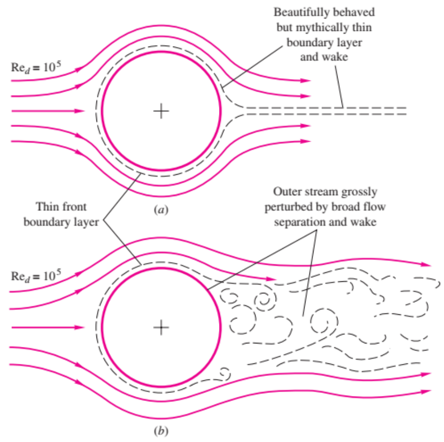


Fig 5.3: Idealized and actual flow pattern for flow past a disk

5.2.2 The drag force

As a result of the disturbance following flowing past an immersed body, a force acting on the body develops, known as the **drag force.** Following Newton's third law, the solid opposes a force of equal magnitude and opposite direction to fluid motion. This is defined as follows:

$$F_d = \frac{1}{2} C_d \rho v^2 A_p \tag{5.1}$$

Where:

C_d is known as the **drag coefficient** and is generally a function of Reynolds number.

A_p is the frontal projected area opposed to fluid flow

For flow past a plate of length L, thickness t and breadth b, the drag coefficient is calculated from:

$$C_d = 1.328 Re_L^{-0.5} \ (Re < 1) \tag{5.2}$$

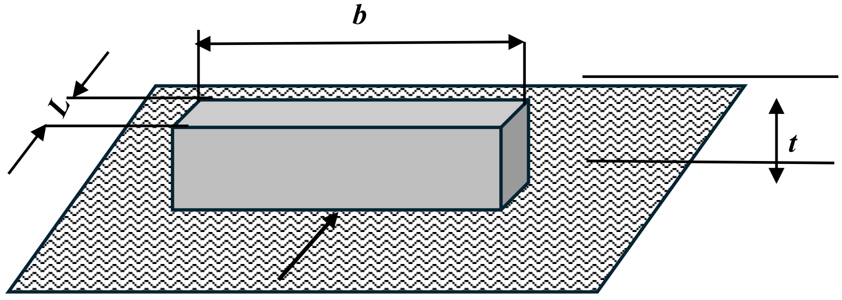


Fig 5.4: Flow past a plate

$C_d = 1 \quad (Re > 2000)$

Where Re_L is the Reynolds number based on the length of the plate.

If $1 < Re < 2000$, special charts relating the drag coefficient to Re must be used.

In that case, $A_p = b.t$

For flow past a cylinder in a direction perpendicular to its axis (Figure 5.3), in laminar flow ($Re < 1$), the drag coefficient is related to Reynolds number by the following expression, where the Reynolds number is based on the diameter of the cylinder.:

$$C_d = 10Re^{-1} \qquad (5.3)$$

For turbulent flow, $Re > 1000$, the drag coefficient is almost constant $C_d \approx 1$.
In that case, the projected area is $2R.h$

For flow past a sphere, in laminar flow ($Re < 1$), the drag coefficient is related to Reynolds number by the following expression, where the Reynolds number is based on the diameter of the sphere:

$$C_d = 24Re^{-1} \qquad (5.4)$$

For turbulent flow, $Re > 2000$, the drag coefficient is almost constant at $C_d \approx 0.44$.
The projected area = πR^2.

Example 5.1
In a river of approximate rectangular shape of width 8 m and water height = 2.8 m, water flows at 36 $m^3.s^{-1}$. It encounters in its flow a large concrete cube of side = 800 mm weighing 1.56 ton. Calculate the drag force acting on the rock.
Solution:

$$A = 8 \times 2.8 = 22.4\ \text{m}^2 \rightarrow v = \frac{36}{22.4} = 1.607\ \text{m.s}^{-1}$$

$$Re = \frac{1000\times1.607\times0.8}{0.001} = 1285600 > 2000$$

$$C_d = 1 \rightarrow \ F_d = \frac{1}{2}C_d\rho v^2 A_p = 0.5 \times 1 \times 1000 \times 1.607^{\,2} \times 0.8^2 = \mathbf{826\ N}$$

Example 5.2
Molasse of specific gravity 1.45 flows in a pipe at 0.2 $m.s^{-1}$ past a spherical pebble of 5 mm diameter. Estimate the drag force due to the presence of the pebble in the fluid path (Viscosity = 7900 Pa.s).
Solution:

$$Re = \frac{1450\times0.2\times0.005}{7.9} = 0.1835 < 1$$

Therefore $C_D = \frac{24}{0.1835} = 130.8$

$A_p = 0.25\pi \times 0.005^2 = 1.9635 \times 10^{-5} \text{m}^2$

$$F_d = \frac{1}{2} 130.8 \times 1450 \times 0.2^2 \times 1.9635 \times 10^{-5} = \mathbf{0.0745\ N}$$

Example 5.3

Water flows in a channel at 2.5 m.s^{-1}, when it crosses a cylindrical obstacle 50 mm in diameter and 300 mm high. Evaluate the drag force.

Solution:

$Re = \frac{1000 \times 2.5 \times 0.05}{0.001} = 1.25 \times 10^5 > 1000$ Hence, $C_D = 1$

$A_p = 0.05 \times 0.3 = 0.015\ \text{m}^2$

$$F_d = \frac{1}{2} \times 1 \times 1000 \times 2.5^2 \times 0.015 = \mathbf{46.875\ N}$$

Example 5.4

The van in the figure is 1904 mm wide and 2170 mm high. It moves at a velocity of 100 km.h^{-1}, facing incoming wind in the opposite direction at 35 km.h^{-1} at 20°C and about 1 atm. Under these conditions, the drag coefficient is assumed to be constant and equals 0.6. Evaluate the drag force acting on the car.

Solution:

$A_p = 1.904 \times 2.17 = 4.132\ \text{m}^2$

The actual velocity of air is the relative velocity between the wind and the car = $100 + 35 = 135\ \text{km.h}^{-1} = \frac{135000}{3600} = 37.5\ \text{m.s}^{-1}$

$\rho = \frac{pM}{10^3 RT} = \frac{1.0132 \times 10^5 \times 29}{1000 \times 8.314 \times (273 + 20)} = 1.206\ \text{kg.m}^{-3}$

$$F_d = \frac{1}{2} \times 0.6 \times 1.206 \times 37.5^2 \times 4.132 = \mathbf{2102\ N}$$

5.2.3 An application: Fall of a sphere in a fluid medium

When a body falls vertically into a fluid, there is a relative motion between the fluid and the sphere so that the drag force, as calculated from the last section, can be applied.

Consider the vertical motion of a spherical body of density ρ_s and diameter D in a fluid medium of density ρ_f. As can be seen in Figure (5.5), this body is under the action of three vertical forces:

- Gravity force F_g, which represents its weight $= \rho_s V g$
- Buoyant force F_b, which, according to Archimede's principle, equals the weight of the displaced fluid $= \rho_f V g$

- Drag force F_d, calculated from Equation (5.3) = $\frac{1}{2}C_d\rho v^2 A_p$

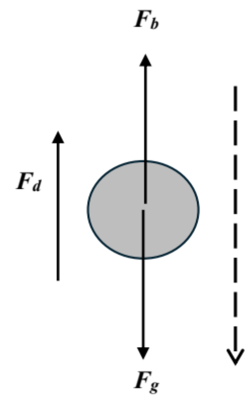


Fig 5.5: Forces acting on a sphere falling through a fluid

The equation of motion of this body can be written as:

$$\rho_s Vg - \rho_f Vg - \frac{1}{2}C_d\rho_f v^2 A_p = m\frac{dv}{dt} \qquad (5.5)$$

In general, the velocity of the body decreases in the first few centimeters to reach a uniform velocity so that $dv/dt = 0$, and the body reaches a **terminal velocity v_t**.

Since the body is spherical in shape, then:

$V = \frac{4\pi}{3}r^3$ and $A_p = \pi r^2$, which on substituting in Equation (5.5) as the body reaches its terminal velocity, yields:

$$v_t^2 = \frac{8r(\rho_s-\rho_f)g}{3C_d\rho_f} \qquad (5.6)$$

The Reynolds number is defined as follows:

$$Re = \frac{\rho_f.v_t.D}{\mu} \qquad (5.7)$$

If the flow is **laminar** $(Re < 1)$, then, according to Equation (5.5), $C_d = 24Re^{-1}$. Substitution of Re from Equation (5.9), and noticing that $r = D/2$, an expression for v_t appears in the following form, known as **Stokes law**:

$$v_t = \frac{gD^2(\rho_p-\rho_f)}{18\mu} \qquad (5.8)$$

If the flow is **turbulent** $(Re > 2000)$, then $C_d \approx 0.44$, and the following form for v_t is obtained:

$$v_t = \sqrt{\frac{3gD(\rho_p - \rho_f)}{\rho_f}} \qquad (5.9)$$

For values of Reynolds numbers ranging from 1 to 2000, charts for C_d as function of Re are rather used.

In calculating the terminal velocity, care should be taken to check the Reynolds number after obtaining the value of v_t to ensure that the right equation has been used.

Example 5.5

10 μm spherical dust particles of density 700 kg.m^{-3} settle in air at 25°C in a chamber 2 m high. At that temperature, the viscosity of air = 0.018 cP. Calculate the time required for these particles to settle.

Solution:

Assuming Stokes law to apply, the density of air is first calculated:

$$\rho = \frac{pM}{10^3 RT} = \frac{1.0132\times10^5\times29}{1000\times8.314\times(273+25)} = 1.186 \text{ kg.m}^{-3}$$

Substituting in equation (5.10):

$$v_t = \frac{9.81\times(10^{-5})^2\times(700-1.186)}{18\times1.8\times10^{-5}} = 0.00212 \text{ m.s}^{-1}$$

Time required to reach bottom of chamber = $\frac{2}{0.00212}$ = $\mathbf{945\ s}$

Check on Re:

$$Re = \frac{1.186\times0.00212\times10^{-5}}{1.8\times10^{-5}} = 1.39\times10^{-3} \ll 1$$. The use of Stokes law is justified.

Example 5.6

Spherical oil drops of diameter 100 μm are allowed to rise in a water tank 3 m high. Their density = 800 kg.m^{-3} and their viscosity = 25 cP. Calculate the time required to free water from oil.

Solution:

Assuming Stokes law to apply:

$$v_t = \frac{9.81\times(10^{-4})^2\times(1000-800)}{18\times0.001} = 1.09\times10^{-3} \text{ m.s}^{-1}$$

Time required for oil to reach the top = $\frac{3}{1.09\times10^{-3}}$ = $\mathbf{2752\ s}$

Check on Re: $Re = \frac{1000\times0.00109\times10^{-4}}{0.001} = 0.109 \ll 1$. The use of Stokes law is justified

Example 5.7

A spherical 2 kg steel ball of density 7850 kg.m^{-3}, is left to fall in water. Calculate its terminal settling velocity.

Solution:

Volume of ball = $\frac{2}{7850} = \frac{4\pi}{3} r^3 \rightarrow r = 0.0393\text{m} \rightarrow D = 0.0786\text{ m}$

If Stokes law is assumed to apply:

$v_t = \frac{9.81 \times (0.0786)^2 \times (7850 - 1000)}{18 \times 0.001} = 415149\text{ m. s}^{-1}$, which is impossible.

So, we assume that equation (5.11) applies:

$$v_t = \sqrt{\frac{3gD(\rho_p - \rho_f)}{\rho_f}} = v_t = \sqrt{\frac{3 \times 9.81 \times 0.0796 \times (7850 - 1000)}{1000}} = \mathbf{4\ m.s^{-1}}$$

Check on Re: $Re = \frac{1000 \times 4 \times 0.257}{0.001} = 1.03 \times 10^6 \gg 2000.$

The use of equation (5.9) is justified

CHAPTER 6

FLOW OF NON-NEWTONIAN FLUIDS

6.1 Introduction

The basic definition of viscosity is based on the following equation, which shows that this parameter is the proportionality constant of the linear dependence of shear stress on shear rate"

$$\tau = \mu.\dot{\gamma} \qquad (1.2)$$

Some liquids, however, do not follow this simple rule, as the relation between τ and $\dot{\gamma}$ is not as simple as that expressed in Equation (1.2). Such fluids are called **Non – Newtonian** and the study of their properties is called **Rheology.** The relation between shear stress and shear rate is called the **constitutive law** of the fluid. The general form of that law is:

$$\tau = f(\dot{\gamma}) \qquad (6.1)$$

The importance of rheology resides in the fact that non – Newtonian fluids are very common in engineering applications, and the application of the conventional equations of flow does not lead to satisfactory results for the prediction of pressure drop.

6.2 Classification of non – Newtonian fluids

There are three main categories of non – Newtonian fluids, namely: Time independent fluids, time dependent fluids and visco-elastic fluids. Since most non – Newtonian fluids encountered in practice are of the first type, this will represent the main topic discussed in this chapter. The main types included in that category are detailed in the following sections.

6.2 Bingham fluids

6.2.1 Constitutive law

Bingham fluids possess a constitutive law close to that of Newtonian fluids (1.2), except for the presence of a constant term in the equation, taking the form:

$$\tau = \tau_0 + k.\dot{\gamma} \qquad (6.2)$$

The constant term τ_0 is known as the **yield stress**, which represents the initial stress necessary to initiate motion. The constant k is known as the **plastic viscosity**, and its physical meaning can be understood by dividing both sides of Equation (6.2) by $\dot{\gamma}$.

$$\frac{\tau}{\dot{\gamma}} = \frac{\tau_0}{\dot{\gamma}} + k$$

The LHS represents the viscosity of the liquid, which is seen to decrease with the shear rate. As $\dot{\gamma} \rightarrow \infty, \frac{\tau}{\dot{\gamma}} \rightarrow k$. This means that the plastic viscosity is the limiting viscosity of the liquid at very high values of shear rate. That is why, it is often referred to as μ_{∞}.

Figures (6.1) and (6.2) show the shear stress – shear rate and the viscosity shear rate relations for Bingham fluids, against Newtonian fluids.

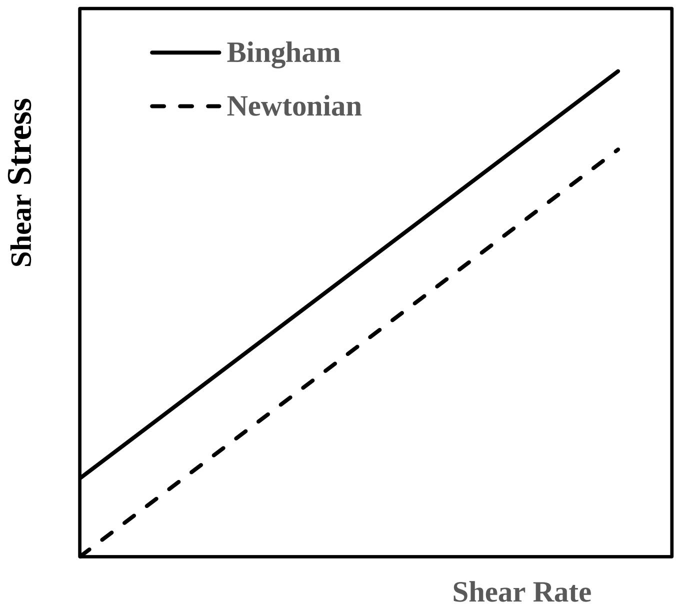


Fig 6.1: Shear stress – Shear rate relation for Bingham fluids

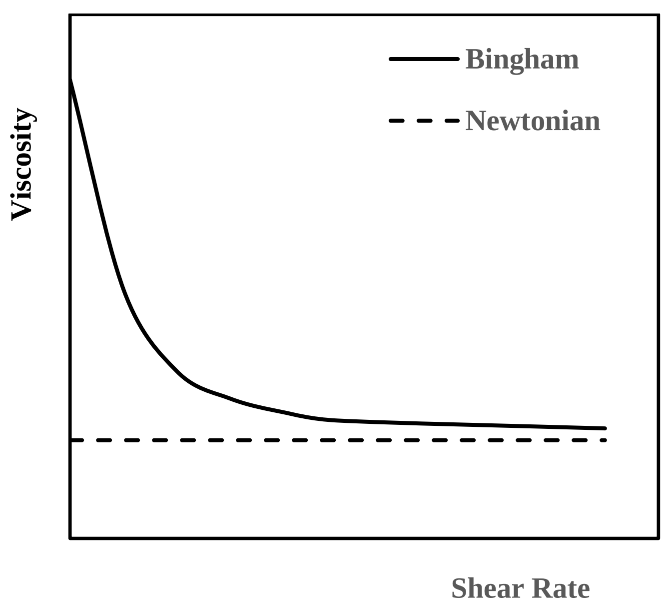


Fig 6.2: Viscosity – Shear rate relation for Bingham fluids

Examples of Bingham behavior are particularly found in the oil drilling industry, where most drilling fluids belong to that type. Other examples include toothpaste, tomato sauce and some types of paints.

6.2.2 Pressure drop correlation for Bingham fluids

To determine the pressure drop in case of Bingham flow, the Darcy equation is used:

$$\frac{\Delta p}{L} = \frac{2f\rho v^2}{D} \qquad (4.6)$$

The main problem is the determination of the friction factor f, since it cannot be obtained using the conventional methods used for Newtonian fluids. Since most Bingham fluid flow is laminar, this will be the only case of interest.

The Reynolds number is defined by Equation (4.1) as is the case in Newtonian fluids:

$$Re = \frac{\rho v D}{\mu_\infty} \qquad (4.1)$$

Another dimensionless group, known as the Hedström number is used in that contest. It is defined as follows:

$$He = \frac{\rho \tau_0 D}{\mu_\infty^2} \qquad (6.3)$$

The friction factor is then calculated from the following expression:

$$f = \frac{46}{Re}\left[1 + \frac{He}{6Re} - \frac{64He^4}{3f^3Re^7}\right] \qquad (6.4)$$

The presence of f in both sides necessitates solving using a trial-and-error technique.

Example 6.1

Acrylic paint is produced in a factory and is allowed to flow in a 50 mm pipe to be distributed into containers at a rate of 500 L.h^{-1}. Its specific gravity = 1.6. This type is known to exhibit Bingham behavior of constitutive law:

$\tau = 0.032 + 1.93\dot{\gamma}$ Pa

Determine the Reynolds and the Hedström numbers, then predict the value of the friction factor. What would the pressure drop over 10 m be?

Solution:

From the constitutive equation, $\tau_0 = 0.032$ Pa and $\mu_\infty = 1.33$ Pa.s

$$A = 0.25\pi \times 0.05^2 = 0.0019635 \text{ m}^2 \rightarrow v = \frac{0.5}{3600 \times 0.0019635} = 0.0707\ m.s^{-1}$$

$Re = \frac{1600 \times 0.0707 \times 0.05}{1.33} = 4.25 \ll 2000$

$He = \frac{1600 \times 0.032 \times 0.05}{1.33^2} = 1.925$

Equation (6.4) is applied:

$$f = \frac{46}{Re}\left[1 + \frac{He}{6Re} - \frac{64He^4}{3f^3Re^7}\right]$$

Hence:

$$f = \frac{46}{4.25}\left[1 + \frac{1.925}{6 \times 4.25} - \frac{64 \times 1.925^4}{3f^3 \times 4.25^7}\right] \rightarrow f = 11.64 - \frac{0.0117}{f^3}$$

Solving by trial, one gets: $\boldsymbol{f = 0.10046 \approx 0.1}$

Applying Equation (4.6):

$$\frac{\Delta p}{10} = \frac{2 \times 0.1 \times 1600 \times 0.0707^2}{0.05}$$

Hence, $\Delta \boldsymbol{p} = \mathbf{319.9}$ **Pa**

6.3 Shear thinning fluids

6.3.1 Constitutive law

Shear thinning (or pseudo-plastic) liquids are characterized by a non – linear shear stress – shear rate relationship. They represent part of a larger group known as **power law fluids**, which follow the following constitutive law:

$$\tau = k.\dot{\gamma}^{\,n} \qquad (6.5)$$

In the case of shear thinning fluids, $n < 1$.

n is known as the **flow index** and as it approaches 1, the fluid behavior approaches Newtonian behavior, whereas k is known as the **consistency index**.

Dividing the previous equation by $\dot{\gamma}$, we get:

$$\mu = \frac{\tau}{\dot{\gamma}} = k.\dot{\gamma}^{\,n-1} \qquad (6.6)$$

Since $n < 1 \rightarrow n - 1 < 0$, and the viscosity – shear rate curve will show a decreasing trend approaching 0 as $\dot{\gamma} \rightarrow \infty$.

Modified lubricating oils and greases used to lubricate bearings represent examples of shear thinning liquids. In case of rapid shaft motion, their viscosity drops considerably. Figures (6.3) and (6.4) illustrate the shear stress – shear rate and viscosity – shear rate behavior of such fluids.

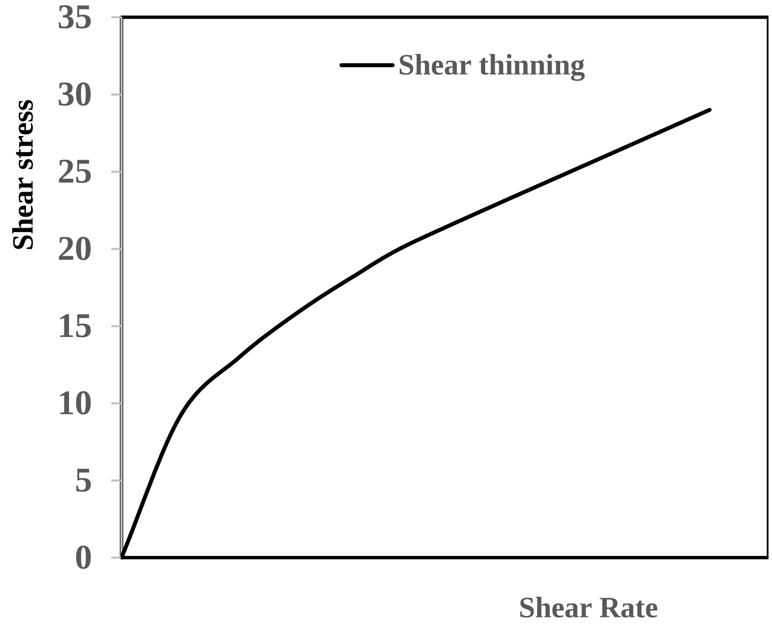


Fig 6.3: Shear stress – Shear rate relation for shear thinning fluids

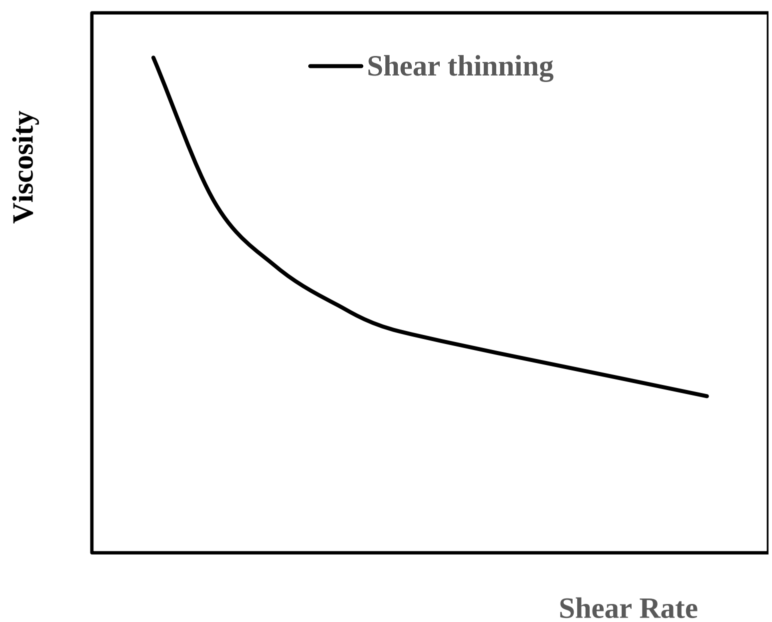


Fig 6.4: Viscosity – Shear rate relation for shear thinning fluids

To test whether a fluid is shear thinning or not and calculate the parameters k and n, a plot is carried out between $\ln \tau$ and $\ln \dot{\gamma}$. The plot should yield a straight line of slope n. The value of k can then be determined by substituting with any point on the line in the following form of equation (6.5):

$$\ln \tau = \ln k + n . \ln \dot{\gamma} \qquad (6.7)$$

6.3.2 Pressure drop correlation for shear thinning fluids

Regardless of whether the flow regime is laminar or turbulent, the following equation can be applied to estimate the pressure drop of shear thinning liquids with a high degree of accuracy:

$$\frac{\Delta p}{L} = \frac{2k}{r}\left[\frac{v.(3n+1)}{n.r}\right]^{n} \qquad (6.8)$$

Example 6.2

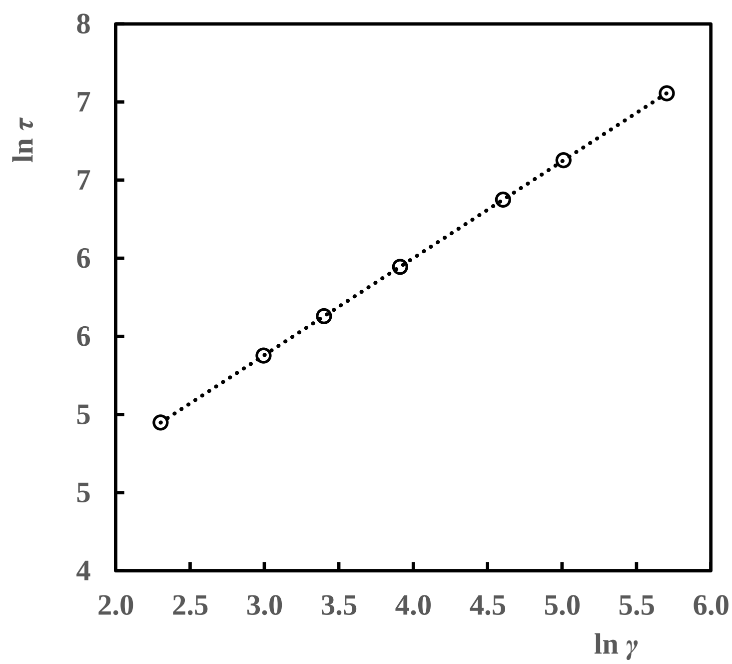
From the following data obtained on a lubricating grease, show that it behaves as a shear thinning fluid and calculate the consistency and flow indices.

Shear strain s^{-1}	10	20	30	50	100	150	300
Shear stress Pa	140	220	280	375	600	750	1150

Solution:

The following table is set and a plot between the two logarithms carried out.

ln γ	2.303	2.996	3.401	3.912	4.605	5.011	5.704
ln τ	4.948	5.378	5.629	5.946	6.376	6.627	7.057



The slope is obtained = $\boldsymbol{n = 0.62}$

Substituting with any point from the given table, like $(30,280)$ in equation (6.5):

$280 = k.\dot{3}0^{\,0.62} \rightarrow \boldsymbol{k = 34}$

Example 6.3

Considers a shampoo product being transported through a straight pipe with a diameter of 250 mm and length of 10 m. The volumetric flow rate is 0.0005 $m^3.s^{-1}$ and the power law index is 0.15, while the consistency index = 48.7. Estimate the pressure drop along the pipe. Then predict the theoretical power for pumping in W.

Solution:

$$A = 0.25\pi \times 0.025^2 = 0.00040875 \text{ m}^2 \rightarrow v = \frac{0.0005}{0.00040875} = 1.223\, m.s^{-1}$$

$$\frac{\Delta p}{10} = \frac{2\times 48.7}{0.0125}\left[\frac{1.223\times(3\times 0.15+1)}{0.15\times 0.0125}\right]^{0.15}$$

$$\Delta p = 188307 \text{ Pa}$$

The theoretical power required for pumping = $Q \times \Delta p = 0.0005 \times 188307 = \mathbf{94\ W}$

6.3.3 velocity profile for laminar flow of shear thinning fluids

The velocity profile of shear thinning liquids in laminar flow is different from that of Newtonian fluids. The profile is shown in Figure (6.5) for a fluid with flow index $n = 0.2$. A comparison with Figure (4.2) shows that the distribution of velocity across the radius of the pipe is no more parabolic.

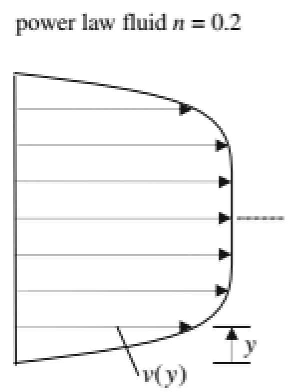


Fig 6.5: Velocity profile for shear thinning fluid

6.4 Shear thickening fluids

6.4.1 Constitutive law

Shear thickening (or dilatant) fluids obey the power law rule (6.6), but with $n > 1$. This means that the shear stress increases at a higher rate with an increase in shear rate. If Equation (6.6) is applied:

$$\mu = \frac{\tau}{\dot{\gamma}} = k.\dot{\gamma}^{\,n-1}$$

The fact that $n > 1$ means that the viscosity will increase with increasing shear rate. Figures (6.6) and (6.7) reveal this behavior.

The most common example of shear thickening liquids is the suspension of starch in water. The more this suspension is agitated, the thicker its consistency gets. However, lately, some new liquids have shown similar behavior, which consist of suspensions of nanoparticles in a liquid, otherwise known as **nanofluids**. These nanofluids are widely used to enhance heat transfer in industrial cooling and heating applications as smart fluids, in nuclear reactors, for extraction of geothermal and other energy sources, in space and defense, in mass transfer applications, in the automotive application as coolants and brake fluids.

6.4.2 Pressure drop correlation for shear thickening fluids

The pressure drop of shear thickening fluids is calculated by the same equation used for shear thinning fluids (Equation 6.8).

6.5 Time dependent fluids

6.5.1 Thixotropy and rhepopepsy

This type of fluid is characterized by the dependence of viscosity on both shear rate and time. This means that at constant shear rate, viscosity of the liquid will vary with time. The fluid is said to be **thixotropic** if viscosity decreases with time at constant shear rate and **rheopeptic** if it increases.

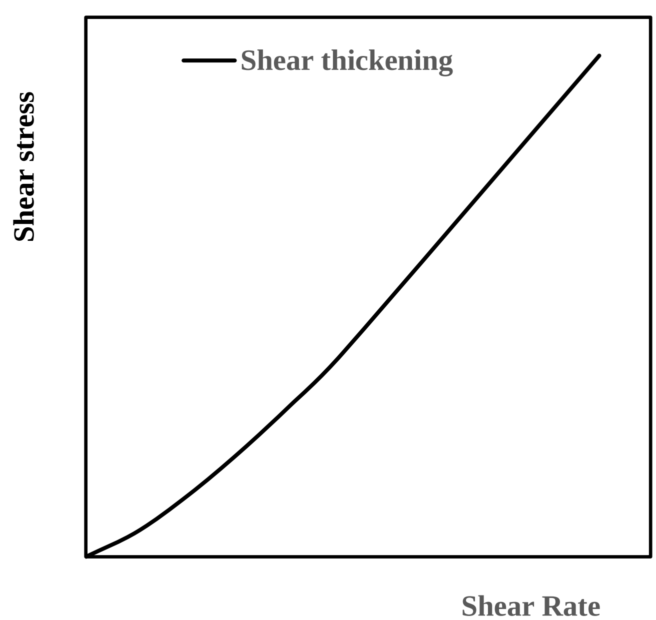


Fig 6.6: Shear stress – Shear rate relation for shear thickening fluids

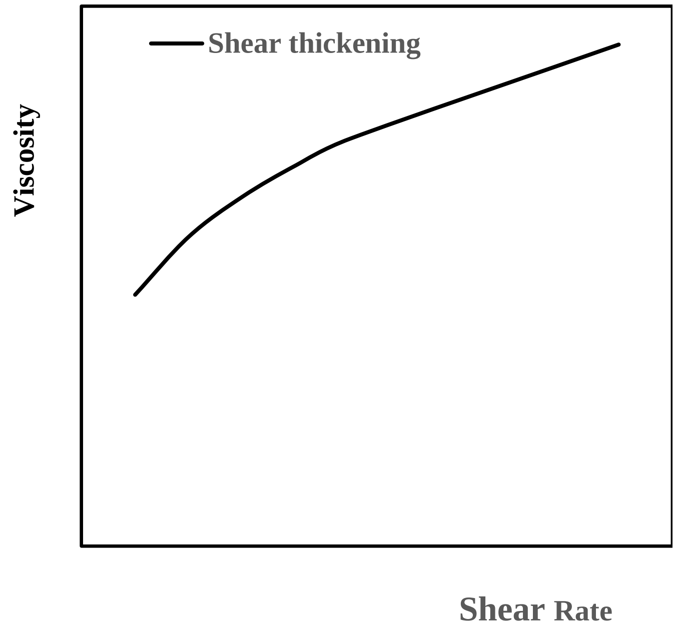


Fig 6.7: Viscosity – Shear rate relation for shear thickening fluids

Thixotropy is much more common than rheopepsy and is encountered in diverse fluids such as drilling fluid, wastewater sludge, some paints, flocculants used in water treatment, some types of clay suspensions and many others. Figure (6.7) illustrates the drop in viscosity at constant shear rate, characteristic of thixotropic fluids. Viscosity decreases and stabilizes at a certain value. If the test is then run backward, by decreasing the shear rate, the viscosity values obtained at any shear rate will always be lower than the forward test values. The corresponding curves for shear stress – shear rate will form a loop known as the **thixotropy hysteresis loop**. (Figure 6.8).

It is important to know the level of thixotropy when choosing the right machinery for manufacturing thixotropic products, given that the mixing equipment must have the necessary power to break it while maintaining a constant speed over time. Once manufactured, the product's temperature drops and its viscosity increases when mixing stops, acquiring the appropriate state for its application.

In their finishing stage, thixotropic products must be stored in tanks fitted with mixing equipment, also known as finishing tanks. The time of mixing must ensure reaching a steady state value for viscosity.

Drilling mud, besides acting as lubricant for the drilling head, must allow for the transportation of rock debris (known as drilling cuttings) to the surface. Because of its thixotropic behavior, the viscosity of drilling mud decreases with time to negatively affect its ability to transport the cuttings. Drilling cuttings are separated from the drilling fluid, which is recycled to the boring head.

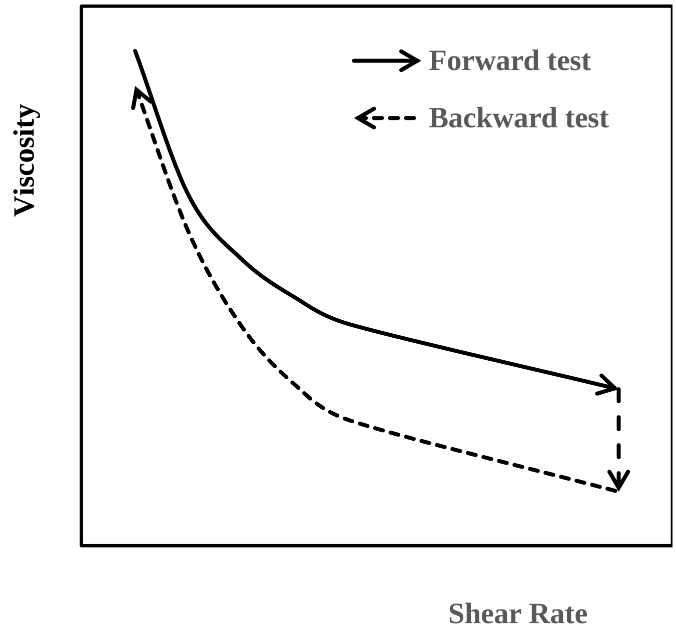


Figure 6.7: Variation of viscosity in thixotropic fluids

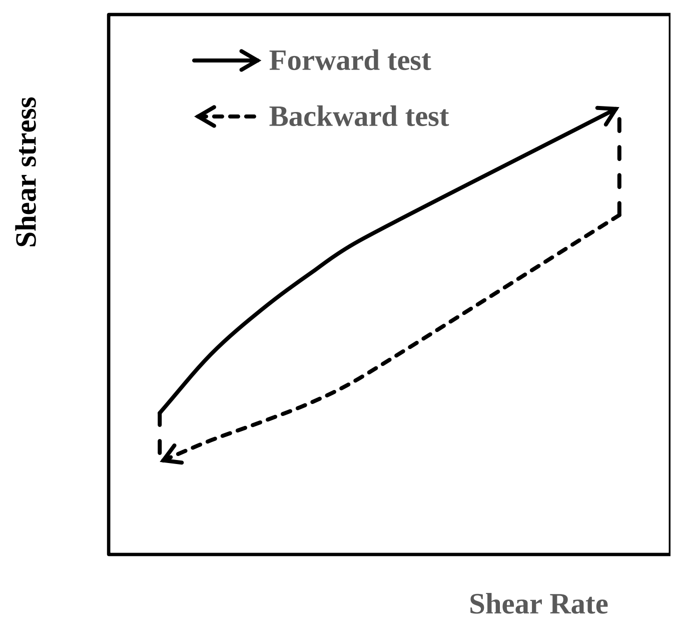


Figure 6.8: Variation of shear stress in thixotropic fluids

6.5.2 Reason for thixotropic behavior

Thixotropy is usually associated with shear thinning behavior. Such liquids possess a certain structure although the bond between molecules is weak. When a thixotropic liquid is agitated to a certain level of shear rate and movement stopped, this structure starts to break down in favor of a new one. If the process is then reversed, by decreasing the shear rate, the new structure will revert partially to the original one. The extent of thixotropy is the extent to which this original structure has been destroyed after waiting for a long time at the constant shear rate, until viscosity values stabilize.

This can be measured by several methods, none of which has proved to be entirely reliable since they depend on indirect estimation of the extent of destruction of the original structure, and to what extent a decrease in shear rate will help recover that structure. Some of these methods are mentioned in what follows:

- **Area of hysteresis loop**: This method has been used for a long time to measure the extent of destruction of the original structure. As it did not always yield reliable results, it is not much in use nowadays.
- **Extent of recovery of viscosity**: In this method, the initial viscosity at the lower shear rate is noted (μ_0) and after reaching a sufficiently high shear rate, the process is reversed by gradually decreasing the shear rate. As the lower value of shear rate is reached, the final viscosity is noted $\left(\mu_f\right)$. The ratio between the final and initial viscosities is taken as a measure of the extent of thixotropy.
- **Extent of recovery of shear stress**: This method is like the previous one except that it relies on measuring shear stress instead of viscosity.
- **The exponential stretching method**: This is a relatively recent method relying on defining a parameter (λ) to measure at each value of shear rate the extent of decrease of viscosity as function of time. A plot between $\ln \lambda$ and time at different values of shear rate is used to obtain almost constant values of time constants that represent the extent of thixotropy.