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Kummer Beta -Weibull Geometric Distribution

A New Generalization of Beta -Weibull Geometric Distribution

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Abstract

A new distribution is defined which is called kummer beta -Weibull geometric (KBWG) distribution. KBWG distribution is considered a new generalization for beta-Weibull geometric distribution. Various properties of KBWG distribution are obtained. Moments and moment generating function are proposed. The method of maximum likelihood estimation is proposed for estimating the model parameters. A Numerical example is explained to illustrate the applications of the Kummer Beta -Weibull Geometric (KBWG) distribution.

Keywords: Kummer beta -Weibull geometric (KBWG) distribution; Fisher information; maximum likelihood estimate; moments; moment generating function.

1. Introduction

Statistical distributions are playing a very important role in the scientific researches. They are very useful in defining, describing and predicting real world phenomena.

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Although many distributions have been created, there are always rooms for developing distributions which are more flexible for fitting specific real world scenarios. There are motivated researchers seeking and developing new and more flexible distributions. As a result, many new distributions and techniques have been developed and studied.

Beta distribution is very versatile and can be used to analyze different types of data sets. Many of the finite range distributions encountered in practice can be easily transformed into the standard beta distribution. Generalized beta distributions have been widely studied in statistics and numerous authors have developed various classes of these distributions. The authors in [1] proposed a general class of distributions for a random variable defined from the logit of the beta random variable by employing two parameters whose role is to introduce skewness and to vary tail weight. Following the authors in [1], who defined the beta normal (BN) distribution, there are many researchers followed the same step to develop new distributions such as the authors in [2] who introduced the beta Gumbel distribution (BGu) and the authors in [3] who proposed beta-exponential distribution.

An extension of the beta-generated method was proposed by the author in [4, 5] by using the Kumaraswamy distribution, of the author in [6], as a generator instead of beta distribution. The Kumaraswamy distributions (KW) is very similar to the beta distribution but has a closed cdf form. Many researchers followed the Kumaraswamy approach to create and develop new distributions such as the authors in [7] who introduced the Kumaraswamy generalized half-normal (K-GHN) distribution.

Kummer beta distribution was proposed, by the authors in [8], on the unit interval (0, 1) with cumulative distribution function and probability density function. The same methodology of the authors in [1] and the authors in [5] can be used to construct a new class of Kummer beta generalized (KBG) distributions. Many researchers followed the approach of Kummer beta to create and develop new distributions such as the authors in [9] who developed the KBG-Weibull (KGBW).

As a result of using Kummer beta approach, new distribution is defined which is called Kummer Beta-Weibull Geometric (KBWG) distribution. Many properties of KBWG distribution are calculated. First, the KBWG density function will be introduced. Second, moments and moment generating function will be obtained. Third, maximum likelihood estimates will be constructed. Fourth, numerical illustration will be presented.

2. Model Derivation

We can use the same methodology of the authors in [1] and the authors in [5] to construct a new class of Kummer beta generalized (KBG) distributions. From an arbitrary parent cdf G(x), the KBG family of cumulative distributions is defined by

$$F(x) = K \int_{0}^{G(x)} x^{a-1}(1-x)^{b-1} e^{-dx} dx \qquad (1)$$

where a>0, b>0, -∞<d<∞, $K^{-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} {}_1F_1(a; a+b; -d)$

and

${}_1F_1(a; a+b; -d) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 t^{a-1}(1-t)^{b-1} e^{-dt} dt = \sum_{i=1}^{\infty} \frac{(a)_i (-d)^i}{(a+b)_i\, i!}$. Taking G(x) in (1) to be the cdf of Weibull geometric distribution which is:

$$G(x) = \frac{1-u_{(\lambda,c)}}{1-pu_{(\lambda,c)}}$$

where $u_{(\lambda,c)} = \exp\{-(\lambda x)^c\}$, p ∈ (0, 1), λ > 0 is a scale parameter, a> 0, b > 0 and c > 0 are shape parameters. A random variable X has Kummer Beta-Weibull Geometric (KBWG) distribution, if it has the fallowing cdf (see fig. 1):

$$F(x) = K \int_0^{\frac{1-u_{(\lambda,c)}}{1-pu_{(\lambda,c)}}} x^{a-1}(1-x)^{b-1} e^{-dx} dx \qquad (2)$$

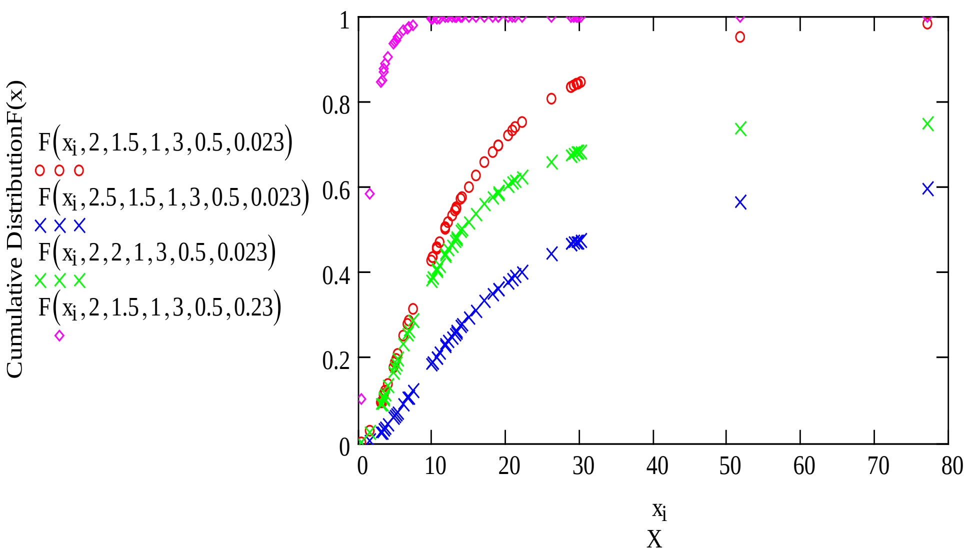

Fig.1. cumulative distribution function of KBWG distribution

also, we can obtain the pdf of KBWG distribution by differentiate (2) as follow:

$$\frac{dF(x)}{dx} = K\left(\frac{1-u_{(\lambda,c)}}{1-pu_{(\lambda,c)}}\right)^{a-1}\left(1-\frac{1-u_{(\lambda,c)}}{1-pu_{(\lambda,c)}}\right)^{b-1} e^{-d\frac{1-u_{(\lambda,c)}}{1-pu_{(\lambda,c)}}}\left(\frac{c\lambda^c(1-p)x^{c-1}u_{(\lambda,c)}}{(1-pu_{(\lambda,c)})^2}\right)$$

$$= K\frac{(1-u_{(\lambda,c)})^{a-1}}{(1-pu_{(\lambda,c)})^{a-1}}\left(\frac{1-pu_{(\lambda,c)}-1+u_{(\lambda,c)}}{1-pu_{(\lambda,c)}}\right)^{b-1} e^{-d\frac{1-u_{(\lambda,c)}}{1-pu_{(\lambda,c)}}}\left(\frac{c\lambda^c(1-p)x^{c-1}u_{(\lambda,c)}}{(1-pu_{(\lambda,c)})^2}\right)$$

$$= K\frac{(1-u_{(\lambda,c)})^{a-1}u_{(\lambda,c)}{}^{b}(1-p)^b x^{c-1}c\lambda^c}{(1-pu_{(\lambda,c)})^{a+b}} e^{-d\frac{1-u_{(\lambda,c)}}{1-pu_{(\lambda,c)}}}$$

then

$$f(x;p,\lambda,c,a,b,d) = Kc\lambda^c(1-p)^b(1-u_{(\lambda,c)})^{a-1}u_{(\lambda,c)}{}^{b}x^{c-1}(1-pu_{(\lambda,c)})^{-(a+b)}e^{-d\frac{1-u_{(\lambda,c)}}{1-pu_{(\lambda,c)}}} \quad (3)$$

where x>0, $u_{(\lambda,c)} = \exp\{-(\lambda x)^c\}$, p ∈ (0, 1), λ > 0 is a scale parameter, a > 0, b > 0 and c>0 are shape parameters,-∞<d<∞, $K^{-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}{}_1F_1(a;a+b;-d)$ and ${}_1F_1(a;a+b;-d) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 t^{a-1}(1-t)^{b-1}e^{-dt}dt = \sum_{i=1}^{\infty}\frac{(a)_i(-d)^i}{(a+b)_i i!}$ (see fig. 2). It can be noticed that equation (3) is probability density function, where $f(x;p,\lambda,c,a,b,d) \geq 0$ for all values of X and $\int f(x;p,\lambda,c,a,b,d)dx = 1$.

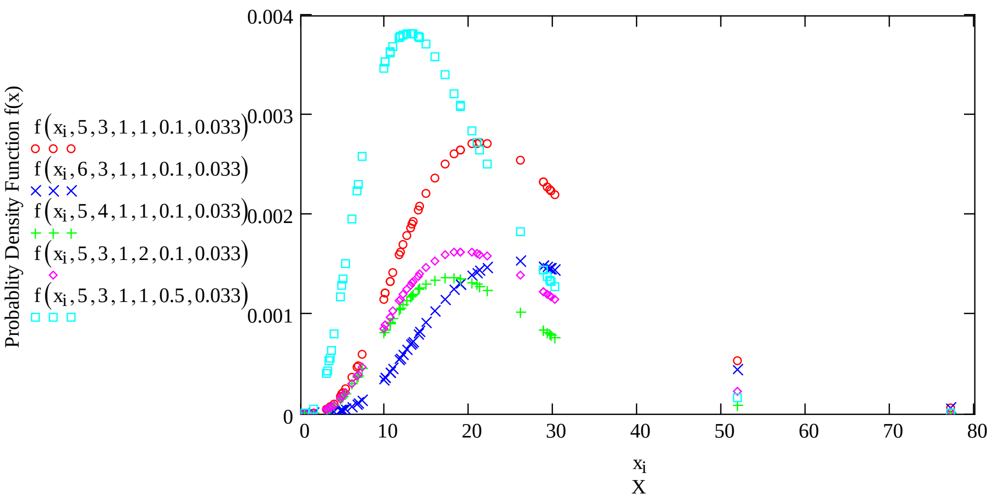

Fig. 2. probability density function of KBWG distribution

The failure rate function corresponding to equation (3) will be (see fig.3)

$$h(x)=\frac{Kc\lambda^{c}(1-p)^{b}\left(1-u_{(\lambda,c)}\right)^{a-1}u_{(\lambda,c)}^{b}x^{c-1}\left(1-pu_{(\lambda,c)}\right)^{-(a+b)}e^{-d\frac{1-u_{(\lambda,c)}}{1-pu_{(\lambda,c)}}}}{1-K\int_{0}^{\frac{1-u_{(\lambda,c)}}{1-pu_{(\lambda,c)}}}x^{a-1}(1-x)^{b-1}e^{-dx}dx}$$

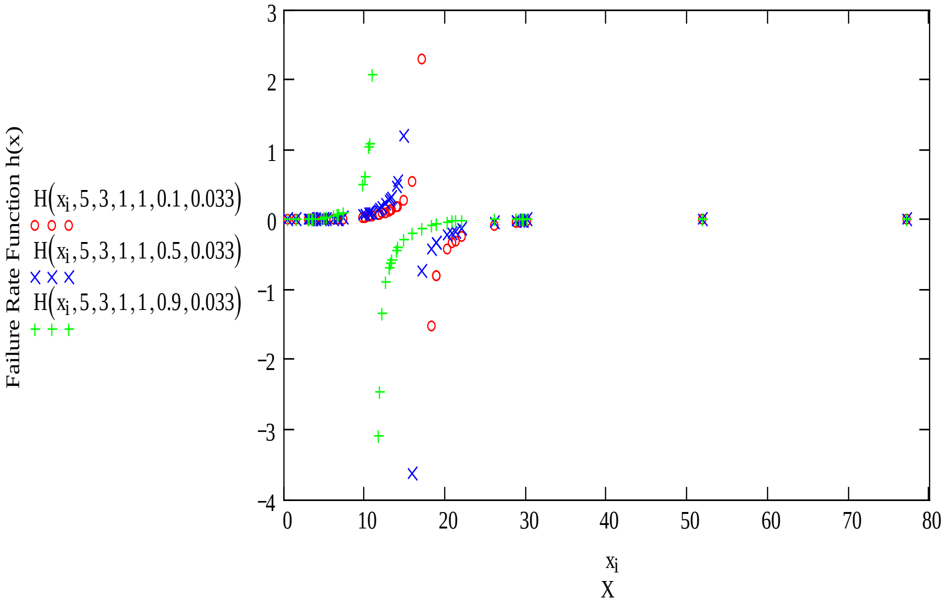

Fig. 3. failure rate function of KBWG distribution

Using (3), some special cases can be obtained as follows (see fig .4):

1) When d=0, the beta-Weibull geometric (BWG) distribution is obtained.

$$f(x;p,\lambda,c,a,b,0)=\frac{c(1-p)^{b}\lambda^{c}x^{c-1}u_{(\lambda,c)}^{b}\left(1-u_{(\lambda,c)}\right)^{a-1}\left(1-pu_{(\lambda,c)}\right)^{-(a+b)}}{B(a,b)}$$

which was introduced by the authors in [10].

2) When c = 1 and d=0, the beta exponential geometric (BEG) distribution is obtained.

$$f(x;p,\lambda,1,a,b,0)=\frac{(1-p)^{b}\lambda u_{(\lambda,1)}^{b}\left(1-u_{(\lambda,1)}\right)^{a-1}\left(1-pu_{(\lambda,1)}\right)^{-(a+b)}}{B(a,b)}$$

which was obtained by the author in [11].

3) When c = b = 1 and d=0, the generalized exponential geometric (GEG) distribution can be obtained.

$$f(x;p,\lambda,1,a,1,0)=\frac{(1-p)\lambda u_{(\lambda,1)}\left(1-u_{(\lambda,1)}\right)^{a-1}\left(1-pu_{(\lambda,1)}\right)^{-(a+1)}}{B(a,1)}$$

which was introduced by the authors in [12].

4) When a = b = 1 and d=0, the WG density function can be obtained as follow

$$f(x;p,\lambda,c,1,1,0)=c(1-p)\ \lambda^{c}x^{c-1}u_{(\lambda,c)}\left(1-pu_{(\lambda,c)}\right)^{-2}$$

which was proposed by the authors in [13].

5) When a = b = c =1 and d=0, we obtain the EG distribution as follow

$$f(x;p,\lambda,1,1,1,0)=\lambda\ (1-p)\ u_{(\lambda,1)}(1-pu)u_{(\lambda,1)}{}^{-2}$$

which was introduced by the authors in [14].

6) When c = 1 and d=0 in addition to p → +0, the KBWG distribution reduces to the beta-exponential (BE) distribution.

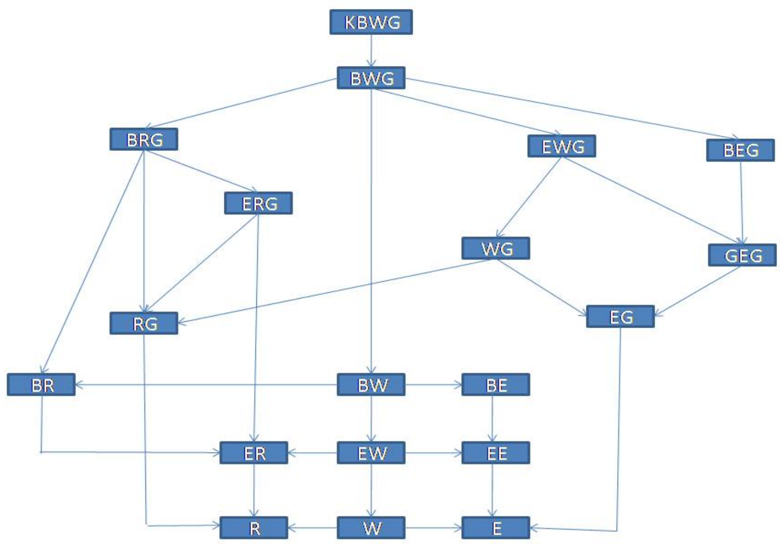

Fig. 4. relationships of the KBWG sub-models

3. Moments and Moment Generating Function

In this section, we derive the r[th] moment for a random variable X having the pdf (3) as follows:

$$E\left(x^r\right)=\int_0^\infty x^r f\left(x;p,\lambda,c,a,b,d\right)dx$$

$$=\int_0^\infty x^r Kc\lambda^c(1-p)^b\left(1-u_{(\lambda,c)}\right)^{a-1}u^b x^{c-1}\left(1-pu_{(\lambda,c)}\right)^{-(a+b)}e^{-d\frac{1-u_{(\lambda,c)}}{1-pu_{(\lambda,c)}}}dx$$

$$=Kc\lambda^c(1-p)^b\int_0^\infty x^r\left(1-u_{(\lambda,c)}\right)^{a-1}u^b x^{c-1}\left(1-pu_{(\lambda,c)}\right)^{-(a+b)}e^{-d\frac{1-u_{(\lambda,c)}}{1-pu_{(\lambda,c)}}}dx$$

$$=Kc\lambda^c(1-p)^b\int_0^\infty x^r\left(1-\exp\left\{-(\lambda x)^c\right\}\right)^{a-1}\exp\left\{-(\lambda x)^c\right\}^b x^{c-1}\left(1-p\exp\left\{-(\lambda x)^c\right\}\right)^{-(a+b)}e^{-d\frac{1-\exp\left\{-(\lambda x)^c\right\}}{1-p\exp\left\{-(\lambda x)^c\right\}}}dx$$

let

$$Y_{(\lambda,c)}=(\lambda x)^c \text{ and } x=\frac{Y_{(\lambda,c)}^{\frac{1}{c}}}{\lambda}$$

then

$$dx=\frac{dY_{(\lambda,c)}}{c\lambda(\lambda x)^{c-1}},$$

and

$$E\left(X^r\right)=Kc\lambda^c\int_0^\infty x^{r+c-1}\left(1-\exp\left\{-Y_{(\lambda,c)}\right\}\right)^{a-1}\exp\left\{-Y_{(\lambda,c)}\right\}^b\left(1-p\exp\left\{-Y_{(\lambda,c)}\right\}\right)^{-(a+b)}e^{-d\frac{1-\exp\left\{-Y_{(\lambda,c)}\right\}}{1-p\exp\left\{-Y_{(\lambda,c)}\right\}}}\frac{dY_{(\lambda,c)}}{c\lambda(\lambda x)^{c-1}}$$

$$=K\int_0^\infty\left(\frac{Y_{(\lambda,c)}^{\frac{1}{c}}}{\lambda}\right)^r\left(1-\exp\left\{-Y_{(\lambda,c)}\right\}\right)^{a-1}\exp\left\{-Y_{(\lambda,c)}\right\}^b\left(1-p\exp\left\{-Y_{(\lambda,c)}\right\}\right)^{-(a+b)}e^{-d\frac{1-\exp\left\{-Y_{(\lambda,c)}\right\}}{1-p\exp\left\{-Y_{(\lambda,c)}\right\}}}dY_{(\lambda,c)}$$

$$=\frac{1}{\lambda^r}K\int_0^\infty Y_{(\lambda,c)}^{\frac{r}{c}}\left(1-\exp\left\{-Y_{(\lambda,c)}\right\}\right)^{a-1}\exp\left\{-Y_{(\lambda,c)}\right\}^b\left(1-p\exp\left\{-Y_{(\lambda,c)}\right\}\right)^{-(a+b)}e^{-d\frac{1-\exp\left\{-Y_{(\lambda,c)}\right\}}{1-p\exp\left\{-Y_{(\lambda,c)}\right\}}}dY_{(\lambda,c)} \quad (4)$$

a may be a real non-integer and an integer, so we have the following two cases:

Case (i): When a is a real non-integer, we have

$$\left[1-\exp\left\{-Y_{(\lambda,c)}\right\}\right]^{a-1}=\Gamma(a)\sum_{m=0}^{\infty}\frac{(-1)^m\exp\left\{-mY_{(\lambda,c)}\right\}}{\Gamma(a-m)m!} \quad (5)$$

applying (5) into (4) gives

$$E\left(X^r\right)=\frac{1}{\lambda^r}K\int_0^\infty Y_{(\lambda,c)}^{\frac{r}{c}}\Gamma(a)\sum_{m=0}^{\infty}\frac{(-1)^m\exp\left\{-mY_{(\lambda,c)}\right\}}{\Gamma(a-m)m!}\exp\left\{-Y_{(\lambda,c)}\right\}^b\left(1-p\exp\left\{-Y_{(\lambda,c)}\right\}\right)^{-(a+b)}e^{-d\frac{1-\exp\left\{-Y_{(\lambda,c)}\right\}}{1-p\exp\left\{-Y_{(\lambda,c)}\right\}}}dY_{(\lambda,c)}$$

$$=\frac{1}{\lambda^r}K\Gamma(a)\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(a-m)m!}\int_0^\infty Y_{(\lambda,c)}^{\frac{r}{c}}\exp\left\{-mY_{(\lambda,c)}\right\}\exp\left\{-Y_{(\lambda,c)}\right\}^b\left(1-p\exp\left\{-Y_{(\lambda,c)}\right\}\right)^{-(a+b)}e^{-d\frac{1-\exp\left\{-Y_{(\lambda,c)}\right\}}{1-p\exp\left\{-Y_{(\lambda,c)}\right\}}}dY_{(\lambda,c)}$$

$$=\frac{1}{\lambda^r}K\Gamma(a)\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(a-m)m!}\int_0^\infty Y_{(\lambda,c)}^{\frac{r}{c}}\left(1-p\exp\left\{-Y_{(\lambda,c)}\right\}\right)^{-(a+b)}e^{\left\{-Y_{(\lambda,c)}\right\}^b-mY_{(\lambda,c)}-d\frac{1-\exp\left\{-Y_{(\lambda,c)}\right\}}{1-p\exp\left\{-Y_{(\lambda,c)}\right\}}}dY_{(\lambda,c)}$$

Case (ii): When a is an integer, it can be shown that

$$\left[1-\exp\left\{-Y_{(\lambda,c)}\right\}\right]^{a-1}=\sum_{m=0}^{a-1}\binom{a-1}{m}(-1)^m\exp\left\{-mY_{(\lambda,c)}\right\}, \quad (6)$$

applying (6) into (4) gives

$$E\left(X^r\right)=\frac{1}{\lambda^r}K\int_0^\infty Y_{(\lambda,c)}^{\frac{r}{c}}\sum_{m=0}^{a-1}\binom{a-1}{m}(-1)^m\exp\left\{-mY_{(\lambda,c)}\right\}\exp\left\{-Y_{(\lambda,c)}\right\}^b\left(1-p\exp\left\{-Y_{(\lambda,c)}\right\}\right)^{-(a+b)}e^{-d\frac{1-\exp\left\{-Y_{(\lambda,c)}\right\}}{1-p\exp\left\{-Y_{(\lambda,c)}\right\}}}dY_{(\lambda,c)}$$

$$=\frac{1}{\lambda^r}K\sum_{m=0}^{a-1}\binom{a-1}{m}(-1)^m\int_0^\infty Y_{(\lambda,c)}^{\frac{r}{c}}\exp\left\{-mY_{(\lambda,c)}\right\}\exp\left\{-Y_{(\lambda,c)}\right\}^b\left(1-p\exp\left\{-Y_{(\lambda,c)}\right\}\right)^{-(a+b)}e^{-d\frac{1-\exp\left\{-Y_{(\lambda,c)}\right\}}{1-p\exp\left\{-Y_{(\lambda,c)}\right\}}}dY_{(\lambda,c)}$$

$$= \frac{1}{\lambda^r} K \sum_{m=0}^{a-1} \binom{a-1}{m} (-1)^m \int_0^\infty Y_{(\lambda,c)}^{\frac{r}{c}} \left(1 - p\exp\left\{-Y_{(\lambda,c)}\right\}\right)^{-(a+b)} e^{\left\{-Y_{(\lambda,c)}\right\}^b - mY_{(\lambda,c)} - d\frac{1-\exp\left\{-Y_{(\lambda,c)}\right\}}{1-p\exp\left\{-Y_{(\lambda,c)}\right\}}} dY_{(\lambda,c)}$$

We can obtain the moment generating function of KBWG from general formula of mgf as follows:

$$M_x(t) = E\left(e^{tx}\right) = \int_0^\infty \exp(tx) f(x; p, \lambda, c, a, b, d) dx$$

$$= Kc\lambda^c (1-p)^b \int_0^\infty \exp(tx) \left(1 - \exp\left\{-(\lambda x)^c\right\}\right)^{a-1} \exp\left\{-(\lambda x)^c\right\}^b x^{c-1} \left(1 - p\exp\left\{-(\lambda x)^c\right\}\right)^{-(a+b)} e^{-d\frac{1-\exp\left\{-(\lambda x)^c\right\}}{1-p\exp\left\{-(\lambda x)^c\right\}}} dx$$

$$= Kc\lambda^c (1-p)^b \int_0^\infty \left(1 - \exp\left\{-(\lambda x)^c\right\}\right)^{a-1} x^{c-1} \left(1 - p\exp\left\{-(\lambda x)^c\right\}\right)^{-(a+b)} e^{\left\{-(\lambda x)^c\right\}^b + tx - d\frac{1-\exp\left\{-(\lambda x)^c\right\}}{1-p\exp\left\{-(\lambda x)^c\right\}}} dx$$

let

$$Y_{(\lambda,c)} = (\lambda x)^c \text{ and } x = \frac{Y_{(\lambda,c)}^{\frac{1}{c}}}{\lambda}$$

then

$$dx = \frac{dY_{(\lambda,c)}}{c\lambda(\lambda x)^{c-1}},$$

and

$$M_x(t) = K(1-p)^b \int_0^\infty \left(1 - \exp\left\{-Y_{(\lambda,c)}\right\}\right)^{a-1} \left(1 - p\exp\left\{-Y_{(\lambda,c)}\right\}\right)^{-(a+b)} e^{\left\{-Y_{(\lambda,c)}\right\}^b + t\frac{Y_{(\lambda,c)}^{\frac{1}{c}}}{\lambda} - d\frac{1-\exp\left\{-Y_{(\lambda,c)}\right\}}{1-p\exp\left\{-Y_{(\lambda,c)}\right\}}} dY_{(\lambda,c)} \qquad (7)$$

a may be a real non-integer and an integer, so we have the following two cases:

Case (i): When a is a real non-integer and applying (5) into (7), the mgf of a random variable X having the KBWG distribution is:

$$M_x(t) = K(1-p)^b \int_0^\infty \Gamma(a) \sum_{m=0}^\infty \frac{(-1)^m \exp\left\{-mY_{(\lambda,c)}\right\}}{\Gamma(a-m)m!} \left(1 - p\exp\left\{-Y_{(\lambda,c)}\right\}\right)^{-(a+b)} e^{\left\{-Y_{(\lambda,c)}\right\}^b + t\frac{Y_{(\lambda,c)}^{\frac{1}{c}}}{\lambda} - d\frac{1-\exp\left\{-Y_{(\lambda,c)}\right\}}{1-p\exp\left\{-Y_{(\lambda,c)}\right\}}} dY_{(\lambda,c)}$$

$$= K(1-p)^b \Gamma(a) \sum_{m=0}^\infty \frac{(-1)^m}{\Gamma(a-m)m!} \int_0^\infty \exp\left\{-mY_{(\lambda,c)}\right\} \left(1 - p\exp\left\{-Y_{(\lambda,c)}\right\}\right)^{-(a+b)} e^{\left\{-Y_{(\lambda,c)}\right\}^b + t\frac{Y_{(\lambda,c)}^{\frac{1}{c}}}{\lambda} - d\frac{1-\exp\left\{-Y_{(\lambda,c)}\right\}}{1-p\exp\left\{-Y_{(\lambda,c)}\right\}}} dY_{(\lambda,c)}$$

$$= K(1-p)^b \Gamma(a)\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(a-m)m!}\int_0^{\infty}\left(1-p\exp\{-Y_{(\lambda,c)}\}\right)^{-(a+b)} e^{\{-Y_{(\lambda,c)}\}^b - mY_{(\lambda,c)} + t\frac{Y_{(\lambda,c)}^{\frac{1}{c}}}{\lambda} - d\frac{1-\exp\{-Y_{(\lambda,c)}\}}{1-p\exp\{-Y_{(\lambda,c)}\}}} dY_{(\lambda,c)}$$

Case (ii): When a is an integerand applying (6) into (7), the mgf of a random variable X having the KBWG distribution is:

$$M_X(t) = K(1-p)^b \int_0^{\infty}\sum_{m=0}^{a-1}\binom{a-1}{m}(-1)^m \exp\{-mY_{(\lambda,c)}\}\left(1-p\exp\{-Y_{(\lambda,c)}\}\right)^{-(a+b)} e^{\{-Y_{(\lambda,c)}\}^b + t\frac{Y_{(\lambda,c)}^{\frac{1}{c}}}{\lambda} - d\frac{1-\exp\{-Y_{(\lambda,c)}\}}{1-p\exp\{-Y_{(\lambda,c)}\}}} dY_{(\lambda,c)}$$

$$= K(1-p)^b \sum_{m=0}^{a-1}\binom{a-1}{m}(-1)^m \int_0^{\infty}\exp\{-mY_{(\lambda,c)}\}\left(1-p\exp\{-Y_{(\lambda,c)}\}\right)^{-(a+b)} e^{\{-Y_{(\lambda,c)}\}^b + t\frac{Y_{(\lambda,c)}^{\frac{1}{c}}}{\lambda} - d\frac{1-\exp\{-Y_{(\lambda,c)}\}}{1-p\exp\{-Y_{(\lambda,c)}\}}} dY_{(\lambda,c)}$$

$$= K(1-p)^b \sum_{m=0}^{a-1}\binom{a-1}{m}(-1)^m \int_0^{\infty}\left(1-p\exp\{-Y_{(\lambda,c)}\}\right)^{-(a+b)} e^{\{-Y_{(\lambda,c)}\}^b - mY_{(\lambda,c)} + t\frac{Y_{(\lambda,c)}^{\frac{1}{c}}}{\lambda} - d\frac{1-\exp\{-Y_{(\lambda,c)}\}}{1-p\exp\{-Y_{(\lambda,c)}\}}} dY_{(\lambda,c)}$$

4. Maximum-likelihood Estimation

Let $x_1, x_2, ...x_n$ be a random sample of size n from KBWG (p, λ, c, a, b,d) distribution (3), then the likelihood function for the vector of parameters $\theta = (p,\lambda,c,a,b,d)^T$ will be:

$$L(\theta) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} Kc\lambda^c(1-p)^b\left(1-u_{(\lambda,c)_i}\right)^{a-1} u_i^b x_i^{c-1}\left(1-pu_{(\lambda,c)_i}\right)^{-(a+b)} e^{-d\frac{1-u_{(\lambda,c)_i}}{1-pu_{(\lambda,c)_i}}} \quad (8)$$

then the log-likelihood function for the vector of parameters $\theta = (p,\lambda,c,a,b,d)^T$, (8) can be written as:

$$\ln L(\theta) = n[\ln(K) + \ln(c) + b\ln(1-p) + c\ln(\lambda)] + (c-1)\sum_{i=1}^{n}\ln(x_i) + b\sum_{i=1}^{n}\ln\left(u_{(\lambda,c)_i}\right) + (a-1)\sum_{i=1}^{n}\ln\left(1-u_{(\lambda,c)_i}\right)$$
$$-(a+b)\sum_{i=1}^{n}\ln\left(1-pu_{(\lambda,c)_i}\right) - d\sum_{i=1}^{n}\frac{1-u_{(\lambda,c)_i}}{1-pu_{(\lambda,c)_i}} \quad (9)$$

where $u_{(\lambda,c)_i} = \exp\{-(\lambda x_i)^c\}$ is a transformed observation. Differentiate (9) with respect to $\theta = (p,\lambda,c,a,b,d)^T$, we have

$$\frac{d\ln L(\theta)}{dc} = \frac{n}{c} + n\ln(\lambda) + \sum_{i=1}^{n}\ln(x_i) - \lambda b\sum_{i=1}^{n}(\lambda x_i)^c\ln(\lambda x_i) + (a-1)\sum_{i=1}^{n}\frac{u_{(\lambda,c)_i}(\lambda x)^c\ln(\lambda x_i)}{1-u_{(\lambda,c)_i}} -$$
$$(10)$$

$$\left(a+b\right)p\sum_{i=1}^{n}\frac{u_{(\lambda,c)_i}\left(\lambda x\right)^{c}\ln\left(\lambda x_i\right)}{1-u_{(\lambda,c)_i}}-d\sum_{i=1}^{n}\frac{u_{(\lambda,c)_i}(\lambda x_i)^{c}\ln(\lambda x_i)(1-pu_{(\lambda,c)_i})+pu_{(\lambda,c)_i}(\lambda x_i)^{c}\ln(\lambda x_i)(1-u_{(\lambda,c)_i})}{\left(1-pu_{(\lambda,c)_i}\right)^2},$$

$$\frac{d\ln L(\theta)}{dp}=-\frac{bn}{1-p}+\left(a+b\right)\sum_{i=1}^{n}\frac{u_{(\lambda,c)_i}}{1-pu_{(\lambda,c)_i}}+d\sum_{i=1}^{n}\frac{\left(1-u_{(\lambda,c)_i}\right)u_{(\lambda,c)_i}}{\left(1-pu_{(\lambda,c)_i}\right)^2},$$

$$\frac{d\ln L(\theta)}{db}=n\ln(1-p)+\frac{n}{K}\frac{dK}{db}+\sum_{i=1}^{n}\ln\left(u_{(\lambda,c)_i}\right)-\sum_{i=1}^{n}\ln\left(1-pu_{(\lambda,c)_i}\right),$$

$$\frac{d\ln L(\theta)}{da}=\frac{n}{K}\frac{dK}{da}+\sum_{i=1}^{n}\ln\left(1-u_{(\lambda,c)_i}\right)-\sum_{i=1}^{n}\ln\left(1-pu_{(\lambda,c)_i}\right),$$

$$\frac{d\ln L(\theta)}{d\lambda}=\frac{nc}{\lambda}-cb\lambda^{c-1}\sum_{i=1}^{n}x_i^{c}+(a-1)c\lambda^{c-1}p\sum_{i=1}^{n}\frac{x_i^{c}u_{(\lambda,c)_i}}{1-u_{(\lambda,c)_i}}-$$

$$(a+b)c\lambda^{c-1}p\sum_{i=1}^{n}\frac{x_i^{c}u_{(\lambda,c)_i}}{1-pu_{(\lambda,c)_i}}-d\sum_{i=1}^{n}\frac{\left(1-pu_{(\lambda,c)_i}\right)\left(u_{(\lambda,c)_i}c\lambda^{c-1}x_i^{c}\right)-pu_{(\lambda,c)_i}c\lambda^{c-1}x_i^{c}\left(1-u_{(\lambda,c)_i}\right)}{\left(1-pu_{(\lambda,c)_i}\right)^2}$$

And

$$\frac{d\ln L(\theta)}{dd}=\frac{n}{K}\frac{dK}{dd}-\sum_{i=1}^{n}\frac{1-u_{(\lambda,c)_i}}{1-pu_{(\lambda,c)_i}}$$

equaling (10) with zero, gives

$$\frac{n}{\hat{c}}+n\ln\left(\hat{\lambda}\right)+\sum_{i=1}^{n}\ln(x_i)-\hat{\lambda}\hat{b}\sum_{i=1}^{n}(\lambda x_i)^{\hat{c}}\ln\left(\hat{\lambda}x_i\right)+$$
$$\left(\hat{a}-1\right)\sum_{i=1}^{n}\frac{\hat{u}_{(\lambda,c)_i}\left(\hat{\lambda}x\right)^{\hat{c}}\ln\left(\hat{\lambda}x_i\right)}{1-\hat{u}_{(\lambda,c)_i}}-\left(\hat{a}+\hat{b}\right)\hat{p}\sum_{i=1}^{n}\frac{\hat{u}_{(\lambda,c)_i}\left(\hat{\lambda}x\right)^{\hat{c}}\ln\left(\hat{\lambda}x_i\right)}{1-\hat{u}_{(\lambda,c)_i}}$$
$$-\hat{d}\sum_{i=1}^{n}\frac{\hat{u}_{(\lambda,c)_i}\left(\hat{\lambda}x\right)^{\hat{c}}\ln(\hat{\lambda}x_i)(1-p\hat{u}_{(\lambda,c)_i})+p\hat{u}_{(\lambda,c)_i}\left(\hat{\lambda}x\right)^{\hat{c}}\ln(\hat{\lambda}x_i)(1-\hat{u}_{(\lambda,c)_i})}{\left(1-p\hat{u}_{(\lambda,c)_i}\right)^2}=0$$
$$-\frac{\hat{b}n}{1-\hat{p}}+\left(\hat{a}+\hat{b}\right)\sum_{i=1}^{n}\frac{\hat{u}_{(\lambda,c)_i}}{1-\hat{p}\hat{u}_{(\lambda,c)_i}}+\hat{d}\sum_{i=1}^{n}\frac{\left(1-\hat{u}_{(\lambda,c)_i}\right)\hat{u}_{(\lambda,c)_i}}{\left(1-p\hat{u}_{(\lambda,c)_i}\right)^2}=0,$$

$$n\ln\left(1-\hat{p}\right)+\frac{n}{K}\frac{dK}{db}+\sum_{i=1}^{n}\ln\left(\hat{u}_{(\lambda,c)_i}\right)-\sum_{i=1}^{n}\ln\left(1-\hat{p}\hat{u}_{(\lambda,c)_i}\right)=0,$$

$$\frac{n}{K}\frac{dK}{da}+\sum_{i=1}^{n}\ln\left(1-\hat{u}_{(\lambda,c)_i}\right)-\sum_{i=1}^{n}\ln\left(1-\hat{p}\hat{u}_{(\lambda,c)_i}\right)=0 \qquad (11)$$

$$\frac{n\hat{c}}{\hat{\lambda}}-\hat{c}\hat{b}\hat{\lambda}^{\hat{c}-1}\sum_{i=1}^{n}x_i^{\hat{c}}+\left(\hat{a}-1\right)\hat{c}\hat{\lambda}^{\hat{c}-1}\hat{p}\sum_{i=1}^{n}\frac{x_i^{c}\hat{u}_{(\lambda,c)_i}}{1-\hat{u}_{(\lambda,c)_i}}-\left(\hat{a}+\hat{b}\right)\hat{c}\hat{\lambda}^{\hat{c}-1}\hat{p}\sum_{i=1}^{n}\frac{x_i^{c}\hat{u}_{(\lambda,c)_i}}{1-p\hat{u}_{(\lambda,c)_i}}$$
$$-\hat{d}\sum_{i=1}^{n}\frac{\left(1-\hat{p}\hat{u}_{(\lambda,c)_i}\right)\left(\hat{u}_{(\lambda,c)_i}\hat{c}\hat{\lambda}^{\hat{c}-1}x_i^{c}\right)-\hat{p}\hat{u}_{(\lambda,c)_i}\hat{c}\hat{\lambda}^{\hat{c}-1}x_i^{c}\left(1-\hat{u}_{(\lambda,c)_i}\right)}{\left(1-\hat{p}\hat{u}_{(\lambda,c)_i}\right)^2}=0$$

$$\textit{and}\quad \frac{n}{K}\frac{dK}{dd}-\sum_{i=1}^{n}\frac{1-\hat{u}_{(\lambda,c)_i}}{1-\hat{p}\hat{u}_{(\lambda,c)_i}}=0$$

where $\hat{u}_{(\lambda,c)_i}=\exp\left\{-\left(\hat{\lambda}x_i\right)^{\hat{c}}\right\}$. Equation (11) can be solved numerically using statistical packages.

The observed information matrix $I\left(\theta\right)$ for parameters $\left(p,\lambda,c,a,b,d\right)$ can be written as follows:

$$I\left(\theta\right)=-E\begin{bmatrix} I_{pp} & I_{p\lambda} & I_{pc} & I_{pa} & I_{pb} & I_{pd}\\ I_{\lambda p} & I_{\lambda\lambda} & I_{\lambda c} & I_{\lambda a} & I_{\lambda b} & I_{\lambda d}\\ I_{cp} & I_{c\lambda} & I_{cc} & I_{ca} & I_{cb} & I_{cd}\\ I_{ap} & I_{a\lambda} & I_{ac} & I_{aa} & I_{ab} & I_{ad}\\ I_{bp} & I_{b\lambda} & I_{bc} & I_{bc} & I_{bb} & I_{bd}\\ I_{dp} & I_{d\lambda} & I_{dc} & I_{da} & I_{db} & I_{dd} \end{bmatrix}$$

where

$$I_{cc}=-\frac{n}{c^2}-b\sum_{i=1}^{n}\left(\lambda x_i\right)^c\ln^2\left(\lambda x_i\right)+\left(a-1\right)\sum_{i=1}^{n}\frac{\left[1-\left(\lambda x_i\right)^c\right]\left(\lambda x_i\right)^c\ln^2\left(\lambda x_i\right)u_{(\lambda,c)_i}}{\left(1-u_{(\lambda,c)_i}\right)}$$
$$+\left(a+b\right)p\sum_{i=1}^{n}\frac{\left[\left(\lambda x_i\right)^c-1\right]\left(\lambda x_i\right)^c\ln^2\left(\lambda x_i\right)u_{(\lambda,c)_i}}{\left(1-pu_{(\lambda,c)_i}\right)}-\left(a-1\right)\sum_{i=1}^{n}\frac{\left(\lambda x_i\right)^{2c}\ln^2\left(\lambda x_i\right)u^2_{(\lambda,c)_i}}{\left(1-u_{(\lambda,c)_i}\right)^2}$$
$$+\left(a+b\right)p^2\sum_{i=1}^{n}\frac{\left(\lambda x_i\right)^{2c}\ln^2\left(\lambda x_i\right)u^2_{(\lambda,c)_i}}{\left(1-pu_{(\lambda,c)_i}\right)^2},$$

$$I_{ca} = \sum_{i=1}^{n} \frac{(\lambda x_i)^c \ln(\lambda x_i) u_{(\lambda,c)_i}}{\left(1 - u_{(\lambda,c)_i}\right)} - p \sum_{i=1}^{n} \frac{(\lambda x_i)^c \ln(\lambda x_i) u_{(\lambda,c)_i}}{\left(1 - p u_{(\lambda,c)_i}\right)},$$

$$I_{cb} = -\sum_{i=1}^{n} (\lambda x_i)^c \ln(\lambda x_i) - p \sum_{i=1}^{n} \frac{(\lambda x_i)^c \ln(\lambda x_i) u_{(\lambda,c)_i}}{\left(1 - p u_{(\lambda,c)_i}\right)},$$

$$I_{cd} = -\sum_{i=1}^{n} \frac{u_{(\lambda,c)_i} (\lambda x_i)^c \ln(\lambda x_i)(1 - p u_{(\lambda,c)_i}) + p u_{(\lambda,c)_i} (\lambda x_i)^c \ln(\lambda x_i)(1 - u_{(\lambda,c)_i})}{\left(1 - p u_{(\lambda,c)_i}\right)^2}$$

$$I_{pp} = -\frac{nb}{(1-p)^2} + (a+b) \sum_{i=1}^{n} \frac{u^2_{(\lambda,c)_i}}{\left(1 - p u_{(\lambda,c)_i}\right)^2},$$

$$I_{p\lambda} = -\frac{(a+b)c}{\lambda} \sum_{i=1}^{n} \frac{(\lambda x_i)^c u_{(\lambda,c)_i}}{\left(1 - p u_{(\lambda,c)_i}\right)} - \frac{(a+b)pc}{\lambda} \sum_{i=1}^{n} \frac{(\lambda x_i)^c u^2_{(\lambda,c)_i}}{(1 - p_i)^2},$$

$$I_{pc} = -(a+b) \sum_{i=1}^{n} \frac{(\lambda x_i)^c \ln(\lambda x_i) u_{(\lambda,c)_i}}{\left(1 - p u_{(\lambda,c)_i}\right)} - (a+b) p \sum_{i=1}^{n} \frac{(\lambda x_i)^c \ln(\lambda x_i) u^2_{(\lambda,c)_i}}{\left(1 - p u_{(\lambda,c)_i}\right)^2},$$

$$I_{pa} = \sum_{i=1}^{n} \frac{u_{(\lambda,c)_i}}{\left(1 - p u_{(\lambda,c)_i}\right)},$$

$$I_{pb} = -\frac{n}{(1-p)} \sum_{i=1}^{n} \frac{u_{(\lambda,c)_i}}{\left(1 - p u_{(\lambda,c)_i}\right)},$$

$$I_{pd} = -\sum_{i=1}^{n} \frac{\left(1 - u_{(\lambda,c)_i}\right) u_{(\lambda,c)_i}}{\left(1 - p u_{(\lambda,c)_i}\right)^2}$$

$$I_{\lambda\lambda} = -\frac{nc}{\lambda^2} + \frac{bc(1-c)}{\lambda^2} \sum_{i=1}^{n} (\lambda x_i)^c + \frac{(c-1)(a-1)c}{\lambda^2} \sum_{i=1}^{n} \frac{(\lambda x_i)^c u_{(\lambda,c)_i}}{\left(1 - u_{(\lambda,c)_i}\right)} - \frac{(a-1)c^2}{\lambda^2} \sum_{i=1}^{n} \frac{(\lambda x_i)^{2c} u_{(\lambda,c)_i}}{\left(1 - u_{(\lambda,c)_i}\right)}$$
$$- \frac{(a-1)c^2}{\lambda^2} \sum_{i=1}^{n} \frac{(\lambda x_i)^{2c} u^2_{(\lambda,c)_i}}{\left(1 - u_{(\lambda,c)_i}\right)^2} + \frac{(1-c)(a+b)pc}{\lambda^2} \sum_{i=1}^{n} \frac{(\lambda x_i)^c u_{(\lambda,c)_i}}{\left(1 - p u_{(\lambda,c)_i}\right)} + \frac{(a+b)pc^2}{\lambda^2} \sum_{i=1}^{n} \frac{(\lambda x_i)^{2c} u_{(\lambda,c)_i}}{\left(1 - p u_{(\lambda,c)_i}\right)}$$
$$+ \frac{(a+b)p^2 c^2}{\lambda^2} \sum_{i=1}^{n} \frac{(\lambda x_i)^{2c} u^2_{(\lambda,c)_i}}{\left(1 - p u_{(\lambda,c)_i}\right)^2},$$

$$I_{\lambda c}=\frac{n}{\lambda}-\frac{bc}{\lambda}\sum_{i=1}^{n}(\lambda x_i)^c\left[\ln(\lambda x_i)^c+c^{-1}\right]+\frac{(a-1)c}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c\ln(\lambda x_i)u_{(\lambda,c)_i}}{\left(1-u_{(\lambda,c)_i}\right)}-\frac{(a-1)}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_{(\lambda,c)_i}}{\left(1-u_{(\lambda,c)_i}\right)}$$

$$-\frac{(a-1)c}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c}\ln(\lambda x_i)u_{(\lambda,c)_i}}{1-u_{(\lambda,c)_i}}+\frac{(a-1)c}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c}\ln(\lambda x_i)u^2_{(\lambda,c)_i}}{\left(1-pu_{(\lambda,c)_i}\right)^2}-\frac{(a+b)pc}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c}\ln(\lambda x_i)u_{(\lambda,c)_i}}{\left(1-pu_{(\lambda,c)_i}\right)}$$

$$-\frac{(a+b)p}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_{(\lambda,c)_i}}{\left(1-pu_{(\lambda,c)_i}\right)}+\frac{(a+b)pc}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c}\ln(\lambda x_i)u_{(\lambda,c)_i}}{\left(1-pu_{(\lambda,c)_i}\right)}+\frac{(a+b)p^2c}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c}\ln(\lambda x_i)u^2_{(\lambda,c)_i}}{\left(1-pu_{(\lambda,c)_i}\right)^2},$$

$$I_{\lambda a}=\frac{c}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_{(\lambda,c)_i}}{\left(1-u_{(\lambda,c)_i}\right)}-\frac{pc}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_{(\lambda,c)_i}}{\left(1-pu_{(\lambda,c)_i}\right)},$$

$$I_{\lambda b}=-\frac{c}{\lambda}\sum_{i=1}^{n}(\lambda x_i)^c-\frac{pc}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_{(\lambda,c)_i}}{\left(1-pu_{(\lambda,c)_i}\right)},$$

$$I_{\lambda d}=-\sum_{i=1}^{n}\frac{\left(1-pu_{(\lambda,c)_i}\right)\left(u_{(\lambda,c)_i}c\lambda^{c-1}x_i^c\right)-pu_{(\lambda,c)_i}c\lambda^{c-1}x_i^c\left(1-u_{(\lambda,c)_i}\right)}{\left(1-pu_{(\lambda,c)_i}\right)^2}$$

$$I_{cc}=-\frac{n}{c^2}-b\sum_{i=1}^{n}(\lambda x_i)^c\ln^2(\lambda x_i)+(a-1)\sum_{i=1}^{n}\frac{\left[1-(\lambda x_i)^c\right](\lambda x_i)^c\ln^2(\lambda x_i)u_{(\lambda,c)_i}}{\left(1-u_{(\lambda,c)_i}\right)}$$

$$+(a+b)p\sum_{i=1}^{n}\frac{\left[(\lambda x_i)^c-1\right](\lambda x_i)^c\ln^2(\lambda x_i)u_{(\lambda,c)_i}}{\left(1-pu_{(\lambda,c)_i}\right)}-(a-1)\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c}\ln^2(\lambda x_i)u^2_{(\lambda,c)_i}}{\left(1-u_{(\lambda,c)_i}\right)^2}$$

$$+(a+b)p^2\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c}\ln^2(\lambda x_i)u^2_{(\lambda,c)_i}}{\left(1-pu_{(\lambda,c)_i}\right)^2},$$

$$I_{ca}=\sum_{i=1}^{n}\frac{(\lambda x_i)^c\ln(\lambda x_i)u_{(\lambda,c)_i}}{\left(1-u_{(\lambda,c)_i}\right)}-p\sum_{i=1}^{n}\frac{(\lambda x_i)^c\ln(\lambda x_i)u_{(\lambda,c)_i}}{\left(1-pu_{(\lambda,c)_i}\right)},$$

$$I_{cb}=-\sum_{i=1}^{n}(\lambda x_i)^c\ln(\lambda x_i)-p\sum_{i=1}^{n}\frac{(\lambda x_i)^c\ln(\lambda x_i)u_{(\lambda,c)_i}}{\left(1-pu_{(\lambda,c)_i}\right)},$$

$$I_{aa}=-n\left[\psi'(a)-\psi'(a+b)\right],$$

$$I_{ab}=n\psi'(a+b)\ \ and\ \ I_{bb}=-n\left[\psi'(b)-\psi'(a+b)\right]$$

5. Numerical Illustration

We will generate 1000 samples of size 50 from the KBWG distribution for certain values of the parameters a, b, c, p, λ and d using proposed random number generator, and then the maximum likelihood estimates for each sample will be obtained, along with the mean, biases, mean square error, skewness, kurtosis of those estimates for different sample sizes.

In table 1, we list the mean, biases, mean square error, skewness, kurtosis for the MLE's of the parameters a, b, c, p, λ and d for 1000 random samples of size 50. The table shows that the mean square error and the bias for the estimates of the parameters a, b, c, p, λ and d decrease as the sample sizes increase. We used a statistical package called mathcad 2001.

Table (1). Mean, Biases, Mean Square Errors, Skewness, Kurtosis for the estimates of KBWG distribution for different values of parameters

$a = 0.9$, $b = 0.8$, C=0.6, λ=0.1, $p = 0.2$, d=0.4						
N		Mean	Bias	MSE	Skewness	Kurtosis
50	a	0.462208	0.437792	0.27286	0.342238	-1.95057
	b	0.734428	0.065572	0.007603	0.173593	3.717151
	C	0.14231	0.45769	0.216887	3.033637	8.513197
	λ	0.055339	0.044661	0.002606	0.347804	-1.85381
	p	0.15433	0.04567	0.014025	1.069642	1.315276
	D	0.197205	0.202795	0.04479	4.080801	22.92882

6. Conclusions

We introduce a new distribution which is called kummer beta-Weibull geometric distribution with six parameter. The new distribution may be considered a generalization of beta-Weibull geometric distribution. We provide a mathematical treatment of the distribution including its density function, moments and moment generating function, and there are still some of its properties need to be investigated. The estimation of the parameters is approached by the method of maximum likelihood and the observed information matrix is calculated. We hope that KBWG distribution exhibit more different shapes of failure rate function. Also we hope that the new model may attract wider applications in statistics.

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