New nonautonomous combined multi-wave solutions for $(2+1)$-dimensional variable coefficients KdV equation

M. S. Osman & J. A. T. Machado
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Abstract  A variety of new types of nonautonomous combined multi-wave solutions of the (2 + 1)-dimensional variable coefficients KdV equation is derived by means of the generalized unified method. These solutions are classified into three categories, namely multi-soliton, periodic and elliptic solutions. The physical insight of the waves is dressed for different values of the free parameters in the obtained solutions.

Keywords  Generalized unified method · Variable coefficients · Combined multi-wave solutions · (2 + 1)-dimensional KdV equation

1 Introduction

Nonlinear evolution equations (NLEE) have been studied in different branches of science such as chemistry, plasma physics, nonlinear optics and fluid mechanics [1–8]. Due to this fact, finding the single exact solutions of NLEE is an important issue and computer symbolic packages were used for mitigating the workload posed by laborious and time-consuming algebraic calculations [9–11].

In the last years, considerable efforts were developed to find the multi-soliton solutions of NLEE for investigating their complexity and dynamics [12–16].

The Hirota’s bilinear approach and its simplified type [17], or the Darboux transformation and the inverse scattering method [18–21] are techniques that have been used with the NLEE for deriving multisolitary wave solutions [22–24].

Herein, we adopt the generalized unified method (GUM) to find and to study multi-wave solutions of the (2+1)-dimensional variable coefficients KdV equation, denoted by (2 + 1) D-vcKdV,

\[ w_t + \alpha(t) w_{xxx} = \beta(t) (w^{-1} w_x)_x, \]  

where \( w = w(x, y, t) \) corresponds to the waves amplitudes (such as shallow water in fluids or electrostatic potential in plasmas) and \( \alpha(t) \) and \( \beta(t) \) denote arbitrary analytic functions. The (2 + 1) D-vcKdV is often used when \( \alpha(t) = 1 \) and \( \beta(t) = 3 \), since it reduces it to the KdV equation when \( x = y \) [25–27]. Boiti et al. [28] first obtained this model based on the weak Lax pair. Lou et al. [29] demonstrated that this model can be constructed by means of the inner parameter-dependent symmetry constraint of the KP equation [12]. Furthermore, it was shown that it is an asymmetric part of the NNV equation [14]. The Painleve’ property of the KP equation has been proved by Dorizzi et al. [30].
The article is arranged as follows: In Sect. 2, different types of multi-wave solutions are extracted using the GUM. The physical meaning of these solutions is also discussed. The main conclusions are provided in Sect. 3.

2 Multi-wave solutions of \((2 + 1)\) D-vcKdV using GUM

In this section, we adopt GUM [31,32] to find double-wave solutions of \((2 + 1)\) D-vcKdV formulated in (1).

After the transformation \(w_{x}(x, y, t) = v_{y}(x, y, t)\), Eq. (1) becomes:

\[
\begin{align*}
w_{t} + \alpha(t) w_{xxx} &= \beta(t) (w v)_{x}, \\
w_{x} &= v_{y}.
\end{align*}
\tag{2}
\]

For double-wave solutions, we assume that

\[
w(x, t) = w_{1}(\xi_{1}, \xi_{2}) = p_{0}(t)
+ \sum_{m=1}^{n_{1}} \sum_{i_{1}+i_{2}=m} p_{i_{1}, i_{2}}(t) g_{1}^{i_{1}}(\xi_{1}) g_{2}^{i_{2}}(\xi_{2}),
\]

\[
v(x, t) = v_{1}(\xi_{1}, \xi_{2}) = q_{0}(t)
+ \sum_{m=1}^{n_{2}} \sum_{i_{1}+i_{2}=m} q_{i_{1}, i_{2}}(t) g_{1}^{i_{1}}(\xi_{1}) g_{2}^{i_{2}}(\xi_{2}),
\]

\[
\begin{align*}
(g_{1}^{i}(\xi_{1}))^p &= \sum_{r=0}^{p} b_{r} g_{1}^{i}(\xi_{1}),
\end{align*}
\]

\[
\begin{align*}
(g_{2}^{i}(\xi_{2}))^p &= \sum_{r=0}^{p} c_{r} g_{2}^{i}(\xi_{2}),
\end{align*}
\tag{3}
\]

where \(\xi_{1} = \alpha_{1} x + \alpha_{2} y + \int^{t} \alpha_{3}(t) dt, \xi_{2} = \beta_{1} x + \beta_{2} y + \int^{t} \beta_{3}(t) dt, \alpha_{j}, \beta_{j}, j = 1, 2\) are arbitrary constants and \(\alpha_{3}(t), \beta_{3}(t), p_{0}(t), p_{i_{1}, i_{2}}(t), q_{0}(t), q_{i_{1}, i_{2}}(t)\) are arbitrary functions.

2.1 Multi-wave solutions when \(p = 1\)

This subsection analyzes the multi-wave solution in the form of soliton or periodic wave solutions.

Let us consider \(p = 1\) in Eq. (3).

For \(p = 1\), the balance condition between the order of Eq. (2) and its nonlinear terms yields \(n_{1} = n_{2} = 2(k - 1), k > 1\), while the consistency condition leads to \(k \leq 3\) [31,32]. Therefore, for \(k \in \{2, 3\}\) solutions exist.

Case 1: If \(k = 2, n_{1} = n_{2} = 2\), then we have

\[
w(x, y, t) = w_{1}(\xi_{1}, \xi_{2}) = p_{0}(t)
+ \sum_{m=1}^{2} \sum_{i_{1}+i_{2}=m} p_{i_{1}, i_{2}}(t) g_{1}^{i_{1}}(\xi_{1}) g_{2}^{i_{2}}(\xi_{2}),
\]

\[
v(x, y, t) = v_{1}(\xi_{1}, \xi_{2}) = q_{0}(t)
+ \sum_{m=1}^{2} \sum_{i_{1}+i_{2}=m} q_{i_{1}, i_{2}}(t) g_{1}^{i_{1}}(\xi_{1}) g_{2}^{i_{2}}(\xi_{2}),
\]

\[
\begin{align*}
g_{1}'(\xi_{1}) &= b_{0} + b_{1} g(\xi_{1}) + b_{2} g^{2}(\xi_{1}),
\end{align*}
\]

\[
\begin{align*}
g_{2}'(\xi_{2}) &= c_{0} + c_{1} g(\xi_{2}) + c_{2} g^{2}(\xi_{2}).
\end{align*}
\tag{4}
\]

By substituting from (4) into (2), we get

\[
\begin{align*}
p_{0}(t) &= \frac{\beta_{2}(\alpha_{1} \beta_{3}(t) - \beta_{1} \alpha_{3}(t))}{2 \alpha_{1} \beta_{1}^{2} \beta(t)},
p_{1,0}(t) &= -\frac{6 b_{1} b_{2} \beta_{2} \alpha(t)}{\beta_{1} \beta(t)},
p_{0,1}(t) &= \frac{6 c_{1} c_{2} \beta_{2} \alpha(t)}{\beta(\xi_{1})},
p_{1,1}(t) &= 0,
p_{0,2}(t) &= \frac{6 c_{2} \beta_{1} \beta_{2} \alpha(t)}{\beta^{2}(t)} ,
p_{2,0}(t) &= -\frac{6 b_{2} \alpha_{1} \beta_{2} \alpha(t)}{\beta^{3}(t)},
\end{align*}
\tag{5}
\]

and

\[
\begin{align*}
q_{0}(t) &= \frac{\alpha_{1} \beta_{1} R_{+} \alpha(t) + \alpha_{1} \beta_{3}(t) + \beta_{1} \alpha_{3}(t)}{2 \alpha_{1} \beta_{1} \beta(t)},
q_{1,0}(t) &= \frac{6 b_{1} b_{2} \alpha_{1} \alpha(t)}{\beta(t)},
q_{0,1}(t) &= \frac{6 c_{1} c_{2} \beta_{1} \alpha(t)}{\beta(t)},
q_{1,1}(t) &= 0,
q_{0,2}(t) &= \frac{6 c_{2} \beta_{1} \alpha(t)}{\beta(t)},
q_{2,0}(t) &= \frac{6 b_{2} \alpha_{1} \alpha(t)}{\beta(t)},
\end{align*}
\tag{6}
\]

where \(R_{\pm} = (c_{1}^{2} + 8 c_{1} c_{0}) \beta_{1}^{2} \pm (b_{1}^{2} + 8 b_{1} b_{0}) \alpha_{1}^{2}\).

Direct calculation of the auxiliary equations in (4) yields

\[
\begin{align*}
g_{1}(\xi_{1}) &= -\frac{b_{1} + R_{1} \tanh \left(\frac{1}{2} R_{1} \xi_{1}\right)}{2 b_{2}},
g_{2}(\xi_{2}) &= -\frac{c_{1} + R_{2} \tanh \left(\frac{1}{2} R_{2} \xi_{2}\right)}{2 c_{2}},
\end{align*}
\tag{7}
\]

where \(R_{1}^{2} = b_{1}^{2} - 4 b_{2} b_{0} > 0\) and \(R_{2}^{2} = c_{1}^{2} - 4 c_{2} c_{0} > 0\).
Substituting (5), (6) and (7) into (3), we get the solution of (2), namely

\[ w(x, y, t) = w_1(\xi_1, \xi_2) = \frac{\beta_2}{\alpha_1 \beta_1 \beta(t)} \left\{ \alpha_1 \beta_1 \left( R_1^2 \alpha_1^2 - R_2^2 \beta_1^2 \right) + \frac{1}{2} (\alpha_1 \beta_3(t) - \beta_1 \alpha_3(t)) \right. \\
- \frac{3}{2} \alpha_1 \beta_1 \left( R_1^2 \alpha_1^2 \tanh^2 \left( \frac{1}{2} R_1 \xi_1 \right) \right. \\
\left. + R_2^2 \beta_1^2 \tanh^2 \left( \frac{1}{2} R_2 \xi_2 \right) \right\}. \]

\[ v(x, y, t) = v_1(\xi_1, \xi_2) = \frac{1}{\alpha_1 \beta_1 \beta(t)} \left\{ - \alpha_1 \beta_1 \left( R_1^2 \alpha_1^2 + R_2^2 \beta_1^2 \right) \right. \\
+ \frac{1}{2} (\alpha_1 \beta_3(t) + \beta_1 \alpha_3(t)) + \frac{3}{2} \alpha_1 \beta_1 \left( \frac{R_1^2 \alpha_1^2 \tanh^2 \left( \frac{1}{2} R_1 \xi_1 \right) - R_2^2 \beta_1^2 \tanh^2 \left( \frac{1}{2} R_2 \xi_2 \right) \right) \right\}. \] (8)

where \( \xi_1 = \alpha_1 x - \frac{\alpha_1 \beta_2}{\beta_1} y + \int \alpha_3(t) \, dt \), \( \xi_2 = \beta_1 x + \beta_2 y + \int \beta_3(t) \, dt \), \( \alpha_1 \), \( \beta_1 \), \( \alpha_2 \), and \( \beta_2 \) are arbitrary constants and \( \alpha_3(t) \), \( \beta_3(t) \), \( \alpha(t) \), \( \beta(t) \) represent arbitrary functions.

Expression (8) is depicted in Figs. 1 and 2 for different \( \alpha(t) \) and \( \beta(t) \).

Figures 1 and 2 illustrate the double-wave solution for two different velocities. These two waves are combined into one rogue wave and then separated into two distinct waves along a characteristic curve resting in the \( xt \)-plane. We notice that after having the rogue wave separated into two waves, the arbitrary functions \( \alpha(t) \) and \( \beta(t) \) do not affect, neither the shape nor the amplitude of the waves (the collisions became elastic) during propagation.

**Case 2:** When \( k = 3, n_1 = n_2 = 4 \), we have

\[ w(x, y, t) = w_1(\xi_1, \xi_2) = p_0(t) \]

\[ + \sum_{m=1}^{4} \sum_{i_1+i_2=m}^{} p_{i_1,i_2}(t) g_1^{i_1}(\xi_1) g_2^{i_2}(\xi_2), \]

\[ v(x, y, t) = v_1(\xi_1, \xi_2) = q_0(t) \]

\[ + \sum_{m=1}^{4} \sum_{i_1+i_2=m}^{} q_{i_1,i_2}(t) g_1^{i_1}(\xi_1) g_2^{i_2}(\xi_2), \]

\[ g_1'(\xi_1) = \sum_{r=0}^{3} b_r g_1^r(\xi_1), \]

\[ g_2'(\xi_2) = \sum_{r=0}^{3} c_r g_2^r(\xi_2). \] (9)

**Fig. 1** The solution \( w(x, y, t) \) when \( y = 0 \): a 3D-plot, b contour plot, for \( \alpha_1 = 2, \beta_1 = 3, \beta_2 = 5 \) and \( b_0 = c_0 = 1, b_1 = c_1 = 0, b_2 = c_2 = -1, \alpha(t) = 2 - \cos(t), \beta(t) = 2 - \sin(3t), \alpha_3(t) = 3 - \cos(t) \) and \( \beta_3(t) = 3 + \cos(t) \)
Repeating the calculations as for case 1, we obtain the solution of (2)

\[
w(x, y, z) = w_1(\xi_1, \xi_2) = \frac{\beta_2}{2\beta_1^2 \beta(t)} \left\{ 4 \beta_1 \alpha(t) (-b_1^2 \alpha_1^2 \right. \\
+ \frac{12 b_3 \exp(2 b_1 \xi_1)}{(-1 + b_3 \exp(2 b_1 \xi_1))^2} \\
+ c_1^2 \frac{\beta_1}{\beta_1} \left( 1 + \frac{12 c_3 \exp(2 c_1 \xi_2)}{(-1 + c_3 \exp(2 c_1 \xi_2))^2} \\
+ \beta_3(t) - \frac{\beta_1 \alpha_3(t)}{\alpha_1} \right) \right\}, \\
v(x, y, z) = v_1(\xi_1, \xi_2) = \frac{1}{2 \beta(t)} \left\{ 4 b_1^2 \alpha_1^2 \right. \\
+ \frac{12 b_3 \exp(2 b_1 \xi_1)}{(-1 + b_3 \exp(2 b_1 \xi_1))^2} + c_1^2 \frac{\beta_1}{\beta_1} \right. \\
\left. + \left( 1 + \frac{12 c_3 \exp(2 c_1 \xi_2)}{(-1 + c_3 \exp(2 c_1 \xi_2))^2} + \frac{\beta_3(t)}{\beta_1} + \frac{\alpha_3(t)}{\alpha_1} \right) \right\}, \tag{10}
\]

where \(\xi_1 = \alpha_1 x - \frac{\alpha_1 \beta_2}{\beta_1} y + \int \alpha_3(t) dt, \xi_2 = \beta_1 x + \beta_2 y + \int \beta_3(t) dt, \) are arbitrary constants and \(\alpha_3(t), \beta_3(t), \alpha(t), \beta(t)\) denote arbitrary functions.

Expression (10) is represented in Figs. 3 and 4 for different \(\alpha(t)\) and \(\beta(t)\).

Figures 3 and 4 depict two periodic kink and anti-kink waves and the core of double-kink waves, respectively. In both cases, we verify that the charts depend on the ratio \(r_{\alpha \beta} = \alpha(t)/\beta(t)\) and that the amplitudes of the waves are bounded for a bounded ratio \(r_{\alpha \beta}\).

2.2 Multi-wave solutions when \(p = 2\)

In this subsection, we consider \(p = 2\) in Eq. (3) to derive multi-waves in the form of elliptic wave solutions. For this purpose, we have \(n_1 = n_2 = 2\) and \(k = 2\) in Eq. (3) yielding

\[
w(x, y, t) = w_1(\xi_1, \xi_2) = p_0(t) \\
+ \sum_{m=1}^{2} \sum_{i_1+i_2=m} p_{i_1,i_2}(t) g_1^{i_1}(\xi_1) g_2^{i_2}(\xi_2), \\
v(x, y, t) = v_1(\xi_1, \xi_2) = q_0(t) \\
+ \sum_{m=1}^{2} \sum_{i_1+i_2=m} q_{i_1,i_2}(t) g_1^{i_1}(\xi_1) g_2^{i_2}(\xi_2), \\
g'(\xi_1) = \sqrt{b_0 + b_2 g_1^2(\xi_1) + b_4 g_1^4(\xi_1)}, \quad g_2(\xi_2) \\
= \sqrt{c_0 + c_2 g_2^2(\xi_2) + c_4 g_2^4(\xi_2)}. \tag{11}
\]

substituting (11) into (2), we get
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\[ p_0(t) = \frac{\beta_2(\alpha(t) (-4b_2\alpha_1^{3}\beta_1 + 4c_2\beta_1^{3}\alpha_1) + \alpha_1\beta_3(t) - \beta_1\alpha_3(t))}{2\alpha_1\beta_1^2\beta(t)}, \quad p_{1,0}(t) = p_{0,1}(t) = p_{1,1}(t) = 0, \]

\[ p_{0,2}(t) = \frac{6c_4\beta_1\beta_2\alpha(t)}{\beta(t)}, \quad p_{2,0}(t) = -\frac{6b_4\alpha_1^2\beta_2\alpha(t)}{\beta^2(t)}, \]

and

\[ q_0(t) = \frac{4\alpha(t) (b_2\alpha_1^{3}\beta_1 + c_2\beta_1^{3}\alpha_1) + \alpha_1\beta_3(t) + \beta_1\alpha_3(t))}{2\alpha_1\beta_1\beta(t)}, \quad q_{1,0}(t) = q_{0,1}(t) = q_{1,1}(t) = 0, \]

\[ q_{0,2}(t) = \frac{6c_4\beta_1^{3}\alpha(t)}{\beta(t)}, \quad q_{2,0}(t) = \frac{6b_4\alpha_1^{2}\alpha(t)}{\beta^2(t)}, \quad \alpha_2 = -\frac{\alpha_1\beta_2}{\beta_1}. \]

\[ (12) \]

**Fig. 3** The solution \( w(x, y, t) \) when \( y = 0 \): a 3D-plot, b contour plot, for \( \alpha_1 = 2, \beta_1 = 3, \beta_2 = 5 \) and \( b_3 = c_3 = -1, b_1 = 2, c_1 = 1, \alpha(t) = 5 + \sin(3t), \beta(t) = 3 - \cos(3t), \alpha_3(t) = 3 + \cos(3t) \) and \( \beta_3(t) = 2 - \sin(3t) \)

**Fig. 4** The solution \( v(x, y, t) \) when \( y = 0 \): a 3D-plot, b contour plot, for \( \alpha_1 = 2, \beta_1 = 3, \beta_2 = 5 \) and \( b_3 = c_3 = -1, b_1 = 2, c_1 = 1, \alpha(t) = 5 + \sin(3t), \beta(t) = 3 - \cos(3t), \alpha_3(t) = 3 + \cos(3t) \) and \( \beta_3(t) = 2 - \sin(3t) \)
Next, the auxiliary equations in (11) will be solved. For certain values of $b_r$ and $c_r$, $r \in \{0, 2, 4\}$, different solutions involving Jacobi elliptic functions are obtained. In this case, if we consider (due to the classification in [33])

\[
b_0 = -\frac{(1 - m_1^2)^2}{4}, \quad b_2 = \frac{1 + m_1^2}{2}, \quad b_4 = -\frac{1}{4},
\]

\[
c_1 = k_1^4 - 2k_1^3 + k_1^2, \quad c_2 = -k_1^2 + 6k_1 - 1, \quad c_4 = -\frac{4}{k_1},
\]

and substitute into (11), we get

\[
g_1(\xi_1) = m_1 \text{cn}(\xi_1, m_1) + \text{dn}(\xi_1, m_1),
\]

\[
g_2(\xi_2) = \frac{k_1 \text{dn}(\xi_2, k_1^2) - \text{cn}(\xi_2, k_1^2)}{k_1 (\text{dn}(\xi_2, k_1^2)^2 + 1)},
\]

where $0 < m_1 < 1$ and $0 < k_1 < 1$ represent the modulus of the Jacobi elliptic functions.

Finally, the general solution of (2), in terms of the Jacobi elliptic functions, is given by

\[
w(x, y, z) = w(\xi_1, \xi_2) = \frac{\beta_2}{\beta_1} \frac{\beta_3}{\beta_1} \left( -2 \alpha_1 \beta_1 ((1 + m_1^2) \alpha_1^2 + 2 (1 - 6k_1 + k_1^2) \beta_1^2) \alpha(t) - \beta_1 \alpha_3(t) \right)
\]

\[
+ \frac{\alpha_1 \beta_3(t)}{\alpha_1 \beta_1} \frac{48 \beta_1^2 \alpha(t) (\text{cn}(\xi_2, k_1^2) - k_1 \text{dn}(\xi_2, k_1^2))^2}{k_1 (\text{dn}(\xi_2, k_1^2)^2 + 1)^2}
\]

\[
+ 3 \alpha_1^2 \text{dn}(\xi_1, m_1) + m_1 \text{cn}(\xi_1, m_1)^2 \alpha(t) \right),
\]

\[
v(x, y, z) = v(\xi_1, \xi_2) = -\frac{3 \alpha_1^2}{2 \beta_1 \beta_1} \left( 2 \alpha_1 \beta_1 ((1 + m_1^2) \alpha_1^2 + 2 (1 - 6k_1 + k_1^2) \beta_1^2) \alpha(t) + \beta_1 \alpha_3(t) \right)
\]

\[
- \frac{3 \alpha_1^2}{2 \beta_1 \beta_1} \frac{16 \beta_1^2 \alpha(t) (\text{cn}(\xi_2, k_1^2) - k_1 \text{dn}(\xi_2, k_1^2))^2}{k_1 \alpha_1^2 \beta_1 (\text{dn}(\xi_2, k_1^2)^2 + 1)^2}
\]

\[
+ (\text{dn}(\xi_1, m_1) + m_1 \text{cn}(\xi_1, m_1)^2 \alpha(t)) \right),
\]

where $\xi_1 = \alpha_1 x - \frac{\alpha_1 \beta_2}{\beta_1} y + \int \alpha_3(t) dt, \xi_2 = \beta_1 x + \beta_2 y + \int \beta_3(t) dt$, $\alpha_1, \beta_1, \beta_2$ are arbitrary constants and $\alpha_3(t), \beta_3(t), \alpha(t), \beta(t)$ denote arbitrary functions.

Expression (16) is represented in Figs. 5 and 6 for different $\alpha(t)$ and $\beta(t)$ when $m_1 = \frac{1}{4}$ and $k_1 = \frac{3}{4}$.

Figure 5a, b presents the fluid-lattice wave solutions and the contour plot for $w(x, y, t)$, respectively. Figure 5b shows a weak spreading of the rogue waves parallel to the $x$-axis.

Figure 6a, b illustrates the fluid-lattice wave solutions and the contour plot for $v(x, y, t)$, respectively. It is visible in Fig. 6b a moderate spreading of the rogue waves parallel to the $x$-axis. In both Figs. 5 and 6, we notice that the rogue waves are created due the interaction between the kinky and anti-kinky periodic waves.

In summary, the results revealed that the GUM method supported by symbolic mathematical packages represents an useful approach to study the $(2 + 1)$D-vcKdV and to analyze the emerging solutions.
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3 Conclusion

In this paper, we investigated analytically new types of polynomial functions including multi-wave solutions of (2+1)-D-vcKdV using the GUM approach. Among distinct solutions, multi-soliton, periodic and elliptic forms were obtained. The GUM procedure is direct and straightforward to use having in mind the help of symbolic computation software. The results show that the arbitrary functions have significant effect on the wave behavior and this property can be used to give a deeper insight into many complex phenomena that occur in different scientific areas. The analysis presented will be useful for studying other relevant physical nonlinear equations. In a future work, it seems important to explore a generating mechanism for deriving the construction of rogue waves from soliton (or multi-soliton) solutions by extending their version from real to complex solutions.

Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interests regarding the publication of this article.

Ethical standards The authors state that this research complies with ethical standards and it does not involve either human participants or animals.

References


Fig. 6 The solution $v(x, y, t)$ when $y = 0$: a 3D-plot, b contour plot, for $\alpha_1 = 2$, $\beta_1 = -3$, $\beta_2 = 5$, $\alpha(t) = 5 + \sin(3t)$, $\beta(t) = 5 - \sin(3t)$, $\alpha_3(t) = 5 - \cos(t)$ and $\beta_3(t) = 3 - \sin(t)$