On multi-soliton solutions for the \((2 + 1)\)-dimensional breaking soliton equation with variable coefficients in a graded-index waveguide

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Abstract

In this work, we construct multi-soliton solutions of the \((2 + 1)\)-dimensional breaking soliton equation with variable coefficients by using the generalized unified method. We employ this method to obtain double- and triple-soliton solutions. Furthermore, we study the nonlinear interactions between these solutions in a graded-index waveguide. The physical insight and the movement role of the waves are discussed and analyzed graphically for different choices of the arbitrary functions in the obtained solutions. The interactions between the solitons are elastic whether the coefficients of the equation are constants or variables.

1. Introduction

In recent years, attention has been paid to nonlinear evolution equations (NLEEs) in different branches of science such as biology, fluid mechanics, plasmas, condensed matter, and nonlinear optics \([1–5]\). Efforts have been dedicated to find the analytic solutions of NLEEs, including lump solutions (which could be obtained by using symbolic computations to many nonlinear wave equations including the Kadomtsev–Petviashvili equation) \([6]\), solitonic and periodic rational solutions \([7–9]\).

In order to study the solutions of NLEEs, some effective methods have been introduced such as a refined invariant subspace method \([10,11]\) (which gives a largest possible solution subspace by choosing a specific solution to a linear ordinary differential equation as a basis solution), improved Hirota method, Hirota’s bilinear method and its simplified form that were used to solve more NLEEs and could lead to establish multiple wave solutions, variable separation method, Darboux transformation, Bäcklund transformation, the inverse scattering method in which they contain different techniques for constructing these solutions, and the multiple exp-function method \([12–25]\).

Hereby, we aim to apply the generalized unified method (GUM) \([26–30]\) to find multi-soliton solutions of the \((2 + 1)\)-dimensional breaking soliton equation with variable coefficients, denoted by 2D-vcBSE, and study these solutions in a graded-index waveguide.

The 2D-vcBSE is given by

\[
\begin{align*}
    u_t + \alpha(t)u_{xxy} + \gamma(t)(uv)_x &= 0, \\
    u_y &= \beta(t)v_x,
\end{align*}
\]

where \(u = u(x, y, t), v = v(x, y, t)\) while \(\alpha(t), \gamma(t)\) and \(\beta(t)\) are arbitrary analytic functions.

The 2D-vcBSE is widely used when \(\gamma(t) = 4\alpha(t)\) and \(\beta(t) = 1\) which describe the \((2 + 1)\)-dimensional interaction of a Riemann wave propagating along the \(y\)-axis with a long wave along the \(x\)-axis \([31,32]\). In this case, Eq. (1) is solved by

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the projective Riccati equation expansion method [33] and the two general solutions are obtained for Eq. (1) by the singular manifold method [32]. Dai derived 2D-vcBSE chaotic behaviors by the mapping method [34].

The structure of this paper is as follows: In Section 2, the 2D-vcBSE is studied and its multi-soliton solutions are derived by using GUM. Also, through the graphic analysis in graded-index waveguide, the properties of these solutions will be analyzed and investigated. Section 3 is devoted to our conclusions.

2. Multi-soliton solutions of 2D-vcBSE by using GUM

In this section, we apply GUM (for details see [26–30]) to find multi-soliton solutions (double- and triple-soliton solutions) of 2D-vcBSE given by Eq. (1).

To obtain N-soliton solutions in the rational functions form we use, for instance, the transformations \( u(x, y, t) = u_{1x}(x, y, t), v(x, y, t) = v_{1x}(x, y, t) \) in Eq. (1). By using these transformations and integrating both sides with respect to \( x \), Eq. (1) can be written as

\[
\begin{align*}
  u_{1x} + \alpha(x) u_{1xy} + \gamma(t) u_{1x} v_{1x} &= 0, \\
  u_{1y} &= \beta(t) v_{1x},
\end{align*}
\]

where the constants of integration are considered to be zero.

We mention that, in Eq. (2) by plugging \( u_{1y} \) from the second equation into the first equation and viewing \( v_{1} \) as a part of the coefficients, we can transform Eq. (2) into a linear partial differential equation of first order, whose general solution can be systematically presented. So, if we take \( v_{1} = \text{a separable function} \) (say \( v_{1} = a(t) b(x) \)), then a general solution to the resulting linear equation could be given explicitly.

2.1. Double-soliton solutions of 2D-vcBSE by using GUM

Here, we use GUM to find two-soliton solutions of Eq. (2). Assume that

\[
\begin{align*}
  u_{1x}(x, y, t) &= U(\xi_{1}, \xi_{2}) = \frac{p_{0}(t) + p_{1}(t) \phi_{1}(\xi_{1}) + p_{2}(t) \phi_{2}(\xi_{2}) + p_{12}(t) \phi_{1}(\xi_{1}) \phi_{2}(\xi_{2})}{q_{0}(t) + q_{1}(t) \phi_{1}(\xi_{1}) + q_{2}(t) \phi_{2}(\xi_{2}) + q_{12}(t) \phi_{1}(\xi_{1}) \phi_{2}(\xi_{2})}, \\
  v_{1x}(x, y, t) &= V(\xi_{1}, \xi_{2}) = \frac{r_{0}(t) + r_{1}(t) \phi_{1}(\xi_{1}) + r_{2}(t) \phi_{2}(\xi_{2}) + r_{12}(t) \phi_{1}(\xi_{1}) \phi_{2}(\xi_{2})}{q_{0}(t) + q_{1}(t) \phi_{1}(\xi_{1}) + q_{2}(t) \phi_{2}(\xi_{2}) + q_{12}(t) \phi_{1}(\xi_{1}) \phi_{2}(\xi_{2})},
\end{align*}
\]

where \( \xi_{1} = \alpha_{1} x + \alpha_{2} y + \int \alpha_{3}(t) dt \), \( \xi_{2} = \beta_{1} x + \beta_{2} y + \int \beta_{3}(t) dt \) and \( p_{0}(t), q_{0}(t), r_{0}(t), p_{1}(t), q_{1}(t), r_{1}(t), p_{2}(t), q_{2}(t), r_{2}(t), p_{12}(t), q_{12}(t), r_{12}(t) \) are arbitrary functions, \( i, o, 1, 2 \).

The auxiliary functions \( \phi_{j}(\xi) \) satisfy the auxiliary equations \( \phi'_{j}(\xi) = c_{j}(t) \phi_{j}(\xi) \), where \( c_{j}(t) \) are arbitrary functions, \( j = 1, 2 \).

By substituting from (3) into (2) and by equating the coefficients of \( \phi_{j}(\xi) \) to be zero, we get a set of algebraic equations. By using any package in symbolic computations (such as the elimination method or other suitable solvable method with the aid of MATHEMATICA or MAPLE), we get

\[
\begin{align*}
  p_{0}(t) &= \frac{A_{-} H_{-} p_{1}(t) q_{2}(t)}{A_{-} H_{+} q_{12}(t)} - \alpha_{1} c_{1}(t) q_{0}(t) \lambda(t), \\
  p_{12}(t) &= q_{2}(t) \left( \frac{A_{-} H_{-} p_{1}(t) q_{2}(t)}{A_{-} H_{+} q_{12}(t)} - \lambda(t) H_{-} \right), \\
  p_{12}(t) &= \frac{A_{-} H_{-} p_{1}(t) q_{2}(t)}{A_{-} H_{+} q_{12}(t)},
\end{align*}
\]

and

\[
\begin{align*}
  r_{1}(t) &= \frac{A_{-} H_{+} q_{12}(t)(6 \alpha_{2} \alpha_{1} c_{1}(t) q_{0}(t) + r_{0}(t) \gamma(t))}{A_{-} H_{-} q_{2}(t) \gamma(t)}, \\
  r_{2}(t) &= q_{2}(t) \left( \frac{r_{0}(t) \gamma(t) + 6 \beta_{2} c_{2}(t) q_{0}(t) \alpha(t)}{q_{0}(t) \gamma(t)} \right), \\
  r_{12}(t) &= q_{12}(t)(6 \alpha_{2} c_{1}(t) q_{0}(t) \alpha(t) + 6 \beta_{2} c_{2}(t) q_{0}(t) \alpha(t) + r_{0}(t) \gamma(t)), \\
  r_{12}(t) &= \frac{q_{0}(t) \gamma(t)}{q_{0}(t) \gamma(t)}, \\
  r_{3}(t) &= -\alpha_{1}^{2} c_{1}(t) \alpha(t),
\end{align*}
\]

where \( A_{\pm} = \alpha_{1}(2 \alpha_{2} \beta_{1} + \alpha_{1} \beta_{2}) c_{1}(t) \pm \beta_{1} \alpha_{1} \beta_{2} + 2 \alpha_{1} \beta_{2} c_{2}(t), H_{\pm} = \alpha_{1} c_{1}(t) \pm \beta_{1} c_{2}(t), \) and \( \lambda(t) = \frac{6 \alpha_{2} \beta_{1}(t) \gamma(t)}{\gamma(t)} \).

By solving the auxiliary equations \( \phi'_{j}(\xi) = c_{j}(t) \phi_{j}(\xi) \), \( j = 1, 2 \) and substituting together with (4)–(5) into (3), we get the solution of Eq. (1) namely

\[
\begin{align*}
  u(x, y, t) = u_{1x}(x, y, t), \\
  u(x, y, t) = U(\xi_{1}, \xi_{2}) = \frac{q_{0}(t)(1 + \frac{A_{-} H_{-} p_{1}(t) q_{2}(t)}{A_{-} H_{+} q_{12}(t)}) + q_{2}(t) e^{2(t) \xi_{2}} + q_{12}(t) e^{2(t) \xi_{1} + c_{2}(t) \xi_{2}}}{q_{0}(t)(1 + \frac{A_{-} H_{-} p_{1}(t) q_{2}(t)}{A_{-} H_{+} q_{12}(t)}) + q_{2}(t) e^{2(t) \xi_{2}} + q_{12}(t) e^{2(t) \xi_{1} + c_{2}(t) \xi_{2}}},
\end{align*}
\]
found that the variable coefficients interaction between kink and anti-kink soliton waves via the solution given by Eq. (7).  

where \( c_1(t) = 2.4 \), \( c_2(t) = 2 \), \( \alpha(t) = 2 - \cos(t) \), \( \beta(t) = 2 \), and \( \gamma(t) = 3 + \sin(t) \).

Fig. 1. (a) 3D-plot for \( u(x, y, t) \) when \( y = 0 \). (b) The contour plot for \( u(x, y, t) \) when \( y = 0 \), \( \alpha_1 = 0.4 \), \( \beta_1 = 0.35 \), \( \alpha_2 = -0.2 \), \( \beta_2 = 0.1 \) and \( \gamma(t) = 3 + \sin(t) \).

\[
\begin{align*}
v(x, y, t) &= v_{14}(x, y, t), \quad v_1(x, y, t) = V(\xi_1, \xi_2) = \\
&= A_- H_- q_2(t)q_0(t)\gamma(t) + R_2(t)q_2(t)e^{\xi_1(t)\xi_2} + q_{12}(t)q_0(t)R_1(t)A_+ H_+ e^{\xi_1(t)\xi_1} \\
&+ \frac{q_0(t)(q_0(t) + q_2(t)e^{\xi_1(t)\xi_2})A_- H_- q_2(t) + q_{12}(t)e^{\xi_1(t)\xi_1}(A_- H_- q_2(t)e^{\xi_1(t)\xi_2} + A_+ H_+ q_0(t)))\gamma(t)}{q_0(t)(q_0(t) + q_2(t)e^{\xi_1(t)\xi_2})A_- H_- q_2(t) + q_{12}(t)e^{\xi_1(t)\xi_1}(A_- H_- q_2(t)e^{\xi_1(t)\xi_2} + A_+ H_+ q_0(t)))\gamma(t)},
\end{align*}
\]

where \( \xi_1 = \alpha_1 x + \beta_1 y - \alpha_2^2 \int \alpha(t)c_1^2(t) \, dt \), \( \xi_2 = \beta_1 x + \beta_2 y - \beta_2^2 \int \alpha(t)c_2^2(t) \, dt \), \( R_1(t) = r_0(t)\gamma(t) + 6\alpha_2 c_1(t)q_0(t)\alpha(t) \), and \( R_2(t) = r_0(t)\gamma(t) + 6\beta_2 c_2(t)q_0(t)\alpha(t) \).

The solution of Eq. (1) given by (6)–(7) is shown in Figs. 1–2 for different values of \( \alpha(t) \), \( \beta(t) \) and \( \gamma(t) \). Fig. 1 represents overtaking interaction between two soliton waves via the solution given by Eq. (6), while Fig. 2 represents interaction between kink and anti-kink soliton waves via the solution given by Eq. (7).

Figs. 1–2 illustrate double-wave solution with two different velocities, along with periodic shape in the \( x-t \) plane. We found that the variable coefficients \( \alpha(t) \), \( \beta(t) \), and \( \gamma(t) \) did not affect the velocities or the amplitude of the waves during the propagation but they affected only the solutions shape. Also, the soliton waves did not affect each other before or after the collisions, so we can conclude that the collisions are elastic.
2.2. Three-soliton solutions of 2D-vcBSE by using GUM

In this section, we find three-soliton solutions of Eq. (2) by using GUM. To this end, we assume that

\[
\begin{align*}
    u_3(x, y, t) &= U(\xi_1, \xi_2, \xi_3) = p_0(t) + \sum_{i=1}^{3} p_i(t) \phi_i(\xi_3) + \sum_{i,j=1}^{3} p_{ij}(t) \phi_i(\xi_3) \phi_j(\xi_3) + p_{123}(t) \phi_1(\xi_3) \phi_2(\xi_3) \phi_3(\xi_3), \\
    v_1(x, y, t) &= V(\xi_1, \xi_2, \xi_3) = q_0(t) + \sum_{i=1}^{3} q_i(t) \phi_i(\xi_3) + \sum_{i,j=1}^{3} q_{ij}(t) \phi_i(\xi_3) \phi_j(\xi_3) + q_{123}(t) \phi_1(\xi_3) \phi_2(\xi_3) \phi_3(\xi_3),
\end{align*}
\]

where \( i < j, \xi_1 = \alpha_1 x + \alpha_2 y + \int \alpha(t) \, dt, \xi_2 = \beta_1 x + \beta_2 y + \int \beta(t) \, dt, \xi_3 = \gamma_1 x + \gamma_2 y + \int \gamma(t) \, dt, \alpha_k, \beta_k, \gamma_k, k = 1, 2, \) are arbitrary constants and \( p_0(t), q_0(t), r_0(t), p_{123}(t), q_{123}(t), r_{123}(t), p_{ij}(t), q_{ij}(t), r_{ij}(t), \) and \( r_{ij}(t), i, j = 1, 2, 3, \) are arbitrary functions. The auxiliary functions \( \phi_i(\xi_3) \) satisfy the auxiliary equations \( \phi_i(\xi_3) = c_i(t) \phi_i(\xi_3), \) where \( c_i(t) \) are arbitrary functions, \( i = 1, 2, 3. \)

By substituting from (8) into (2) and by a similar way as we did in the last section (2.1), we get

\[
\begin{align*}
    p_0(t) &= \frac{G_+ H p_1(t) q_{23}(t)}{A^2 q_{123}(t) R_2 \beta_1^2 \gamma_2^2} + 6 \gamma_1 q_0(t) c_3(t) \lambda(t), \\
    p_2(t) &= \frac{A_+ R_1 q_{23}(t)(6 \beta_1^2 \gamma_2^2 R_2 A_+ \beta_1^2 \gamma_2^2 R_2 q_3(t) q_{123}(t) + G_+ H p_1(t) q_{23}(t))}{A^2 G_+ A_- \beta_1^2 \gamma_2^2 R_2 q_3(t) q_{123}(t)}, \\
    p_3(t) &= \frac{q_3(t)}{A_+ A_-} \left( \frac{G_+ H p_1(t) q_{23}(t)}{A^2 R_2 \beta_1^2 \gamma_2^2 q_{123}(t) q_0(t)} + 12 \gamma_1 c_3(t) \lambda(t) \right), \\
    p_{13}(t) &= \frac{q_{13}(t)}{A_+ A_-} \left( \frac{H p_1(t) q_{13}(t)}{A^2 \beta_1^2 \gamma_2^2 q_0(t)} + \frac{6 A_2^2 \gamma_1 R_2 q_3(t) c_3(t) \lambda(t)}{G_+ q_23(t)} \right), \\
    p_{12}(t) &= A^2 R_1 \left( \frac{6 \beta_1^2 \gamma_2^2 q_3(t) q_{123}(t) q_0(t) \lambda(t)}{A^2 G_+ G_+ q_{23}(t) q_0(t)} + G_+ p_1(t) q_{23}(t) \right), \\
    p_{23}(t) &= q_{23}(t) \left( \frac{A^2 G_+ G_+ H R_1(t) q_{23}(t) q_0(t) \lambda(t)}{A^2 \beta_1^2 \gamma_2^2 R_2 q_{123}(t) q_0(t)} + 6 \lambda(t) c_3(t) A_+ \right), \\
    p_{123}(t) &= \frac{G_+ H p_1(t) q_{23}(t)}{A^2 \beta_1^2 \gamma_2^2 R_2 q_3(t) q_{123}(t) q_0(t)} + 6 \lambda(t) A_+ A_+ q_{123}(t), \\
    r_0(t) &= \frac{G_+ H r_1(t) q_{23}(t)}{A^2 q_{123}(t) R_2 \beta_1^2 \gamma_2^2} - \frac{1}{\gamma(t)} (6 \gamma_2 q_0(t) c_3(t) \alpha(t)), \\
    r_3(t) &= \frac{G_+ H r_1(t) q_{23}(t)}{A^2 \beta_1^2 \gamma_2^2 R_2 q_{123}(t) q_0(t)} a(t), \\
    r_2(t) &= \frac{A_+ R_1 q_{23}(t)(6 \beta_1^2 \gamma_2^2 R_2 q_0(t) q_{123}(t) q_0(t) \lambda(t) + G_+ H r_1(t) q_{23}(t) \gamma(t))}{A^2 G_+ A_- \beta_1^2 \gamma_2^2 R_2 q_3(t) q_{123}(t) \gamma(t)} + 6 A_2^2 R_2 c_2(t) \beta_1^2 \beta_2 \gamma_2^2 q_{123}(t) \gamma(t) \alpha(t)), \\
    r_{12}(t) &= A^2 R_1 (G_+ H r_1(t) q_{23}(t) \gamma(t) + 6 A_2^2 R_2 c_2(t) \beta_1^2 \beta_2 \gamma_2^2 q_{123}(t) \gamma(t) \alpha(t)), \\
    r_{13}(t) &= \frac{q_{13}(t)}{A_+ A_-} \left( \frac{G_+ H r_1(t) q_{23}(t)}{A^2 \beta_1^2 \gamma_2^2 R_2 q_{123}(t) q_0(t) \lambda(t) + 6 \beta_2 c_2(t) \alpha(t)} \right), \\
    r_{23}(t) &= q_{23}(t) \left( \frac{A^2 G_+ G_+ H r_1(t) q_{23}(t) q_0(t) \lambda(t)}{A^2 \beta_1^2 \gamma_2^2 R_2 q_{123}(t) q_0(t)} + G_+ q_{23}(t) \gamma(t) \right), \\
    r_{123}(t) &= \frac{G_+ H r_1(t) q_{23}(t)}{A^2 \beta_1^2 \gamma_2^2 R_2 q_3(t) q_{123}(t) q_0(t)} + 6 \lambda(t) B_+ \alpha(t), \\
    q_1(t) &= \frac{A^2 \beta_1^2 \gamma_2^2 R_2 q_0(t) q_{123}(t)}{G_+ H q_{23}(t)}, \\
    q_2(t) &= \frac{A_+ R_1 q_{23}(t) q_0(t)}{A_- G_- q_3(t)}, \\
    q_{12}(t) &= \frac{A^2 R_1 \gamma_1 \gamma_2 \gamma_2 \gamma_2 q_0(t) q_{123}(t)}{G_- H q_3(t)}, \\
    q_{13}(t) &= \frac{A_- R_1 q_{23}(t) q_0(t)}{A_+ G_+ q_3(t)}, \\
    \alpha_3(t) &= \alpha_1 \gamma_1 \gamma_2 c_3(t) \alpha(t), \\
    \beta_3(t) &= -\beta_1^2 \beta_2 c_2(t) \alpha(t), \\
    \gamma_3(t) &= -\gamma_1^2 \gamma_2 c_2(t) \alpha(t), \\
    c_1(t) &= -\frac{\gamma_1 c_3(t)}{\alpha_1}, \\
    \alpha_2 &= -\frac{\gamma_1^2 \gamma_2}{\gamma_1}, \\
    R_1 &= c_2(t) \beta_1(2 \beta_2 \gamma_1 + \beta_1 \gamma_2) + c_3(t) \gamma_1(2 \beta_2 \gamma_1 + 2 \beta_1 \gamma_2), \\
    \lambda(t) &= \frac{\alpha(t) \beta(t)}{\gamma(t)}, \\
    R_2 &= c_2(t) \beta_1(-2 \beta_2 \gamma_1 + \beta_1 \gamma_2) + c_3(t) \gamma_1(\beta_2 \gamma_1 - 2 \beta_1 \gamma_2).
\end{align*}
\]

\[ (11) \]
\[ A_{\pm} = \beta_1 c_2(t) \pm \gamma_1 c_3(t), \quad G_+ = R_2 + 2 \gamma_1 (2 \beta_1 \gamma_2 - \beta_2 \gamma_1) c_3(t), \quad G_- = R_1 - 2 \gamma_1 (2 \beta_1 \gamma_2 + \beta_2 \gamma_1) c_3(t), \quad \text{and} \quad B_{\pm} = \beta_2 c_2(t) \pm \gamma_2 c_3(t). \]

By solving the auxiliary equations \( \phi'_l(\xi_j) = c_1(t) \phi_l(\xi_j) \), \( l = 1, 2, 3 \) and substituting together with (9)–(11) into (8), we get the solution of Eq. (2) which is very lengthy to be written here.

The solution (8) of Eq. (2) when \( N = 3 \) is shown in Figs. 3–4 for different values of \( \alpha(t), \beta(t) \) and \( \gamma(t) \). The same discussion as in the case of the two-soliton solutions holds in the case of three-soliton solutions, but they will not be repeated here. The obtained conclusions in Figs. 3–4 are similar with that in the two-soliton situation.

### 3. Conclusion

Here, we have investigated the \((2 + 1)\)-dimensional breaking soliton equation with variable coefficients (2D-vcBSE) via the generalized unified method (GUM). Based on GUM, multi-soliton solutions of 2D-vcBSE have been obtained. Soliton evolution and interactions have been graphically presented and analyzed in a graded-index waveguide. We found that the soliton waves keep their velocities and amplitude invariant during the propagation along with periodic shape in the \( x-t \) plane. So, the collisions between these waves are elastic. Our results show the diversity of the spatial and space–time structures of multi-soliton waves in nonlinear dynamic systems. At the same time, we also hope that our results will provide some valuable information in the field of nonlinear science.
References


