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On Dynamical Games

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Abstract: Two examples of dynamical games are studied. The first is a dynamic game representing water allocation to two different crops. The second is the spatial evolutionary game introduced by Hofbauer and Sigmund [1]. We make use of the factorization method to find exact traveling wave solutions of this system. Then motivated by this game, we give a generalization to a class of 2-dimensional systems .

Key words: Dynamical games;factorization method;spatial evolutionary game

1 Introduction

Game theory was first introduced by von Neumann and Morgenstern in 1944 as a mathematical model. It is the study of ways in which strategic interactions among rational players produce outcomes with respect to the preferences of the players. Each player in a game faces a choice among two or more possible strategies. A strategy is a predetermined program of play that tells the player what actions to take in response to every possible strategy other players may use [2]. Water allocation for different purposes was studied by several authors as a static problem and recently by Daene C. McKinney and Andre G. Savitsky [3]. We study the dynamic game representing water allocation to two different crops and the spatial evolutionary game introduced by Hofbauer as a dynamical game [1]. We use the factorization method [4-8] to find exact solutions for the resulting partial differential equation.

2 A dynamic game for water allocation for two crops

Water allocation for different purposes is an important problem. However most of its studies are static while the problem is dynamic. Here we introduce a dynamic element of this problem. The profit function of one crop consuming quantity x of water is [3]

$$\Pi = p[\alpha + \beta x + \gamma \ln(x)] - cx, \tag{1}$$

where p, c are the price and the production cost of the crop and α, β, γ are constants. For more than one crop, the profit function is

$$\Pi = \sum_j \{p_j[\alpha_j + \beta_j x_j + \gamma_j \ln(x_j)] - c_j x_j\}, \tag{2}$$

subject to the constraint $\sum_j x_j = 1$, where the total quantity of allowed water has been normalized to one. For the case of two crops one has

$$\Pi = p_1[\alpha_1 + \beta_1 x + \gamma_1 x] - c_1 x + p_2[\alpha_2 + \beta_2(1-x) + \gamma_2(1-x)] - c_2(1-x). \tag{3}$$

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The optimal water allocation is obtained by $\frac{\partial \Pi}{\partial x}=0$ which gives

$$x = \frac{-b_1 \pm \sqrt{b_1^2-4p_1\gamma_1 a_1}}{2a_1}, \quad (4)$$

$$a_1 = c_1-c_2+p_2\beta_2-p_1\beta_1, \quad (5)$$

$$b_1 = -a_1-p_2\gamma_2-p_1\gamma_1 \quad (6)$$

provided that the condition

$$0<x<1, \quad (7)$$

is satisfied. The corresponding dynamical game is given by [9, 10]

$$x_{t+1}=x_t+a\frac{\partial \Pi}{\partial x_t},$$

this is called a bounded rationality system and a is a constant. For the dynamic water allocation problem one gets

$$x_{t+1}=x_t+a[p_{1t}(\beta_1+\frac{\gamma_1}{x_t})-p_{2t}(\beta_2+\frac{\gamma_2}{1-x_t})-(c_1-c_2)]. \quad (8)$$

The equilibrium solution of the system (6) is given by (4) and the condition (5) guarantees that it is locally asymptotically stable where $p_1(p_2)$ is the limit of the sequence p_{1t} (p_{2t}) respectively as t increases.

3 Spatial games

Spatial games have been proposed by Hofbauer [1]. For a matrix game with two strategies they typically take the form

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u(1-u)(a+bu), \quad (9)$$

where u is the fraction of adopters of the first strategy and a, b are constants. Here we apply a method of factorizing the wave equation corresponding to (7) to get an exact solution.

Using the coordinate transformation $z=x-\omega t$ (ω is the propagation speed) in equation (7) we obtain the following nonlinear ordinary differential equation

$$u''+cu'+u(1-u)(a+bu)=0, \quad (10)$$

where $'$ means the derivative with respect to z.Using operator notation, equation (8) takes the form

$$\left[D^2+cD+F(u)\right]u=0, \quad F(u)=(1-u)(a+bu). \quad (11)$$

The factorization of (9) leads to

$$\left[D-\varphi_2(u)\right]\left[D-\varphi_1(u)\right]u=0, \quad (12)$$

and then

$$u''-[\varphi_2+\varphi_1+\frac{d\varphi_1}{du}u]u'+\varphi_1\varphi_2 u=0. \quad (13)$$

Comparing (11) and (8) we obtain the conditions on φ_1 and φ_2 as:

$$-(\varphi_2+\varphi_1+\frac{d\varphi_1}{du}u)=\omega, \quad \varphi_1\varphi_2=F(u). \quad (14)$$

Now, we choose φ_1 and φ_2 such that

$$\varphi_1(u)=\lambda(a+bu), \quad \varphi_2(u)=\frac{1}{\lambda}(1-u). \quad (15)$$

Substituting from (13) into the first equation in (12), we obtain

$$\lambda_{\pm}=\frac{-\omega\pm\sqrt{\omega^2-4a}}{2}, \quad \omega^2>4a, \quad (16)$$

and the corresponding first order differential equation is

$$[D - \lambda_{\pm}(a + bu)]\, u = 0. \tag{17}$$

By direct integration of (15), we get

$$u^{\pm}(z) = \frac{a}{-b + \exp(-\lambda_{\pm} a(z + z_0))}, \tag{18}$$

where z_0 is the integration constant.

One particular solution is obtained from (16) as

$$u_1(z) = a\left(-b + \exp[-\lambda_{\pm} a(z - z_0)]\right)^{-1} \tag{19}$$

and equation (15) is transformed to the Riccati equation

$$u' - \lambda_{\pm}(a + bu)u = 0. \tag{20}$$

The two-parameter solution is obtained from (17), (18) as [11] :

$$u_{\mu}(z) = u_1 - \frac{a e^{\lambda_{\pm} a(z - z_0)}(1 - b e^{\lambda_{\pm} a(z - z_0)})^{-1}}{1 - \mu(1 - b e^{\lambda_{\pm} a(z - z_0)})}\ . \tag{21}$$

It is clear from Eq. (21) that when $|\mu|$ runs from zero to infinity, the solution goes from the trivial solution $u = 0$ to the particular solution $u = u_1$.

Motivated by the above case we introduce a generalization of a class of two dimensional systems using the factorization method. The four possible generalizations are

$$[D - f_1(u)][D - g_1(v)]u = 0,\ [D - f_2(u)][D - g_2(v)]v = 0, \tag{22}$$

$$[D - f_1(u)][D - g_1(v)]u = 0,\ [D - g_2(v)][D - f_2(u)]v = 0, \tag{23}$$

$$[D - g_1(v)][D - f_1(u)]u = 0,\ [D - f_2(u)][D - g_2(v)]v = 0, \tag{24}$$

$$[D - g_1(v)][D - f_1(u)]u = 0,\ [D - g_2(v)][D - f_2(u)]v = 0. \tag{25}$$

And for example, the 1^{st} and 3^{rd} systems (22), (24) corresponds to the equations

$$u'' - [g_1(v) + f_1(u)]\, u' + [-\frac{dg_1}{dv} v' + f_1(u) g_1(v)]u = 0, \tag{26}$$

$$v'' - \left[g_2(v) + f_2(u) + v\frac{dg_2}{dv}\right] v' + [f_2(u) g_2(v)]v = 0, \tag{27}$$

and

$$u'' - \left[f_1(u) + (v g_1(v))'\right] u' + u f_1(u) g_1(v) = 0, \tag{28}$$

$$v'' - \left[f_2(u) + (v g_2(v))'\right] v' + v f_2(u) g_2(v) = 0. \tag{29}$$

The second system has the solution:

$$\int \frac{du}{u f_1(u)} = \zeta + c_1, \quad \int \frac{dv}{v g_2(v)} = \zeta + c_2, \quad \int \frac{f_2(u)}{u} du = \int \frac{g_1(v)}{v} dv, \tag{30}$$

where $\zeta = x - \omega t$.

As an example consider the system

$$\begin{array}{l} u'' - u'(a_1 v + b_1 + 1 - 2u) + u(1-u)(a_1 v + b_1) = 0, \\ v'' - v'(a_2 u + b_2 + 1 - 2v) + v(1-v)(a_2 u + b_2) = 0, \end{array} \tag{31}$$

which corresponds to

$$g_1(v) = (a_1 v + b_1), \quad f_1(u) = (1-u), \quad f_2(u) = (a_2 u + b_2), \quad g_2(v) = (1-v),$$

and the particular solution is given as:

$$u(x,t) = \frac{1}{1 + \exp[A(\omega t - x)]}, \quad v(x,t) = \frac{1}{1 + \exp[B(\omega t - x)]}, \tag{32}$$

where a_1, b_1, a_2, b_2 are constants, admitting the relation

$$a_2 u + b_2 \ln(u) = a_1 v + b_1 \ln(v),$$

and A, B are arbitrary constants.

As another example consider the system

$$\begin{array}{l} u'' - u'(2 + 3u^2 + v) + u(1+u^2)(v+1) = 0, \\ v'' - v'(2 - u + 3v^2) + v(1+v^2)(1-u) = 0, \end{array} \tag{33}$$

which corresponds to

$$g_1(v) = (v+1), \quad f_1(u) = (1+u^2), \quad f_2(u) = (1-u), \quad g_2(v) = (1+v^2),$$

and the particular solution takes the form

$$u(x,t) = \pm\frac{1}{\sqrt{-1 + A\exp 2[(\omega t - x)]}}, \quad v(x,t) = \pm\frac{1}{\sqrt{-1 + B\exp[2(\omega t - x)]}},$$

where A and B are arbitrary constants.

4 Conclusions

Dynamical games representing water allocation to two different crops and the spatial evolutionary game introduced by Hofbauer and Sigmund are studied. We used the factorization method to find exact solution for the spatial evolutionary game introduced by Hofbauer and Sigmund and the two parameter solutions have been found. A generalization to a class of two dimensional system has been introduced and applied to a two examples.

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