Chaos, Solitons and Fractals 41 (2009) 1550–1556

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

journal homepage: www.elsevier.com/locate/chaos

Exact solution for the generalized Telegraph Fisher's equation

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ARTICLE INFO

Article history:
Accepted 23 June 2008

ABSTRACT

In this paper, we applied the factorization scheme for the generalized Telegraph Fisher's equation and an exact particular solution has been found. The exact particular solution for the generalized Fisher's equation was obtained as a particular case of the generalized Telegraph Fisher's equation and the two-parameter solution can be obtained when $n = 2$.

1. Introduction

The theory of Einsteins Brownian motion provides a microscopic model of diffusion. It is known that quantum mechanics is related to diffusion by a formal analytic continuation. Hence, the relationship between Brownian motion and quantum mechanics is vague [1–4]. Ord showed that a random walk model of Brownian motion produces the diffusion equation [5–7] or the Telegraph equation to describe particle density. It can also be shown that the correlations in the space-time geometry of Brownian particles obey both the Schroedinger and Dirac equations [8]. For more details see Fig. 6 given by Ord [8].

The standard diffusion equation (which is also known as Fick's second law) depends on the continuity equation and the Fick's first law which are

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x}, \quad J(x,t) = -D\frac{\partial u(x,t)}{\partial x}, \tag{1}$$

where $J(x,t)$ is the current of the diffusing object (flux), (which may be technology, concepts, etc.) $u(x,t)$ is the distribution function of the diffusing quantity, and D is the diffusion constant. The resulting standard equation is

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2}. \tag{2}$$

A basic weakness of this equation is that the flux $J(x,t)$ reacts instantaneously to the gradient of $u(x,t)$; consequently an unbounded propagation speed is assumed.

A way to go round this unphysical effect is to solve the diffusion Eq. (2) by the use of Fourier and Laplace transforms. Applying the Fourier transform in the spatial one-dimension to Eq. (2) and denoting by

$$F_x\{p(x,t)\} = p(k,t), \tag{3}$$

we get

$$\begin{aligned} F_x\left\{\frac{\partial}{\partial t}u(x,t)\right\} &= \frac{\partial}{\partial t}F_x\{u(x,t)\} \\ &= \frac{\partial}{\partial t}u(x,t) \end{aligned} \tag{4}$$

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doi:10.1016/j.chaos.2008.06.018

and

$$F_x\left\{\frac{\partial^2}{\partial x^2}u(x,t)\right\} = -D(2\pi k)^2 F_x\{u(x,t)\}$$
$$= -D(2\pi k)^2 u(x,t), \tag{5}$$

thus

$$u(x,t) = -D(2\pi k)^2 u(x,t). \tag{6}$$

This linear differential equation has the solution

$$u(x,t) = u(k,t=0)\mathrm{e}^{-D(2\pi k)^2 t}. \tag{7}$$

Now we take the inverse Fourier transform of Eq. (7)

$$u(x,t) = F_k^{-1}\{u(k,t=0)\mathrm{e}^{-D(2\pi k)^2 t}\} \tag{8}$$

Using the convolution theorem we get

$$u(x,t) = F_k^{-1}\{u(k,t=0)\} * F_k^{-1}\{\mathrm{e}^{-D(2\pi k)^2 t}\}, \tag{9}$$

where * is the convolution operator

$$(g*h)(y) = \int_{-\infty}^{\infty} g(x)h(y-x)\mathrm{d}x. \tag{10}$$

Now $F_k^{-1}\{u(k,t=0)\} = u(x,t=0)$ and

$$F_k^{-1}\{\mathrm{e}^{-D(2\pi k)^2 t}\} = \frac{1}{\sqrt{4\pi Dt}}\mathrm{e}^{-\frac{x^2}{4Dt}} \tag{11}$$

therefore

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} u(\tilde{x},0)\mathrm{e}^{-\frac{(x-\tilde{x})^2}{4Dt}}. \tag{12}$$

This equation is also known as the Poisson integral.

The convolution kernel is the Gaussian bell-curve with width of $\sqrt{4Dt}$. Consider now an initial condition in the form of a δ-function, $u(x,0) = \delta(x)$. Then

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}}\mathrm{e}^{-\frac{x^2}{4Dt}}. \tag{13}$$

So a δ-function diffuses out as a Gaussian. If one now considers an arbitrary condition $u(x,0)$ as a sum (integral) of δ-functions, one can see that the solution $u(x,t)$ is the sum (integral) of the diffused δ-functions, which have become Gaussian's. This helps to get an intuitive understanding of Eq. (12). It is obvious that no matter how large x is and how small t may be, then $u(x,t)$ is nonzero which violates the fact that all physical propagation speeds $(\frac{\triangle x}{\triangle t})$ are finite. This is specially true in biological and economics systems where it is known that in many cases propagation speed is typically small. This, mathematically speaking, is due to the fact that Eq. (2) is a parabolic partial differential equation.

To overcome this weakness (Fick's law), Cattaneo in 1948 proposed a modified approach [9–11]. He replaced the constitutive Eq. (1) by

$$J(x,t) + \tau\frac{\partial J(x,t)}{\partial t} = -D\frac{\partial u}{\partial x}(x,t), \tag{14}$$

where now the flux relaxes, with same given characteristic time constant τ. Combining (14) with the equation of continuity (1), one obtains the modified diffusion equation or (Telegraph equation)

$$\frac{\partial u(x,t)}{\partial t} + \tau\frac{\partial^2 u(x,t)}{\partial t^2} = D\frac{\partial^2 u(x,t)}{\partial x^2} \tag{15}$$

for constant D and τ.

The corresponding Telegraph reaction diffusion (TRD) equation of (15) is given by [12], this equation is used in many applications of sciences [13–18], and given as

$$\tau\frac{\partial^2 u(x,t)}{\partial t^2} + \left(1 - \tau\frac{\mathrm{d}f}{\mathrm{d}u}\right)\frac{\partial u(x,t)}{\partial t} = D\nabla^2 u(x,t) + f(u(x,t)), \tag{16}$$

where $f(u)$ is a polynomial function in u.

The time constant τ can be related to the memory effect of the flux J as a function of the distribution u as shown in what follows.

Another motive for TRD comes from media with memory [14] where the flux J is related to the density u through a relaxation function $K(t)$ as follows

$$J(x,t) = -\int_0^t K(t-t')u_x(x,t')\mathrm{d}t'. \tag{17}$$

We will see that, with a suitable choice for $K(t)$, the standard Telegraph equation is obtained; Indeed, let us compute the left-hand side of the Eq. (14). For our generalization (17), we get

$$J(x,t) + \tau\frac{\partial J(x,t)}{\partial t} = -\left(\tau\frac{\partial}{\partial t} + 1\right)\int_0^t K(t-t')\frac{\partial u(x,t')}{\partial x}\mathrm{d}t'. \tag{18}$$

Hence by comparing with Eq. (14) it appears clear that we must have

$$\tau K(0) = D, \quad \tau\frac{\partial}{\partial t}K(t) + K(t) = 0.$$

Solving this differential equation, we obtain the relaxation function that makes the non-local theory of transport compatible with the Cattaneo equation

$$K(t) = \frac{D}{\tau}\exp\left(-\frac{t}{\tau}\right).$$

This further supports that Telegraph diffusion equation is more suitable for economic and biological systems than the usual one since, e.g., it is known that we take our decisions according to our previous experiences so memory effects are quite relevant.

In this paper, we concentrate our work on the following spacial form of Eq. (16), namely

$$\tau\frac{\partial^2 u(x,t)}{\partial t^2} + \left(1 - \tau\frac{\mathrm{d}f}{\mathrm{d}u}\right)\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + (1-u^n), \tag{19}$$

which we will call the generalized Telegraph Fisher's equation (GTFE).

2. Factorization procedure for nonlinear ordinary second-order differential equations

Factorization of second-order linear differential equations is a well established technique to find solutions in an algebraic manner [19–24]. Rosu and Cornejo found one particular solution once the nonlinear equation is factorized with the use of two first-order differential operators [19]. They used the method for equations of types:

$$u'' + \gamma u' + f(u) = 0 \tag{20}$$

and

$$u'' + g(u)u' + f(u) = 0 \tag{21}$$

where $'$ means the derivative $D = \frac{\mathrm{d}}{\mathrm{d}z}$, $g(u)$ and $f(u)$ are polynomials in u. We concentrate our work in this paper on equation of the type (21).

Now, Eq. (21) can be factorized as

$$[D - \varphi_2(u)][D - \varphi_1(u)]u = 0, \tag{22}$$

which leads to the equation

$$u'' - \frac{\mathrm{d}\varphi_1}{\mathrm{d}u}uu' - \varphi_1 u' - \varphi_2 u' + \varphi_1\varphi_2 u = 0, \tag{23}$$

or

$$u'' - \left(\varphi_1 + \varphi_2 + \frac{\mathrm{d}\varphi_1}{\mathrm{d}u}u\right)u' + \varphi_1\varphi_2 u = 0. \tag{24}$$

Comparing (24) and (21) we find

$$g(u) = -\left(\varphi_1 + \varphi_2 + \frac{\mathrm{d}\varphi_1}{\mathrm{d}u}u\right) \quad \text{and} \quad f(u) = \varphi_1\varphi_2 u. \tag{25}$$

If Eq. (21) can be factorized as in Eq. (22), then a first particular solution can be easily found by solving

$$[D - \varphi_1(u)]u = 0.$$

3. The generalized Telegraph Fisher's equation

We can write the GTFE as

$$\tau u_{tt} + \left(1 - \tau \frac{\mathrm{d}f}{\mathrm{d}u}\right) u_t = u_{xx} + f(u), \quad f(u) = u(1 - u^n). \tag{26}$$

It is clear that when $\tau = 0$, Eq. (26) reduces to the generalized Fisher equation [19].

Using the coordinate transformation $z = x - ct$ (c is the propagation speed) in Eq. (26) we obtain the following nonlinear ordinary differential equation

$$u'' + \left[\frac{c(1-\tau)}{(1-c^2\tau)} + \frac{c\tau(1+n)}{(1-c^2\tau)} u^n\right] u' + \frac{1}{(1-c^2\tau)} u(1-u^n) = 0, \quad 1 - c^2\tau > 0, \tag{27}$$

or

$$u'' + [A + Bu^n]u' + Cu(1 - u^n) = 0, \tag{28}$$

where

$$A = \frac{c(1-\tau)}{(1-c^2\tau)}, \quad B = \frac{c\tau(1+n)}{(1-c^2\tau)}, \quad C = \frac{1}{(1-c^2\tau)}. \tag{29}$$

Eq. (27) in standard form (21) takes the form (28), where

$$g(u) = A + Bu^n. \tag{30}$$

Using operator notation, Eq. (28) takes the form

$$\left[D^2 + g(u)D + \frac{Cf(u)}{u}\right] u = 0, \tag{31}$$

The factorization of (31) leads to

$$[D - \varphi_2(u)][D - \varphi_1(u)]u = 0, \tag{32}$$

and then

$$u'' - [\varphi_2 + \varphi_1 + \frac{\mathrm{d}\varphi_1}{\mathrm{d}u} u]u' + \varphi_1\varphi_2 u = 0. \tag{33}$$

Comparing (33) and (28) we obtain the conditions on φ_1 and φ_2 as

$$-(\varphi_2 + \varphi_1 + \frac{\mathrm{d}\varphi_1}{\mathrm{d}u} u) = g(u), \quad \varphi_1\varphi_2 = \frac{Cf(u)}{u}, \tag{34}$$

therefore

$$\varphi_1\varphi_2 = C(1 - u^n) = C(1 + u^{\frac{n}{2}})(1 - u^{\frac{n}{2}}), \quad C = ab. \tag{35}$$

Now, choosing φ_1 and φ_2 such that

$$\varphi_1(u) = a(1 - u^{\frac{n}{2}}), \quad \varphi_2(u) = b(1 + u^{\frac{n}{2}}), \tag{36}$$

where C is an arbitrary constant that must be determined.

Substituting from (36) into (34) we obtain

$$(a + b) + \left(b - a - \frac{an}{2}\right) u^{\frac{n}{2}} = (A + Bu^n), \tag{37}$$

then

$$(a + b) = A, \quad (b - a - \frac{an}{2}) = 0, \tag{38}$$

which implies that

$$a = \frac{2A}{n+4}, \quad b = \frac{C(n+4)}{2A},$$

then

$$a = \pm\sqrt{C}\lambda_n, \quad b = \pm\sqrt{C}\lambda_n^{-1}, \quad \lambda_n = \left(\frac{n}{2} + 1\right)^{-\frac{1}{2}}, \tag{39}$$

and

$$\varphi_1(u) = \pm\sqrt{C}\lambda_n(1 - u^{\frac{n}{2}}), \quad \varphi_2(u) = \pm\sqrt{C}\lambda_n^{-1}(1 + u^{\frac{n}{2}}). \tag{40}$$

The corresponding factorization is

$$[D \mp \sqrt{C}\lambda_n^{-1}(1 + u^{\frac{n}{2}})][D \mp \sqrt{C}\lambda_n(1 - u^{\frac{n}{2}})]u = 0, \tag{41}$$

and the compatible first-order differential equation is

$$[D \pm \sqrt{C}\lambda_n(u^{\frac{n}{2}} - 1)]u = 0. \tag{42}$$

By direct integration we get

$$u^{\pm}(z) = \left[1 \pm \exp\sqrt{C}(\lambda_n - \lambda_n^{-1})(z - z_0)\right]^{-\frac{2}{n}}, \tag{43}$$

where z_0 is the integration constant.

The solution Eq. (43) in hyperbolic form is given as

$$u^{+}(z) = \left[\frac{1}{2} - \frac{1}{2}\tanh\left[\frac{(\lambda_n - \lambda_n^{-1})}{2\sqrt{(1 - c^2\tau)}}(z - z_0)\right]\right]^{\frac{2}{n}}, \tag{44}$$

$$u^{-}(z) = \left[\frac{1}{2} - \frac{1}{2}\coth\left[\frac{(\lambda_n - \lambda_n^{-1})}{2\sqrt{(1 - c^2\tau)}}(z - z_0)\right]\right]^{\frac{2}{n}}. \tag{45}$$

Putting $\tau = 0$ in (44) and (45) we find an exact particular solution for the generalized Fisher's equation [19,25]. In Fig. 1, we show a plot of the solution u^{+} for different values of τ and $n = 3$:

Considering now the factorization of Eq. (32) by choosing

$$\varphi_1(u) = a(1 + u^{\frac{n}{2}}), \quad \varphi_2(u) = b(1 - u^{\frac{n}{2}}). \tag{46}$$

The change of order of the factorization brackets gives

$$[D \mp \sqrt{C}\lambda_n^{-1}(1 - u^{\frac{n}{2}})][D \mp \sqrt{C}\lambda_n(1 + u^{\frac{n}{2}})]u = 0, \tag{47}$$

and therefore the compatibility is with the different first-order equation

$$[D \pm \sqrt{C}\lambda_n(1 + u^{\frac{n}{2}})]u = 0, \tag{48}$$

then the direct integration gives the solution

$$u^{\pm}(z) = (-1)^{\frac{2}{n}}[1 \mp \exp[\sqrt{C}(\lambda_n - \lambda_n^{-1})(z - z_0)]]^{-\frac{2}{n}} \tag{49}$$

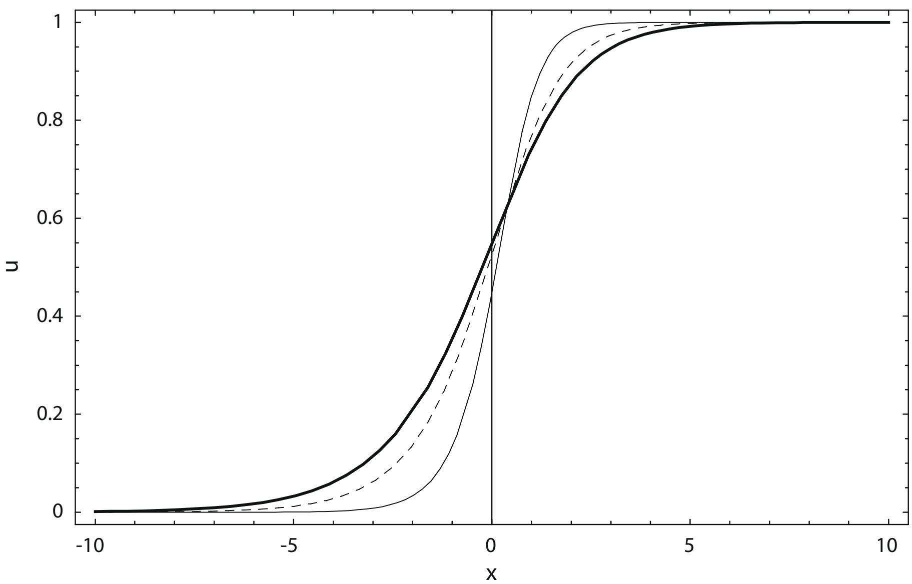

Fig. 1. The solution u^{+} of Eq. (44) for different values of τ, namely, the normal, dashed and solid graphs represent the solutions for $\tau = 20, 10, 0$, respectively, at $t = 2$, $c = 0.2$, $z_0 = 0$, and $n = 3$.

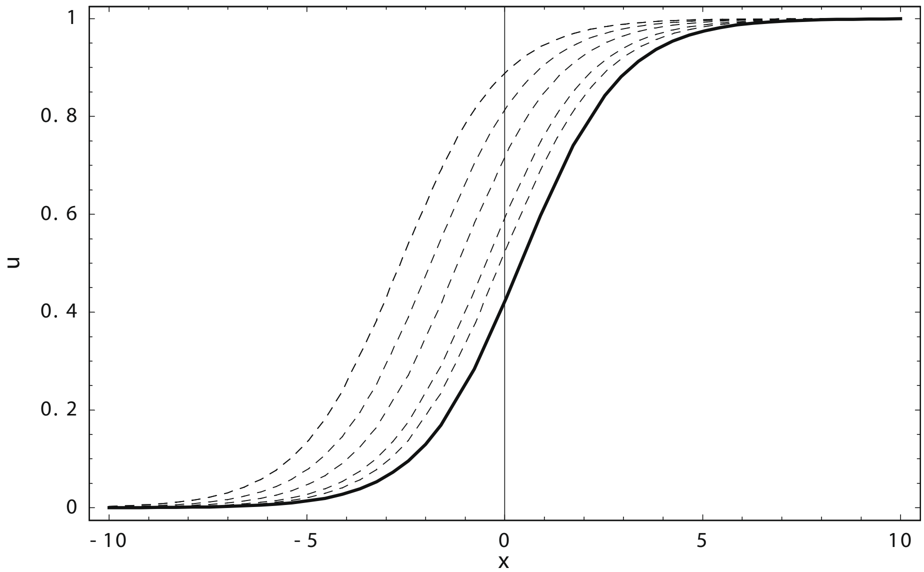

Fig. 2. The solutions u_1 and u_μ for $\mu = 1.1, 1.2, 1.3, 2, 3$, at $t = 2$, $c = 0.2$, $z_0 = 0$, $\tau = 5$ and $n = 2$.

or

$$u^{+}(z) = \left[\frac{1}{2} - \frac{1}{2}\tanh\left[\frac{-(\lambda_n - \lambda_n^{-1})}{2\sqrt{(1-c^2\tau)}}(z - z_0)\right]\right]^{\frac{2}{n}}, \tag{50}$$

$$u^{-}(z) = \left[\frac{1}{2} - \frac{1}{2}\coth\left[\frac{-(\lambda_n - \lambda_n^{-1})}{2\sqrt{(1-c^2\tau)}}(z - z_0)\right]\right]^{\frac{2}{n}}. \tag{51}$$

Following the work of Reyes and Rosu [26–28], we find a two-parameter solution for the special case when $n = 2$. In this case $\varphi_1(u) = a(1+u)$ and $\varphi_2(u) = b(1-u)$, one particular solution is obtained from (43) as

$$u_1(z) = \left[1 \pm \exp\left(-\frac{(z - z_0)}{\sqrt{2(1-c^2\tau)}}\right)\right]^{-1}, \tag{52}$$

and Eq. (42) is transformed to the following Riccati equation

$$u' \pm \sqrt{C}\lambda_n(u-1)u = 0. \tag{53}$$

The two-parameter solution is obtained from (52) and (53) as

$$u_\mu(z) = u_1(z) + \frac{\exp\left(\frac{(z-z_0)}{\sqrt{2(1-c^2\tau)}}\right)\left[1 \pm \exp(\frac{(z-z_0)}{\sqrt{2(1-c^2\tau)}})\right]^{-1}}{\left[\mu\left(1 \pm \exp(\frac{(z-z_0)}{\sqrt{2(1-c^2\tau)}})\right) - 1\right]}. \tag{54}$$

It is clear from Fig. 2 that when $|\mu|$ runs from zero to infinity, the above parametric solution goes from the trivial solution $u = 0$ to the the particular solution $u = u_1$.

4. Conclusions

We applied the factorization scheme to the generalized Telegraph Fisher's equation and exact particular solution have been found. The exact particular solution for the generalized Fisher's equation was obtained as a particular case of the generalized Telegraph Fisher's equation. We found that the factorization technique is easier and more efficient than other methods used to find particular solutions [29].

Acknowledgements

The authors would like to thank the referee for very valuable comments that have considerably improve the paper. This work was Supported by College of Sciences-Research Center Project No. (Math/2008/02).

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