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EXACT SOLUTION FOR THE MODIFIED EQUATION FOR SPATIAL POPULATION DYNAMICS

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Abstract

We present a modification of the second order nonlinear partial differential equation that describe the dynamics of population reproduction and by using the factorization method, we find a particular exact solution for the modified equation.

1. Introduction

The dynamics of population reproduction and spatial distribution was studied in terms of a two-dimensional continuous flow model by

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Beckmann [5-8] by constructing a second order nonlinear partial differential equation [6]. Ahmed and Abdusalam have found that, delay is an important factor in studying the model proposed by Beckmann [6]. We have used the telegraph reaction diffusion to modify the equation for spatial population dynamics [1-4]. Rosu and Cornejo [10, 13] have proved that for some nonlinear second order ordinary differential equations it is a very simple task to find one particular solution once the nonlinear equation is factorized with the use of two first order differential operators. As we know that there is no exact solution for the model and its modification [4, 6]. The paper is organized as follows: In Section 2, we summarize and generalize the factorization scheme of ordinary differential equations with polynomial nonlinearities, and this leads to an easy finding of analytical solutions. In Section 3, we use the factorization method to find an explicit particular solution for our modified equation.

2. Factorization Procedure for Nonlinear Ordinary Second Order Differential Equations

Factorization of second order linear differential equations is a well established technique to find solutions in an algebraic manner [11, 12, 14]. Rosu and Cornejo found one particular solution once the nonlinear equation is factorized with the use of two first order differential operators [9, 13]. They use the method for equations of types:

$$u'' + \gamma u' + f(u) = 0, \tag{1}$$

and

$$u'' + g(u)u' + f(u) = 0, \tag{2}$$

where γ is a constant, $g(u)$ and $f(u)$ are polynomials in u.

We concentrate our work in this paper on a new type of differential equation, namely:

$$h(u)u'' + g(u)u' + f(u) = 0, \tag{3}$$

where $'$ means the derivative $D = \frac{d}{dz}$, $h(u)$, $g(u)$ and $f(u)$ are polynomials in u.

Now, equation (3) can be factorized as

$$[h(u)D - \varphi_2(u)][D - \varphi_1(u)]u = 0, \tag{4}$$

which leads to the equation

$$h(u)u'' - h(u)\frac{d\varphi_1}{du}uu' - h(u)\varphi_1 u' - \varphi_2 u' + \varphi_1\varphi_2 u = 0, \tag{5}$$

or

$$h(u)u'' - \left(h(u)\varphi_1 + \varphi_2 + h(u)\frac{d\varphi_1}{du}u\right)u' + \varphi_1\varphi_2 u = 0. \tag{6}$$

Comparing (6) and (3), we find

$$g(u) = -\left(h(u)\varphi_1 + \varphi_2 + h(u)\frac{d\varphi_1}{du}u\right), \tag{7}$$

and

$$f(u) = \varphi_1\varphi_2 u. \tag{8}$$

If $f(u)$ is a polynomial function, then $g(u)$ will have the same order as the bigger of the factorizing functions $\varphi_1(u)$ and $\varphi_2(u)$, and will also be a function of the constant parameters provided by the function $f(u)$.

3. Modified Equation for Spatial Population Dynamics

In this section, we will obtain an exact particular solution for our modified equation for spatial population dynamics [4] by using the method presented in Section 2.

The modified equation for spatial population dynamics is given by [4]:

$$\tau u_{tt} + \left(1 - \tau\frac{df}{du}\right)u_t = -m(b - 2u)u_{xx} + f(u),$$

$$f(u) = u(a - bu + u^2), \tag{9}$$

where b, a, m are positive constants, u is the population density and τ is a time constant. It is clear that when $\tau = 0$, equation (9) reduces to the Beckmann [5-8] second order nonlinear partial differential equation [6].

Using the coordinate transformation $z = x - ct$ (c is the propagation speed) in equation (9), we obtain the following nonlinear ordinary differential equation

$$[b_0 + b_1 u]u'' + [a_0 + a_1 u + a_2 u^2]u' + f(u) = 0, \tag{10}$$

where

$$b_0 = -(mb + \tau c^2),\ b_1 = 2m,\ a_0 = c(1 - \tau a),\ a_1 = -2\tau cb,\ a_2 = -3c\tau. \tag{11}$$

The standard form of equation (10) is

$$h(u)u'' + g(u)u' + f(u) = 0, \tag{12}$$

where $h(u)$, $g(u)$ and $f(u)$ are polynomials and given as

$$h(u) = b_0 + b_1 u, \quad g(u) = a_0 + a_1 u + a_2 u^2, \quad f(u) = u(a - bu + u^2). \tag{13}$$

Equation (12) can be written using operator notations in the form

$$[h(u)D^2 + g(u)D + F(u)]u = 0, \tag{14}$$

where $F(u) = \dfrac{f(u)}{u}$.

The factorization of (14) leads to

$$[h(u)D - \varphi_2(u)][D - \varphi_1(u)]u = 0, \tag{15}$$

and then,

$$h(u)u'' + \left[- \varphi_2 - h(u)\varphi_1 - h(u)\frac{d\varphi_1}{du} u\right]u' + \varphi_1\varphi_2 u = 0. \tag{16}$$

Comparing (16) and (14), we obtain the conditions on φ_1 and φ_2 as

$$- \varphi_2 - h(u)\varphi_1 - h(u)\frac{d\varphi_1}{du} u = g(u), \quad \varphi_1\varphi_2 = F(u). \tag{17}$$

Therefore

$$\varphi_1\varphi_2 = a - bu + u^2 = (u - \alpha_1)(u - \alpha_2).$$

Now, choosing φ_1 and φ_2 such that where λ is an arbitrary constant which must be determined and

$$\alpha_1^{\pm} = \frac{b \pm \sqrt{b^2 - 4a}}{2}, \quad \alpha_2^{\pm} = b - \frac{b \pm \sqrt{b^2 - 4a}}{2}, \quad b^2 \geq 4a. \tag{19}$$

The first condition in equation (17) leads to

$$-\frac{1}{\lambda}(u - \alpha_2) - (b_0 + b_1 u)[\lambda(u - \alpha_1)] - \lambda(b_0 + b_1 u)u = a_0 + a_1 u + a_2 u^2,$$

or

$$\left(\frac{\alpha_2}{\lambda} + \lambda\alpha_1 b_0\right) + \left(\alpha_1 b_1 \lambda - 2\lambda b_0 - \frac{1}{\lambda}\right)u + (-2\lambda b_1)u^2 = a_0 + a_1 u + a_2 u^2,$$

then

$$\left(\frac{\alpha_2}{\lambda} + \lambda\alpha_1 b_0\right) = a_0, \quad \left(\alpha_1 b_1 \lambda - 2\lambda b_0 - \frac{1}{\lambda}\right) = a_1, \quad (-2\lambda b_1) = a_2. \tag{20}$$

From (19) and (20), we find

$$\lambda^{\pm} = \frac{c(1 - a\tau) \pm \sqrt{c\left(c + 4mab - 2ac\tau - 4ac\tau + a^2 c\tau^2\right)}}{2\alpha_1(c^2\tau - mbc)}, \tag{21}$$

and equation (15) takes the form

$$\left[h(u)D - \frac{1}{\lambda}(u - \alpha_1)\right][D - \lambda(u - \alpha_1)]u = 0. \tag{22}$$

The compatible first order equation is

$$u' - \lambda(u - \alpha_1)u = 0. \tag{23}$$

Integration of (23) and use of (19, 21) give the following particular solution of (9):

$$u^{\pm}(z) = \alpha_1^{\pm}\left[1 + \tanh\left[\frac{\lambda^{\pm}\alpha_1^{\pm}}{2}(z - z_0)\right]\right], \tag{24}$$

where z_0 is the integration constant.

In Figures 1 and 2 we show a plot of solutions (24) for different values of τ, namely $\tau = 0, 2, 2.5$ at $t = 1$.

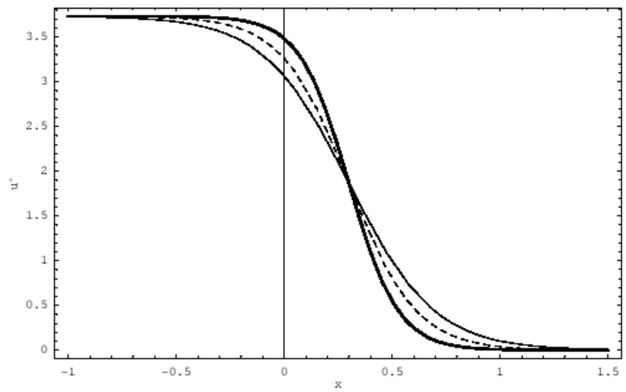

Figure 1. The solution $u^{\pm}(x, t)$ of (24) for different values of τ, namely, the normal, dashed and solid graphs represent the solutions for $\tau = 0, 2, 2.5$ respectively at $t = 1, c = 0.3, z_0 = 0$, $b = 4, a = 1$ and $m = 0.2$.

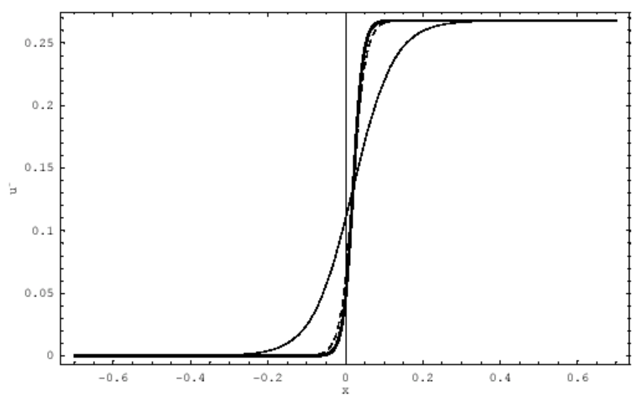

Figure 2. The solution $u^{-}(x, t)$ of (24) for different values of τ, namely, the normal, dashed and solid graphs represent the solutions for $\tau = 0, 2, 2.3$ respectively at $t = 0.1, c = 0.2, z_0 = 0, b = 4, a = 1$ and $m = 0.2$.

4. Conclusions

In this paper, the factorization method that was proposed by Rosu and Cornejo [13] has been extended to a more general form of second order nonlinear partial differential equation. The general form was applied to our modified equation for the spatial population dynamics. An exact particular solution has been obtained for the equation. An exact solution for Martian J. Beckmann equation was obtained as a special case of our solution.

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