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Bayesian and Non-Bayesian Estimation for the Bivariate Inverse Weibull Distribution Under Progressive Type-II Censoring

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Abstract
Recently, bivariate inverse Weibull distribution was derived; many of its properties have been discussed. Progressive Type-II censoring for bivariate inverse Weibull distribution has been proposed. The problem of estimating the unknown parameters of this distribution in the presence of progressive Type-II censoring by both Maximum likelihood and Bayesian estimation methods is considered in this paper. Moreover, asymptotic and bootstrap confidence intervals for the model parameters are obtained. Simulation study and a real data set are presented to illustrate the proposed procedure.

Keywords Bivariate inverse Weibull distribution · Maximum likelihood estimation · Prior distribution · Bayesian estimation · Progressive Type-II censoring · Bootstrap confidence Intervals

1 Introduction

Many times, the life failure data of interest is bivariate in nature. Any study on twins or on failure data recoded twice on the same system naturally leads to bivariate data. For example, Hougaard et al. [1] studied data on life length of Danish twins and Lin et al. [2] considered a data of colon cancer and the time from treatment to death. Paired data could consist of blindness in the left right eye, failure time of the left right kidney or age at death of parent/child in a genetic study. Eliwa and El-Morshedy [3] proposed

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the bivariate Gumbel-G family, and they discussed Bayesian and maximum likelihood techniques to estimate parameters of this model. Almetwally et al. [4] introduced Farlie–Gumbel–Morgenstern (FGM) bivariate Weibull distribution, and they discussed maximum likelihood, inference function for margins and semi-parametric methods to estimate parameters of this model. Parameter estimation of bivariate Fréchet distribution based on Farlie–Gumbel–Morgenstern and Ali–Mikhail–Haq copulas has been introduced by Almetwally and Muhammed [5].

Yousaf et al. [6] discussed Bayesian and classical inferences for the Chen distribution assuming upper record values. Sultana et al. [7] investigated the estimation problems of the unknown parameters of the Kumaraswamy distribution under type I progressive hybrid censoring. El-Sherpieny et al. [8] obtained progressive Type-II hybrid censored samples based on maximum product spacing and maximum likelihood estimation method for power Lomax distribution. Almetwally et al. [9] discussed adaptive type-II progressive censoring schemes of maximum product spacing for Weibull distribution. Tien [10] defined and detailed the concept of a serve good, which can be thought of as a physical good or product enveloped by a services-oriented layer that makes the good smarter or more adaptable and customizable for a particular use.

Recently, Muhammed [11] introduced a new bivariate inverse Weibull (BIW) distribution, whose marginals were inverse Weibull (IW) distribution. This new BIW distribution was obtained using a method similar to that used to obtain the Marshall–Olkin bivariate exponential model [12]. The proposed BIW distribution was constructed from three independent IW distributions using a maximization process. This new distribution is a singular distribution, and it can be used quit conveniently if there are ties in the data, and it often used to fit paired data in survival studies where there is a possibility of simultaneous occurrence of both the events. Muhammed [11] provided the following interpretations for the BIW model

Competing risks model Assume a system has two components, labeled 1 and 2, and the survival time of component i is denoted by $X_i, i = 1, 2$. It is considered that there are three independent causes of failures, which may affect the system. Only component 1 can fail due to cause 1, and similarly only component 2 can fail due to cause 2, while both the components can fail at the same time due to cause 3. Let U_i be the lifetime of cause $i, i = 1,\ 2,\ 3$. If U_1, U_2 and U_3 follow an IW distribution, then (X_1, X_2) follow the BIW model.

Shock model Suppose there are three independent sources of shocks; say 1, 2 and 3. Suppose these shocks are affecting a system with two components, say 1 and 2. It is assumed that the shock from source 1 reaches the system and destroys component 1 immediately, the shock from source 2 reaches the system and destroys component 2 immediately, while if the shock from source 3 hits the system it destroys both components immediately. Let U_i denote the inter-interval times between the shocks in source $i, i = 1, 2$, which follow the distribution IW. If X_1, X_2 denote the survival times of the components, then (X_1, X_2) follows the BIW model.

Stress model Suppose a system has two components. Each component is subject to individual independent stress say U_1 and U_2 respectively. The system has an overall stress U_3 which has been transmitted to both the components equally, independent of their individual stresses. Therefore, the observed stress at the two components are $X_1 = \max(U_1, U_3)$ and $X_2 = \max(U_2, U_3)$ respectively.

Maintenance model Suppose a system has two components and it is assumed that each component has been maintained independently and also there is an overall maintenance. Due to component maintenance, suppose the lifetime of the individual components is increased by U_i amount and because of the overall maintenance, the lifetime of each component is increased by U_3 amount. Therefore, the increased lifetimes of the two components are $X_1 = \max(U_1, U_3)$ and $X_2 = \max(U_2, U_3)$ respectively.

There are several mechanisms that can lead to censored data. So-called Type-II censoring that describes the experiment would be continued until a fixed proportion of items have failed. For a bivariate population censoring may be effective on the ordered variable only as example; densities of several metals alloy specimens are measured, and each specimens are set in operation simultaneously, the population parameters may be estimated sequentially as a specimens fail, but at each stage the available sample will be censored with respect to the order variate, time to failure, while the associated density is completely known. A generalization of this Type-II censoring is called progressive Type-II censoring, which arises as follows: of the n items placed on a life test, suppose R_1. functioning items are randomly removed from the test right after the first failure. Similarly, immediately after observing the second failure, R_2 items are randomly removed from the remaining $n - R_1 - 2$ items on the test, and so on until each item is taken care of either due to its failure or due to its removal from the test. The data obtained in this manner are said to be progressively Type-II censored data.

This paper considers bivariate Marshall–Olkin family and derives the likelihood function under progressive Type-II censoring in general. The derived likelihood function is applied on the bivariate inverse Weibull distribution. The paper is organized as follows. In Sect. 2, the Marshall–Olkin bivariate IW is introduced. Section 3 introduces point and interval estimation of the model parameters under progressive Type-II censoring. Bayesian Inference is presented in Sect. 4. In Sects. 5 and 6 respectively, a simulation study and data analysis are described. Finally, we conclude the paper in Sect. 7.

2 Model Assumption and Data Description

In this section, we discussed the bivariate inverse Weibull model and progressive censored samples for bivariate model.

2.1 Bivariate Inverse Weibull Model

Recently, Muhammed [11] defined the Bivariate inverse Weibull Distribution as following:

Suppose U_1, U_2 and U_3 are three independent random variables, and

$$U_i \sim \text{IW}(\lambda_i, \alpha) \text{ for } i = 1, 2, 3,$$

where IW(λ, α) denotes an inverse Weibull distribution with the shape parameter α and scale parameter λ. The cdf and pdf for the inverted Weibull random variable respectively, are

$$F_{IW}(x;\lambda,\alpha) = e^{-\lambda x^{-\alpha}}, \quad f_{IW}(x;\lambda,\alpha) = \alpha\lambda x^{-\alpha-1}e^{-\lambda x^{-\alpha}}, x > 0; \lambda, \alpha > 0.$$

Now, define $X_1 = \max(U_1, U_3)$ and $X_2 = \max(U_2, U_3)$ then it is said that the bivariate vector (X_1, X_2) has bivariate inverse Weibull distribution with parameters $(\lambda_1, \lambda_2, \lambda_3, \alpha)$ denoted by $BIW(\lambda_1, \lambda_2, \lambda_3, \alpha)$.

It should be noted that the random variables U_1, U_2 and U_3 have a common shape parameter. This ensure that the marginal distributions of X_1 and X_2 are $IW(\lambda_1 + \lambda_3, \alpha)$ and $(\lambda_2 + \lambda_3, \alpha)$ respectively, furthermore the distribution of $\max(X_1, X_2)$ is IW $(\lambda_1 + \lambda_2 + \lambda_3, \alpha)$ In addition, when $\lambda_3 = 0$ the two random variables X_1 and X_2 will be independent, hence λ_3 can be considered as a correlation control parameter. Then, the joint cdf of (X_1, X_2) is given as follows

$$F_{BIW}(x_1, x_2) = \prod_{i=1}^{3} F_{IW}(x_i; \lambda_i, \alpha)$$

where $x_3 = \min(x_1, x_2)$ Then,

$$F_{BIW}(x_1, x_2) = \begin{cases} F_1(x_1, x_2) & if \quad 0 < x_1 < x_2 < \infty \\ F_2(x_1, x_2) & if \quad 0 < x_2 < x_1 < \infty \\ F_3(x) & if \quad 0 < x_1 = x_2 = x < \infty, \end{cases} \tag{1}$$

where

$$\begin{aligned} F_1(x_1, x_2) &= F_{IW}(x_1; \lambda_1 + \lambda_3, \alpha) F_{IW}(x_2; \lambda_2, \alpha) \\ F_2(x_1, x_2) &= F_{IW}(x_1; \lambda_1, \alpha) F_{IW}(x_2; \lambda_2 + \lambda_3, \alpha) \\ \text{and } F_3(x) &= F_{IW}(x; \lambda_1 + \lambda_2 + \lambda_3, \alpha). \end{aligned}$$

The joint pdf of (X_1, X_2) is given as follows

$$f_{BIW}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & if \quad 0 < x_1 < x_2 < \infty \\ f_2(x_1, x_2) & if \quad 0 < x_2 < x_1 < \infty \\ f_3(x) & if \quad 0 < x_1 = x_2 = x < \infty, \end{cases} \tag{2}$$

where

$$\begin{aligned} f_1(x_1, x_2) &= f_{IW}(x_1; \lambda_1 + \lambda_3, \alpha) f_{IW}(x_2; \lambda_2, \alpha) \\ f_2(x_1, x_2) &= f_{IW}(x_1; \lambda_1, \alpha) f_{IW}(x_2; \lambda_2 + \lambda_3, \alpha) \\ \text{and } f_3(x) &= \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} f_{IW}(x; \lambda_1 + \lambda_2 + \lambda_3, \alpha) \end{aligned}$$

Proof The expressions for $f_1(.,.)$ and $f_2(.,.)$ can be obtained simply by taking $\frac{\partial^2}{\partial x_1 \partial x_2} F_{X_1,X_2}(x_1,x_2)$ for $x_1 < x_2$ and $x_2 < x_1$ respectively. But $f_3(.)$ cannot be obtained in the same way. Using the fact that

$$\int_0^\infty \int_0^{x_2} f_1(x_1,x_2)dx_1dx_2 + \int_0^\infty \int_0^{x_1} f_2(x_1,x_2)dx_2dx_1 + \int_0^\infty f_3(x)dx = 1,$$

$$\int_0^\infty \int_0^{x_2} f_1(x_1,x_2)dx_1dx_2 = \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3} \text{ and } \int_0^\infty \int_0^{x_1} f_2(x_1,x_2)dx_2dx_1 = \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}$$

Hence, we obtain $\int_0^\infty f_3(x)dx = \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}$. □

Therefore, the results follow.

2.2 Data Description

The likelihood function based on a Progressive Type-II censoring for bivariate distributions has introduced firstly by Balakrishnan and Kim [13]. Along the same line, Almetwally [14] and El-Sherpieny et al. [15] have used the likelihood function based on a Progressive Type-II censoring for FGM bivariate Weibull distribution. In this section we will generalize the likelihood function for bivariate Marshal–Olkin distributions.

Suppose that there are n independent pairs of components $(X_{1i}, X_{2i}), i = 1, \ldots, n$ under experiment, and each of them has $BIW(\lambda_1, \lambda_2, \lambda_3, \alpha)$ lifetime distribution.

Based on a Type-II progressive censoring scheme $(n, m, R_1, \ldots, R_m)$, we have the following observations;

$$\big[(x_{11:m:n}, x_{[21:m:n]}), (x_{12:m:n}, x_{[22:m:n]}\big), \ldots, (x_{1m:m:n}, x_{[2m:m:n]})],$$

where $X_{1i:m:n}$ be the ith order statistic of X_1 and $X_{[2i:m:n]}$ be its concomitant of X_2, $i = 1, \ldots, m$.

Then the joint probability of $(X_{1i:m:n}, X_{[2i:m:n]}), i = 1, \ldots, m$ is given by

$$\begin{aligned} L(\theta) &= \prod_{i=1}^m f_{(X_{1i:m:n}, X_{[2i:m:n]})}(x_{1i:m:n}, x_{[2i:m:n]})\big[S_{X_1}(x_{1i:m:n})\big]^{R_i} \\ &= C\prod_{i=1}^m \big[f_1(x_{1i:m:n}, x_{[2i:m:n]})\big]^{\delta_{1i}} \big[f_2(x_{1i:m:n}, x_{[2i:m:n]})\big]^{\delta_{2i}} \\ &\quad \big[f_3(x_{1i:m:n}, x_{[2i:m:n]})\big]^{\delta_{3i}} \big[S_{X_1}(x_{1i:m:n})\big]^{R_i} \end{aligned} \quad (3)$$

where $C = n(n - R_1 - 1)\ldots(n - R_1 - R_2 - \ldots - m + 1)$, $f_1(.)$, $f_2(.)$, $f_3(.)$ are as given in (2) and $S_{X_1}(.)$ is the survival function of X_1. Also δ_{ji}, $j = 1, 2, 3$ are event indicators such that

$$\delta_{1i} = \begin{cases} 1, & X_{1i:m:n} < X_{[2i:m:n]} \\ 0, & otherwise \end{cases},$$

$$\delta_{2i} = \begin{cases} 1, & X_{1i:m:n} > X_{[2i:m:n]} \\ 0, & otherwise \end{cases},$$

$$\delta_{3i} = \begin{cases} 1, & X_{1i:m:n} = X_{[2i:m:n]} \\ 0, & otherwise \end{cases},$$

That produce

$$m_1 = \sum_{i=1}^{m} \delta_{1i},\ m_2 = \sum_{i=1}^{m} \delta_{2i} \text{ and } m_3 = \sum_{i=1}^{m} \delta_{3i} \text{ such that } m = m_1 + m_2 + m_3.$$

Throughout this paper it is assumed that $n, m, R_1, \ldots, R_m$ are predetermined and fixed.

Follows are some special cases from progressive Type-II censoring that applied on bivariate Marshal–Olkin family of Distributions:

(I) *Complete Case* If $R_1 = \ldots = R_m = 0$ and $n = m$

Then (3) reduced to

$$L(\theta) = \prod_{i=1}^{n} [f_1(x_{1i}, x_{2i})]^{\delta_{1i}} [f_2(x_{1i}, x_{2i})]^{\delta_{2i}} [f_3(x_{1i}, x_{2i})]^{\delta_{3i}}$$

where

$$n_1 = \sum_{i=1}^{n} \delta_{1i},\ n_2 = \sum_{i=1}^{n} \delta_{2i} \text{ and } n_3 = \sum_{i=1}^{n} \delta_{3i} \text{ such that } n = n_1 + n_2 + n_3.$$

(II) *Type-II censoring case* if $R_1 = R_2 = \ldots = R_{\mathrm{m}-1} = 0,\ and\ R_{\mathrm{m}} = n - m$

Then (3) reduced to

$$L(\theta) = C\big[S_{X_1}(x_{m:m:n})\big]^{n-m} \prod_{i=1}^{m} \big[f_1(x_{1i:m:n}, x_{[2i:m:n]})\big]^{\delta_{1i}} \big[f_2(x_{1i:m:n}, x_{[2i:m:n]})\big]^{\delta_{2i}} \big[f_3(x_{1i:m:n}, x_{[2i:m:n]})\big]^{\delta_{3i}}.$$

3 Maximum Likelihood Estimation

In this section, we discussed the maximum likelihood estimation of parameter for BIW based on progressively Type-II censored sample. Assume $(x_{11:m:n}, x_{[21:m:n]}) < (x_{12:m:n}, x_{[22:m:n]}) < \ldots (x_{1m:m:n}, x_{[2m:m:n]})$ denote progressively Type-II censored sample from $BIW(\lambda_1, \lambda_2, \lambda_3, \alpha)$ distribution whose pdf and cdf are given in (2) and (3), for simplicity assume $x_{1i} = x_{i:m:n}$ and $x_{2i} = x_{[2i:m:n]}$.

The log-likelihood function $l(\theta) = LogL(\theta)$ is then given as

$$
\begin{aligned}
l(\theta) \propto\ & m_1 \log \lambda_2 + m_1 \log(\lambda_1 + \lambda_2) + m_2 \log \lambda_1 + m_2 \log(\lambda_2 + \lambda_3) + m_3 \log \lambda_3 \\
& + m_3 \log(\lambda_1 + \lambda_2 + \lambda_3) + (2m_1 + 2m_2 + m_3) \log \alpha - (\lambda_1 + \lambda_3) \sum_{i=1}^{m} \delta_{1i} x_{1i}^{-\alpha} \\
& - (\lambda_2 + \lambda_3) \sum_{i=1}^{m} \delta_{2i} x_{2i}^{-\alpha} - (\lambda_1 + \lambda_2 + \lambda_3) \sum_{i=1}^{m} \delta_{3i} x_{1i}^{-\alpha} - \lambda_1 \sum_{i=1}^{m} \delta_{2i} x_{1i}^{-\alpha} \\
& - \lambda_2 \sum_{i=1}^{m} \delta_{1i} x_{2i}^{-\alpha} - (\alpha + 1) \left[\sum_{i=1}^{m} (\delta_{1i} + \delta_{2i}) log x_{1i} x_{2i} + \sum_{i=1}^{m} \delta_{3i} log x_{1i} \right] \\
& + \sum_{i=1}^{n} R_i \log\left[1 - e^{-(\lambda_1+\lambda_3) x_{1i}^{-\alpha}} \right].
\end{aligned} \tag{4}
$$

where $\theta = (\lambda_1, \lambda_2, \lambda_3, \alpha)$.

Calculating the first partial derivatives of (4) with respect to $\lambda_1, \lambda_2, \lambda_3$ and α and equating each to zero, we get the likelihood equations as following

$$
\begin{aligned}
& \frac{m_2}{\hat{\lambda}_1} + \frac{m_1}{\hat{\lambda}_1 + \hat{\lambda}_3} + \frac{m_3}{\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3} - A(\hat{\alpha}) + B\left(\hat{\lambda}_1, \hat{\lambda}_3, \hat{\alpha}\right) = 0, \\
& \frac{m_1}{\hat{\lambda}_2} + \frac{m_2}{\hat{\lambda}_2 + \hat{\lambda}_3} + \frac{m_3}{\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3} - C(\hat{\alpha}) = 0, \\
& \frac{m_3}{\hat{\lambda}_3} + \frac{m_1}{\hat{\lambda}_1 + \hat{\lambda}_3} + \frac{m_2}{\hat{\lambda}_2 + \hat{\lambda}_3} + \frac{m_3}{\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3} - D(\hat{\alpha}) + B\left(\hat{\lambda}_1, \hat{\lambda}_3, \hat{\alpha}\right) = 0, \\
& \frac{2m_1 + 2m_2 + m_3}{\hat{\alpha}} - E\left(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\alpha}\right) - G\left(\hat{\lambda}_1, \hat{\lambda}_3, \hat{\alpha}\right) + H = 0.
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
A(\alpha) &= \sum_{i=1}^{m} \delta_{1i} x_{1i}^{-\alpha} + \sum_{i=1}^{m} \delta_{2i} x_{1i}^{-\alpha} + \sum_{i=1}^{m} \delta_{3i} x_{1i}^{-\alpha}, \\
B\left(\hat{\lambda}_1, \hat{\lambda}_3, \hat{\alpha}\right) &= \sum_{i=1}^{m} R_i \frac{x_{1i}^{-\hat{\alpha}} e^{-\left(\hat{\lambda}_1+\hat{\lambda}_3\right) x_{1i}^{-\hat{\alpha}}}}{1 - e^{-\left(\hat{\lambda}_1+\hat{\lambda}_3\right) x_{1i}^{-\hat{\alpha}}}}, \\
C(\alpha) &= \sum_{i=1}^{m} \delta_{3i} x_{1i}^{-\alpha} + \sum_{i=1}^{m} \delta_{1i} x_{2i}^{-\alpha} + \sum_{i=1}^{m} \delta_{2i} x_{2i}^{-\alpha}, \\
D(\alpha) &= \sum_{i=1}^{m} \delta_{1i} x_{1i}^{-\alpha} + \sum_{i=1}^{m} \delta_{2i} x_{2i}^{-\alpha} + \sum_{i=1}^{m} \delta_{3i} x_{1i}^{-\alpha},
\end{aligned}
$$

$$E\left(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\alpha}\right) = (\hat{\lambda}_1 + \hat{\lambda}_3)\sum_{i=1}^{m}\delta_{1i}x_{1i}^{-\hat{\alpha}}logx_{1i} + \hat{\lambda}_1\sum_{i=1}^{m}\delta_{2i}x_{1i}^{-\hat{\alpha}}logx_{1i}$$
$$+ \hat{\lambda}_2\sum_{i=1}^{m}\delta_{1i}x_{2i}^{-\hat{\alpha}}logx_{1i} + (\hat{\lambda}_2 + \hat{\lambda}_3)\sum_{i=1}^{m}\delta_{2i}x_{2i}^{-\hat{\alpha}}logx_{1i}$$
$$+ (\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3)\sum_{i=1}^{m}\delta_{1i}x_{1i}^{(-\hat{\alpha})}logx_1 i,$$
$$G\left(\hat{\lambda}_1, \hat{\lambda}_3, \hat{\alpha}\right) = \sum_{i=1}^{m} R_i\frac{(\hat{\lambda}_1 + \hat{\lambda}_3)x_{1i}^{-\hat{\alpha}}e^{-(\hat{\lambda}_1+\hat{\lambda}_3)x_{1i}^{-\hat{\alpha}}}logx_{1i}}{1 - e^{-(\hat{\lambda}_1+\hat{\lambda}_3)x_{1i}^{-\hat{\alpha}}}},$$
$$\text{and } H = \sum_{i=1}^{m}(\delta_{1i} + \delta_{2i})logx_{1i}x_{2i} + \sum_{i=1}^{m}\delta_{3i}logx_{1i}.$$

To obtain a solution for a set of Eq. (5), we can use the Newton–Raphson algorithm to solve the equations simultaneously, to get the desired MLEs of $\lambda_1, \lambda_2, \lambda_3$ and α.

3.1 Asymptotic Confidence Intervals

To set confidence intervals for the unknown parameters use the asymptotic normal distribution of the MLEs. In relation to asymptotic variance–covariance matrix of the MLEs of the parameters, it can be approximated by inverting the Fisher information matrix F, where it is consist of the negative derivatives of the natural logarithm of the likelihood function evaluated at $(\hat{\lambda}_1, \hat{\lambda}_{2,} \hat{\lambda}_3, \hat{\alpha})$ the MLEs of the parameters.

Now, from the log-likelihood function in (4), we have

$$I_{11} = \frac{\partial^2 l}{\partial\lambda_1^2} = \frac{-m_2}{\lambda_1^2} - \frac{m_1}{(\lambda_1 + \lambda_3)^2} - \frac{m_3}{(\lambda_1 + \lambda_2 + \lambda_3)^2} - \varphi(\lambda_1, \lambda_3, \alpha),$$
$$I_{12} = \frac{\partial^2 l}{\partial\lambda_1\partial\lambda_2} = -\frac{m_3}{(\lambda_1 + \lambda_2 + \lambda_3)^2},$$
$$I_{13} = \frac{\partial^2 l}{\partial\lambda_1\partial\lambda_3} = -\frac{m_1}{(\lambda_1 + \lambda_3)^2} - \frac{m_3}{(\lambda_1 + \lambda_2 + \lambda_3)^2},$$
$$I_{14} = \frac{\partial^2 l}{\partial\lambda_1\partial\alpha} = \tau(\alpha) - \varsigma(\lambda_1, \lambda_3, \alpha)$$
$$I_{22} = \frac{\partial^2 l}{\partial\lambda_2^2} = \frac{-m_1}{\lambda_2^2} - \frac{m_2}{(\lambda_2 + \lambda_3)^2} - \frac{m_3}{(\lambda_1 + \lambda_2 + \lambda_3)^2},$$
$$I_{33} = \frac{\partial^2 l}{\partial\lambda_3^2} = -\frac{m_1}{(\lambda_1 + \lambda_3)^2} - \frac{m_2}{(\lambda_2 + \lambda_3)^2} - \frac{m_3}{\lambda_3^2} - \frac{m_3}{(\lambda_1 + \lambda_2 + \lambda_3)^2},$$
$$I_{44} = \frac{\partial^2 l}{\partial\alpha^2} = -\frac{2m_1 + 2m_2 + m_3}{\alpha^2} + \psi(\lambda_1, \lambda_2, \lambda_3, \alpha) + \omega(\lambda_1, \lambda_3, \alpha),$$

$$\text{and } I_{24} = \frac{\partial^2 l}{\partial\lambda_2\partial\alpha} = 0 = \frac{\partial^2 l}{\partial\lambda_3\partial\alpha} = I_{34},$$

where

$$\varphi(\lambda_1, \lambda_3, \alpha) = \sum_{i=1}^{m} R_i \frac{x_{1i}^{-2\alpha} e^{-(\lambda_1+\lambda_3)x_{1i}^{-\alpha}}}{1 - e^{-(\lambda_1+\lambda_3)x_{1i}^{-\alpha}}},$$

$$\psi(\lambda_1, \lambda_2, \lambda_3, \alpha) = (\lambda_1 + \lambda_3)\sum_{i=1}^{m} \delta_{1i} x_{1i}^{-\alpha}(logx_{1i})^2 + \lambda_1 \sum_{i=1}^{m} \delta_{2i} x_{1i}^{-\alpha}(logx_{1i})^2$$
$$+ \lambda_2 \sum_{i=1}^{m} \delta_{1i} x_{2i}^{-\alpha}(logx_{1i})^2 + (\lambda_2 + \lambda_3)\sum_{i=1}^{m} \delta_{2i} x_{2i}^{-\alpha}(logx_{1i})^2$$
$$+ (\lambda_1 + \lambda_2 + \lambda_3)\sum_{i=1}^{m} \delta_{1i} x_{1i}^{-\alpha}(logx_{1i})^2,$$

$$\omega(\lambda_1, \lambda_3, \alpha) = (\lambda_1 + \lambda_3)\sum_{i=1}^{m} R_i \frac{x_{1i}^{-\alpha}(logx_{1i})^2 e^{-(\lambda_1+\lambda_3)x_{1i}^{-\alpha}}\left(1 - e^{-(\lambda_1+\lambda_3)x_{1i}^{-\alpha}}\right)\left(x_{1i}^{-\alpha} - 1\right)}{\left(1 - e^{-(\lambda_1+\lambda_3)x_{1i}^{-\alpha}}\right)^2}$$
$$+ (\lambda_1 + \lambda_3)\sum_{i=1}^{m} R_i \frac{x_{1i}^{-\alpha} e^{-(\lambda_1+\lambda_3)x_{1i}^{-\alpha}}}{\left(1 - e^{-(\lambda_1+\lambda_3)x_{1i}^{-\alpha}}\right)^2},$$

$$\tau(\alpha) = \sum_{i=1}^{m} \delta_{1i} x_{1i}^{-\alpha} logx_{1i} + \sum_{i=1}^{m} \delta_{2i} x_{1i}^{-\alpha} logx_{1i} + \sum_{i=1}^{m} \delta_{3i} x_{1i}^{-\alpha} logx_{1i},$$

$$\varsigma(\lambda_1, \lambda_3, \alpha) = (\lambda_1 + \lambda_3)\sum_{i=1}^{m} R_i \frac{x_{1i}^{-2\alpha} logx_{1i} e^{-2(\lambda_1+\lambda_3)x_{1i}^{-\alpha}}}{\left(1 - e^{-(\lambda_1+\lambda_3)x_{1i}^{-\alpha}}\right)^2}$$
$$+ \sum_{i=1}^{m} R_i \frac{x_{1i}^{-\alpha} logx_{1i} e^{-(\lambda_1+\lambda_3)x_{1i}^{-\alpha}}[(\lambda_1 + \lambda_3)x_{1i} - 1]\left(1 - e^{-(\lambda_1+\lambda_3)x_{1i}^{-\alpha}}\right)}{\left(1 - e^{-(\lambda_1+\lambda_3)x_{1i}^{-\alpha}}\right)^2}.$$

Hence, the asymptotic variance–covariance matrix can be given as follows

$$F^{-1} = \begin{pmatrix} -I_{11} & -I_{12} & -I_{13} & -I_{14} \\ -I_{21} & -I_{22} & -I_{23} & -I_{24} \\ -I_{31} & -I_{32} & -I_{33} & -I_{34} \\ -I_{41} & -I_{42} & -I_{43} & -I_{44} \end{pmatrix}^{-1}_{\Big|\left(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\alpha}\right)}.$$

Now, a $100(1-\gamma)\%$ approximate confidence intervals for $\lambda_1, \lambda_2, \lambda_3$ and α are given respectively, as following

$$\hat{\lambda}_1 \pm z_{\frac{\gamma}{2}}\sqrt{v_{11}},\ \hat{\lambda}_2 \pm z_{\frac{\gamma}{2}}\sqrt{v_{22}},\ \hat{\lambda}_3 \pm z_{\frac{\gamma}{2}}\sqrt{v_{33}} \text{ and } \hat{\alpha} \pm z_{\frac{\gamma}{2}}\sqrt{v_{44}}. \quad (6)$$

where v_{11}, v_{22}, v_{33} and v_{44} are the elements on the main diagonal of the variance–covariance matrix F^{-1} and $z_{\frac{\gamma}{2}}$ is the percentile of the standard normal distribution with right tail $\frac{\gamma}{2}$.

3.2 Bootstrap Confidence Intervals

In this section, we use the parametric bootstrap method to construct confidence intervals for the unknown parameters $\lambda_1, \lambda_2, \lambda_3$ and α. We introduce two parametric bootstrap methods, percentile bootstrap confidence interval (B-PCI) discussed by Efron [16] and Tibshirani and Efron [17] and bootstrap-t confidence interval (B-TCI) discussed by Hall [18] and Kreiss and Paparoditis [19]. The following steps are followed to obtain samples for both methods:

1. Obtain the MLEs $\hat{\Theta} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\alpha}))$ for the unknown parameters $\Theta = (\lambda_1, \lambda_2, \lambda_3, \alpha)$ based on the original progressively Type II censored sample
$$(x_{1i}, x_{2i}) = (x_{11:m:n}, x_{[21:m:n]}) < (x_{12:m:n}, x_{[22:m:n]}) < \ldots < (x_{1m:m:n}, x_{[2m:m:n]}).$$
2. Using $\hat{\Theta}$, generate a bootstrap sample $(x_{1i}, x_{2i})^*, i = 1, 2, \ldots m$ where $(x_{1i}, x_{2i})^* \sim BIW$ distribution.
3. As in step 1 based on $(x_{1i}, x_{2i})^*, i = 1, 2, \ldots m$ compute the bootstrap sample estimates of $\hat{\Theta}$ say, $\hat{\Theta}^*$.
4. Repeat the above steps 2 and 3 $N = 1000$ times, then we have N estimate of Θ.
5. Order the bootstrap replications of $\hat{\Theta}^*$ such that $\hat{\Theta}_1^* < \hat{\Theta}_2^* < \ldots < \hat{\Theta}_N^*$.

Percentile bootstrap confidence interval (B-PCI)

Let $G(\xi) = P\left(\hat{\Theta}^* \le \xi\right)$ be cdf of $\hat{\Theta}^*$. Define $\hat{\Theta}^* = G^{-1}(\xi)$ for given ξ. The approximate bootstrap $100(1-\gamma)\%$ confidence interval of $\hat{\Theta}^*$ is given by $(\hat{\Theta}^*_{\frac{\gamma}{2}}, \hat{\Theta}^*_{1-\frac{\gamma}{2}})$.

Bootstrap-t confidence interval (B-TCI)

In step 3 get $\hat{\Theta}^*$ and also calculate $var(\hat{\Theta}^*)$ using the observed fisher information matrix.

Compute the statistic $T_j^* = \frac{\hat{\Theta}_j^* - \hat{\Theta}}{\sqrt{var\left(\hat{\Theta}_j^*\right)}}, \; j = 1, \ldots N$.

Arrange the bootstrap replications of T^* such that $T_1^* < T_2^* < \ldots < T_N^*$.

Let $H(\xi) = P(T^* \le \xi)$ be cdf of T^*. For a given ξ define

$$\hat{\Theta}_{boot-t} = \hat{\Theta} + \sqrt{var\left(\hat{\Theta}\right)} H^{-1}(\xi).\frac{dy}{dx}$$

The approximate $100(1-\gamma)\%$ bootstrap confidence interval of $\hat{\Theta}$ will be

$$\left(\hat{\Theta}_{boot-t}\left(\frac{\gamma}{2}\right),\; \hat{\Theta}_{boot-t}\left(1-\frac{\gamma}{2}\right)\right).$$

4 Bayes Estimation

In this section we consider the Bayesian analysis for the BIW distribution under progressive Type II censoring. We obtained the Bayes estimators under the squared error loss function

4.1 Prior Assumptions

When the shape parameter α is known, we assume the same conjugate prior on λ_1, λ_2 and λ_3 as considered by Kundu and Gupta [20].

Assume λ_1, λ_2 and λ_3 are independent, and distributed as gamma as following

$$\pi_i(\lambda_i) = \frac{b^{a_i}}{\Gamma(a_i)} \lambda_i^{a_i - 1} e^{-b\lambda_i}, \quad i = 1, 2, 3, \; \lambda_i > 0.$$

The joint prior density of λ_1, λ_2 and λ_3,

$$\pi_0(\lambda_1, \lambda_2, \lambda_3) = \prod_{i=1}^{3} \frac{b^{a_i}}{\Gamma(a_i)} \lambda_i^{a_i - 1} e^{-b\lambda_i} .. \tag{7}$$

4.2 Posterior Analysis and Bayesian Inference

Assume we have a bivariate sample from BIW $(\lambda_1, \lambda_2, \lambda_3, \alpha)$ under progressive Type II censoring and it is denoted as following

$$D = \left[(x_{11:m:n}, x_{[21:m:n]}), (x_{12:m:n}, x_{[22:m:n]} \right), \ldots, (x_{1m:m:n}, x_{[2m:m:n]})].$$

Let $= m_1 + m_2 + m_3, \lambda = \lambda_1 + \lambda_2 + \lambda_3, \lambda_{13} = \lambda_1 + \lambda_3$ and $\lambda_{23} = \lambda_2 + \lambda_3$.
Then the Likelihood function given in (3) can be written as

$$\begin{aligned} L(D \backslash \Theta) &= Exp(\log L(D \backslash \Theta)) \\ L(D \backslash \Theta) &= \sum_{k=1}^{m_3} \sum_{j=1}^{m_1} \sum_{s=1}^{m_2+k} \prod_{i=1}^{m} \sum_{l=1}^{R_i} \binom{m_1}{j} \binom{m_3}{k} \binom{m_2+k}{s} \binom{R_i}{l} \lambda_1^{m_2+m_3+j-k} \lambda_2^{m_1+s} \cdot \lambda_3^{m+k-s-j} \\ & Exp(-\lambda_{13} Z_1(\alpha) - \lambda_1 Z_2(\alpha) - \lambda Z_3(\alpha) - \lambda_2 Z_4(\alpha) - \lambda_{23} Z_5(\alpha) - (\alpha+1) Z_6), \end{aligned} \tag{8}$$

where

$$Z_1(\alpha) = \sum_{i=1}^{m} \delta_{1i} x_{1i}^{-\alpha} + l x_{1i}^{-\alpha}, \; Z_2(\alpha) = \sum_{i=1}^{m} \delta_{2i} x_{1i}^{-\alpha}, \; Z_3(\alpha) = \sum_{i=1}^{m} \delta_{3i} x_{1i}^{-\alpha},$$

$$Z_4(\alpha) = \sum_{i=1}^{m} \delta_{1i} x_{2i}^{-\alpha}, \; Z_5(\alpha) = \sum_{i=1}^{m} \delta_{2i} x_{2i}^{-\alpha},$$

$$\text{and } Z_6 = \sum_{i=1}^{m} (\delta_{1i} + \delta_{2i}) log x_{1i} x_{2i} + \delta_{3i} log x_{1i}.$$

Since

$$f(D,\Theta) = \pi_0(\Theta)L(D\backslash\Theta) \text{ and } f(D) = \int f(D\backslash\Theta)d\Theta = \int \pi_0(\Theta)L(D\backslash\Theta)d\Theta.$$

Hence the joint posterior density function of $\Theta = (\lambda_1, \lambda_2, \lambda_3, \alpha)$ given the data D, denoted by $\pi_1(\Theta\backslash D)$ can be written as

$$\begin{aligned}
\pi_1(\Theta\backslash D) &= \frac{f(D,\Theta)}{f(D)} \\
\pi_1(\Theta\backslash D) &\propto \sum_{j=1}^{m_1}\sum_{s=1}^{m_2}\prod_{i=1}^{m}\sum_{l=1}^{R_i} w_{lijs}\cdot Gamma\big[\lambda_1; a_{1j}, b_1 + T_1(\alpha)\big] \\
&\quad \cdot Gamma[\lambda_2; a_{2s}, b_{2s} + T_2(\alpha)] \cdot Gamma[\lambda_3; a_{3s}, b_3 + T_3(\alpha)], \qquad (9)
\end{aligned}$$

where $w_{lijs} = \frac{C_{ljs}}{\sum_{j=1}^{m_1}\sum_{s=1}^{m_2}\prod_{i=1}^{m}\sum_{l=1}^{R_i} C_{lijs}}$, and $C_{ljks} = \binom{m_1}{j}\binom{m_2}{s}\binom{R_i}{l}\cdot\frac{\Gamma(a_{1j})}{[b_1+T_1(\alpha)]^{a_{1j}}}\cdot\frac{\Gamma(a_{2s})}{[b_{2s}+T_2(\alpha)]^{a_{2s}}}\cdot\frac{\Gamma(a_{3sj})}{[b_3+T_3(\alpha)]^{a_{3sj}}}$, $T_1(\alpha) = Z_1(\alpha) + Z_2(\alpha) + Z_3(\alpha)$, $T_2(\alpha) = Z_3(\alpha) + Z_4(\alpha) + Z_5(\alpha)$, $T_3(\alpha) = Z_1(\alpha) + Z_3(\alpha) + Z_5(\alpha)$, $a_{1j} = a_1 + m_2 + j$, $a_{2s} = a_2 + m_1 + s$, and $a_{3sj} = a_3 + m - s - j$.

Therefore, under the assumption of independence of λ_1, λ_2 and λ_3 and α is assumed to be known. It is possible to get the Bayes estimators of λ_1, λ_2 and λ_3 explicitly under the square error loss function using (9) and they will be as follows

$$\begin{aligned}
\check{\lambda}_1 &= \frac{1}{b_1 + T_1(\alpha)}\sum_{j=1}^{m_1}\sum_{s=1}^{m_2}\prod_{i=1}^{m}\sum_{l=1}^{R_i} w_{lijs}\cdot a_{1j}, \\
\check{\lambda}_2 &= \frac{1}{b_2 + T_2(\alpha)}\sum_{j=1}^{m_1}\sum_{s=1}^{m_2}\prod_{i=1}^{m}\sum_{l=1}^{R_i} w_{lijs}\cdot a_{2s},
\end{aligned}$$

and

$$\check{\lambda}_3 = \frac{1}{b_3 + T_3(\alpha)}\sum_{j=1}^{m_1}\sum_{s=1}^{m_2}\prod_{i=1}^{m}\sum_{l=1}^{R_i} w_{lijs}\cdot a_{3sj}.$$

Now, if we assume that α is unknown then the Bayes estimates for the vector of unknown parameters $\Theta = (\lambda_1, \lambda_2, \lambda_3, \alpha)$ under the square error loss function can be obtained by using Markov Chain Monte Carlo (MCMC) technique to generate samples from the posterior distributions and then compute the Bayes estimators of the individual parameters. To generate samples from the proposed family, we use the Metropolis–Hastings method (Metropolis et al. [21]) with normal proposal distribution).

5 Simulation Study

In this section; a Monte Carlo simulation is done to compare the performance of the different progressive Type-II schemes. In our study, we consider the following censoring schemes

- Scheme I: $R_1 = R_2 = \ldots = R_{m-1} = 0,\ and\ R_m = n - m.$
- Scheme II: $R_1 = n - m\ and\ R_2 = R_3 = \ldots = R_{m-1} = 0.$
- Scheme III: $R_1 = R_2 = \ldots = R_{m-1} = 1,\ and\ R_m = n - 2m - 1.$
- Scheme IV: $R_1 = n - 2m - 1\ and\ R_2 = R_3 = \ldots = R_m = 1.$

For estimating the parameters of BIW distribution using R language.

Simulation Algorithm the simulation experiments were carried out based on the following data generated form IW distributions, where U_1, U_2 and U_3 are distributed as IW with α shape parameters and λ_i scale parameter; $i = 1, 2, 3$ the values of the parameters λ_1, λ_2 and λ_3 and α is chosen as the following cases for the generated random variables:

Case 1: ($\alpha = 2, \lambda_1 = 1.4, \lambda_2 = 1.5, \lambda_3 = 1.3$).
Case 2: ($\alpha = 0.75, \lambda_1 = 1.25, \lambda_2 = 0.85, \lambda_3 = 1.5$).

For different sample size $n =$ 30, 50 and 100. The comparison is performed by calculate in the Bias, the MSE and the length of confidence interval (L.CI) for each method as following

$$Bias = (\hat{\delta} - \delta). \tag{10}$$

$$MSE = Mean(\hat{\delta} - \delta)^2. \tag{11}$$

and

$$L \cdot CI = Upper \cdot CI - Lower \cdot CI \tag{12}$$

We restricted the number of repeated-samples to 10,000. From Tables 1, 2, 3, 4, 5 and 6 we can conclude the following:

i. When the sample size increase, the MSE, Bias and CI length of the considered parameters are decreases.
ii. When then number of stages (m) increase, the value of the Bias, MSE, Length of CI are also decrease for the parameters of BIW distribution
iii. For fixed values of the sample size, Scheme II gives more an accurate result than other schemes in decreasing the MSE, Bias and length of CI
iv. The Bayesian estimates have more relative efficiency than MLE for most parameters of BIW distribution.
v. With regard to the interval estimation for parameters of BIW distribution, the Bootstrap CIs is more efficient than the traditional method ACI.
vi. The B-TCI has more efficiency than B-PCI for most parameters.

Table 1 Estimation of the parameters of BIW distribution under different schemes when $n = 30$: Case 1

m	Scheme		MLE					Bayes				
			Bias	MSE	ACI	B-TCI	B-PCI	Bias	MSE	ACI	B-TCI	B-PCI
10	I	$\hat{\alpha}$	0.2475	0.2854	1.8566	0.1863	0.1793	0.2158	0.2748	1.8820	0.0591	0.0588
		$\hat{\lambda}_1$	0.1710	0.9481	3.7595	0.1850	0.1892	0.0762	0.2312	1.8705	0.1125	0.1128
		$\hat{\lambda}_2$	0.2816	0.7747	3.2705	0.2558	0.2588	0.3906	0.5667	2.5353	0.1030	0.1004
		$\hat{\lambda}_3$	0.0909	0.8536	3.6059	0.2142	0.2193	0.2712	0.3642	2.1240	0.1133	0.1101
10	II	$\hat{\alpha}$	0.2054	0.2301	1.7003	0.1596	0.1617	0.2651	0.2340	1.5940	0.0523	0.0508
		$\hat{\lambda}_1$	0.1230	0.6013	3.0027	0.2286	0.2336	0.3087	0.4342	2.2934	0.0962	0.0967
		$\hat{\lambda}_2$	0.3850	1.3476	4.2951	0.2632	0.2517	0.3982	0.6334	2.7147	0.1358	0.1328
		$\hat{\lambda}_3$	0.0827	0.4894	2.7244	0.2101	0.2097	0.2536	0.3558	2.1270	0.0837	0.0854
10	III	$\hat{\alpha}$	0.2445	0.2823	1.8500	0.1503	0.1547	0.2534	0.2160	1.5351	0.0580	0.0586
		$\hat{\lambda}_1$	0.1324	0.9519	3.7910	0.1719	0.1679	0.1438	0.2148	1.7356	0.1173	0.1170
		$\hat{\lambda}_2$	0.3557	1.1101	3.8897	0.2650	0.2675	0.4120	0.6022	2.5908	0.1232	0.1241
		$\hat{\lambda}_3$	0.1524	0.9432	3.7617	0.1842	0.1878	0.2124	0.2723	1.8779	0.1187	0.1161
10	IV	$\hat{\alpha}$	0.2534	0.2616	1.7426	0.1254	0.1221	0.2276	0.1969	1.4984	0.0543	0.0560
		$\hat{\lambda}_1$	0.1760	0.7557	3.3389	0.1861	0.1840	0.2323	0.4017	2.3194	0.1076	0.1056
		$\hat{\lambda}_2$	0.3870	1.1980	4.0154	0.2202	0.2240	0.3117	0.5523	2.6534	0.1279	0.1307
		$\hat{\lambda}_3$	0.1068	0.6660	3.1733	0.2244	0.2270	0.2647	0.5407	2.6983	0.1017	0.1053
15	I	$\hat{\alpha}$	0.1765	0.1665	1.4427	0.1345	0.1355	0.2007	0.1614	1.3711	0.0442	0.0444
		$\hat{\lambda}_1$	0.1247	0.5180	2.7801	0.1763	0.1813	0.0603	0.2173	1.8001	0.0922	0.0944
		$\hat{\lambda}_2$	0.1988	0.4848	2.6171	0.2254	0.2305	0.3475	0.4656	2.3136	0.0831	0.0808
		$\hat{\lambda}_3$	0.1139	0.4855	2.6961	0.1869	0.1821	0.1638	0.2395	1.8170	0.0858	0.0857

Table 1 continued

m	Scheme		MLE					Bayes				
			Bias	MSE	ACI	B-TCI	B-PCI	Bias	MSE	ACI	B-TCI	B-PCI
15	II	$\hat{\alpha}$	0.1364	0.1362	1.3451	0.1251	0.1260	0.1484	0.1210	1.2397	0.0428	0.0430
		$\hat{\lambda}_1$	0.0920	0.4308	2.5488	0.1978	0.2010	0.2464	0.3129	1.9784	0.0809	0.0825
		$\hat{\lambda}_2$	0.2586	0.6963	3.1115	0.2721	0.2748	0.3842	0.6330	2.7450	0.0960	0.0961
		$\hat{\lambda}_3$	0.0836	0.3850	2.4114	0.1961	0.1963	0.1822	0.2832	1.9698	0.0780	0.0791
15	III	$\hat{\alpha}$	0.1284	0.1264	1.3000	0.1204	0.1189	0.1707	0.1729	1.4911	0.0418	0.0417
		$\hat{\lambda}_1$	0.1175	0.4755	2.6650	0.1741	0.1724	0.2353	0.3884	2.2698	0.0845	0.0855
		$\hat{\lambda}_2$	0.1958	0.4652	2.5624	0.1723	0.1684	0.2349	0.3390	2.0954	0.0813	0.0807
		$\hat{\lambda}_3$	0.0787	0.3670	2.3559	0.1770	0.1793	0.2551	0.3800	2.2072	0.0737	0.0745
25	I	$\hat{\alpha}$	0.0957	0.0738	0.9976	0.0940	0.0922	0.1031	0.0616	0.8892	0.0318	0.0319
		$\hat{\lambda}_1$	0.1085	0.2814	2.0364	0.1691	0.1676	0.0540	0.1877	1.6836	0.0656	0.0631
		$\hat{\lambda}_2$	0.1181	0.2717	1.9911	0.1605	0.1637	0.1922	0.2181	1.6767	0.0639	0.0653
		$\hat{\lambda}_3$	0.0757	0.2190	1.8114	0.1460	0.1519	0.0998	0.1500	1.4744	0.0583	0.0585
25	II	$\hat{\alpha}$	0.0885	0.0764	1.0269	0.0983	0.0993	0.0992	0.0725	0.9864	0.0340	0.0345
		$\hat{\lambda}_1$	0.1172	0.2912	2.0659	0.1808	0.1809	0.1852	0.2414	1.7928	0.0660	0.0658
		$\hat{\lambda}_2$	0.1414	0.3201	2.1484	0.2062	0.2043	0.1978	0.3059	2.0496	0.0685	0.0673
		$\hat{\lambda}_3$	0.0911	0.2543	1.9451	0.1511	0.1461	0.2377	0.2128	1.5574	0.0600	0.0607

Table 2 Estimation of the parameters of BIW distribution under different schemes when $n = 50$: Case 1

n = 50			MLE					Bayes				
M	Scheme		Bias	MSE	ACI	B-TCI	B-PCI	Bias	MSE	ACI	B-TCI	B-PCI
15	I	$\hat{\alpha}$	0.1623	0.1462	1.3578	0.1258	0.1233	0.1857	0.1418	1.2909	0.0429	0.0441
		$\hat{\lambda}_1$	0.0793	0.5806	2.9723	0.1797	0.1877	0.1747	0.2332	1.7735	0.0965	0.0978
		$\hat{\lambda}_2$	0.1735	0.4780	2.6249	0.1887	0.1844	0.2553	0.3054	1.9308	0.0849	0.0843
		$\hat{\lambda}_3$	0.0674	0.5210	2.8187	0.1537	0.1554	0.0820	0.1657	1.5710	0.0902	0.0910
15	II	$\hat{\alpha}$	0.1321	0.1324	1.3298	0.1272	0.1280	0.0752	0.1089	1.2658	0.0420	0.0417
		$\hat{\lambda}_1$	0.0484	0.3441	2.2927	0.1773	0.1810	0.1567	0.2240	1.7597	0.0711	0.0718
		$\hat{\lambda}_2$	0.2295	0.6420	3.0108	0.2812	0.2777	0.4077	0.6721	2.8021	0.0983	0.1028
		$\hat{\lambda}_3$	0.0726	0.3013	2.1339	0.1718	0.1703	0.2247	0.2290	1.6646	0.0673	0.0677
15	III	$\hat{\alpha}$	0.1515	0.1391	1.3367	0.1273	0.1273	0.1194	0.1215	1.2900	0.0425	0.0412
		$\hat{\lambda}_1$	0.0732	0.5085	2.7818	0.1943	0.1934	0.1403	0.2817	2.0167	0.0855	0.0877
		$\hat{\lambda}_2$	0.1700	0.3522	2.2299	0.1923	0.1894	0.2295	0.2838	1.8941	0.0721	0.0749
		$\hat{\lambda}_3$	0.0399	0.5030	2.7773	0.1669	0.1651	0.1333	0.2067	1.7124	0.0920	0.0918
15	IV	$\hat{\alpha}$	0.1368	0.1344	1.3341	0.1021	0.1004	0.2088	0.1444	1.2486	0.0419	0.0435
		$\hat{\lambda}_1$	0.0323	0.3770	2.4049	0.1693	0.1708	0.1107	0.2837	2.0492	0.0773	0.0773
		$\hat{\lambda}_2$	0.1521	0.4772	2.6428	0.1807	0.1780	0.2717	0.3959	2.2320	0.0808	0.0821
		$\hat{\lambda}_3$	0.0808	0.3422	2.2725	0.1513	0.1479	0.1870	0.2561	1.8494	0.0728	0.0725
25	I	$\hat{\alpha}$	0.1001	0.0748	0.9983	0.0983	0.0974	0.0774	0.0686	0.9856	0.0329	0.0328
		$\hat{\lambda}_1$	0.1033	0.2564	1.9441	0.1488	0.1478	0.0612	0.1479	1.4959	0.0596	0.0598
		$\hat{\lambda}_2$	0.1186	0.2221	1.7889	0.1508	0.1467	0.1542	0.1636	1.4732	0.0555	0.0547
		$\hat{\lambda}_3$	0.0336	0.2124	1.8026	0.1292	0.1268	0.1070	0.1257	1.3319	0.0567	0.0568

Table 2 continued

n = 50			MLE					Bayes				
M	Scheme		Bias	MSE	ACI	B-TCI	B-PCI	Bias	MSE	ACI	B-TCI	B-PCI
25	II	$\hat{\alpha}$	0.0880	0.0743	1.0119	0.1012	0.1017	0.1338	0.0879	1.0426	0.0324	0.0318
		$\hat{\lambda}_1$	0.0594	0.2018	1.7462	0.1632	0.1630	0.0688	0.1908	1.6821	0.0546	0.0562
		$\hat{\lambda}_2$	0.1468	0.3455	2.2322	0.1874	0.1896	0.2121	0.2776	1.9000	0.0715	0.0730
		$\hat{\lambda}_3$	0.0440	0.1933	1.7156	0.1482	0.1488	0.0474	0.1498	1.5134	0.0546	0.0544
25	III	$\hat{\alpha}$	0.0883	0.0759	1.0232	0.0794	0.0778	0.1257	0.0796	0.9934	0.0320	0.0320
		$\hat{\lambda}_1$	0.0511	0.2191	1.8248	0.1363	0.1308	0.1252	0.1978	1.6784	0.0588	0.0585
		$\hat{\lambda}_2$	0.0956	0.2241	1.8185	0.1563	0.1552	0.0717	0.2284	1.7844	0.0574	0.0586
		$\hat{\lambda}_3$	0.0719	0.1929	1.6991	0.1170	0.1162	0.1192	0.1595	1.4991	0.0533	0.0549
35	I	$\hat{\alpha}$	0.0692	0.0520	0.8521	0.0795	0.0780	0.0674	0.0470	0.8122	0.0268	0.0265
		$\hat{\lambda}_1$	0.0807	0.1776	1.6220	0.1488	0.1516	0.1281	0.1573	1.4789	0.0497	0.0501
		$\hat{\lambda}_2$	0.0923	0.1667	1.5599	0.1393	0.1452	0.1091	0.1355	1.3853	0.0495	0.0489
		$\hat{\lambda}_3$	0.0447	0.1480	1.4988	0.1312	0.1355	0.0817	0.1171	1.3092	0.0476	0.0477
35	II	$\hat{\alpha}$	0.0567	0.0490	0.8393	0.0822	0.0823	0.0836	0.0520	0.8356	0.0270	0.0269
		$\hat{\lambda}_1$	0.0562	0.1665	1.5850	0.1257	0.1228	0.1163	0.1135	1.2455	0.0498	0.0488
		$\hat{\lambda}_2$	0.1252	0.2235	1.7881	0.1574	0.1569	0.1651	0.1953	1.6149	0.0593	0.0586
		$\hat{\lambda}_3$	0.0447	0.1259	1.3805	0.1385	0.1379	0.0934	0.1289	1.3660	0.0456	0.0451

Table 3 Estimation of the parameters of BIW distribution under different schemes when $n = 100$: Case 1

n = 100			MLE					Bayes				
m	Scheme		Bias	MSE	ACI	B-TCI	B-PCI	Bias	MSE	ACI	B-TCI	B-PCI
30	I	$\hat{\alpha}$	0.0815	0.0617	0.9205	0.0998	0.0991	0.0617	0.0619	0.9893	0.0285	0.0285
		$\hat{\lambda}_1$	0.0257	0.2514	1.9641	0.1369	0.1410	0.0175	0.1276	1.4058	0.0616	0.0642
		$\hat{\lambda}_2$	0.0868	0.1422	1.4393	0.1349	0.1344	0.0847	0.1382	1.2879	0.0438	0.0435
		$\hat{\lambda}_3$	0.0458	0.2264	1.8573	0.1239	0.1222	0.0973	0.1069	1.2298	0.0587	0.0591
30	II	$\hat{\alpha}$	0.0423	0.0537	0.8933	0.0862	0.0870	0.0878	0.0569	0.8740	0.0284	0.0287
		$\hat{\lambda}_1$	0.0255	0.1509	1.5204	0.1488	0.1531	0.0598	0.1440	1.5359	0.0484	0.0491
		$\hat{\lambda}_2$	0.0883	0.2503	1.9313	0.1854	0.1850	0.1196	0.2423	1.8228	0.0604	0.0605
		$\hat{\lambda}_3$	0.0261	0.1459	1.4944	0.1402	0.1396	0.0909	0.1348	1.4560	0.0472	0.0483
30	III	$\hat{\alpha}$	0.0625	0.0556	0.8919	0.0879	0.0892	0.0702	0.0542	0.8744	0.0279	0.0276
		$\hat{\lambda}_1$	0.0307	0.2299	1.8768	0.1533	0.1505	0.0370	0.1546	1.5421	0.0600	0.0592
		$\hat{\lambda}_2$	0.0716	0.1429	1.4559	0.1341	0.1327	0.1319	0.1353	1.3525	0.0459	0.0471
		$\hat{\lambda}_3$	0.0474	0.2014	1.7504	0.1452	0.1415	0.0550	0.1364	1.4388	0.0551	0.0567
30	IV	$\hat{\alpha}$	0.0592	0.0504	0.8496	0.0778	0.0825	0.0816	0.0461	0.7820	0.0265	0.0254
		$\hat{\lambda}_1$	0.0384	0.1698	1.6090	0.1442	0.1460	0.0805	0.1320	1.3961	0.0511	0.0518
		$\hat{\lambda}_2$	0.0659	0.1653	1.5736	0.1327	0.1332	0.1224	0.1250	1.3066	0.0513	0.0525
		$\hat{\lambda}_3$	0.0415	0.1412	1.4646	0.1303	0.1310	0.0955	0.1148	1.2809	0.0457	0.0458
40	I	$\hat{\alpha}$	0.0540	0.0405	0.7608	0.0742	0.0737	0.0908	0.0418	0.7424	0.0236	0.0237
		$\hat{\lambda}_1$	0.0310	0.1571	1.5495	0.1227	0.1243	0.0229	0.0915	1.1883	0.0478	0.0478
		$\hat{\lambda}_2$	0.0682	0.1108	1.2779	0.1137	0.1167	0.0613	0.0887	1.1483	0.0406	0.0408
		$\hat{\lambda}_3$	0.0332	0.1281	1.3975	0.1136	0.1129	0.0931	0.0990	1.1844	0.0454	0.0469

Table 3 continued

n = 100			MLE					Bayes				
m	Scheme		Bias	MSE	ACI	B-TCI	B-PCI	Bias	MSE	ACI	B-TCI	B-PCI
40	II	$\hat{\alpha}$	0.0401	0.0411	0.7792	0.0728	0.0734	0.0720	0.0388	0.7223	0.0250	0.0256
		$\hat{\lambda}_1$	0.0475	0.1218	1.3561	0.1089	0.1068	0.1337	0.0911	1.0661	0.0428	0.0445
		$\hat{\lambda}_2$	0.0539	0.1492	1.4999	0.1392	0.1424	0.1375	0.1409	1.3763	0.0453	0.0451
		$\hat{\lambda}_3$	0.0260	0.0995	1.2329	0.1064	0.1089	0.0680	0.0797	1.0793	0.0379	0.0368
40	III	$\hat{\alpha}$	0.0568	0.0396	0.7475	0.0784	0.0808	0.0443	0.0411	0.7798	0.0230	0.0229
		$\hat{\lambda}_1$	0.0360	0.1433	1.4780	0.1381	0.1408	0.0607	0.1166	1.3241	0.0460	0.0467
		$\hat{\lambda}_2$	0.0603	0.1129	1.2965	0.1354	0.1315	0.0807	0.1019	1.3129	0.0420	0.0408
		$\hat{\lambda}_3$	0.0360	0.1264	1.3873	0.1091	0.1109	0.0496	0.0809	1.1035	0.0424	0.0424
40	IV	$\hat{\alpha}$	0.0526	0.0380	0.7364	0.0635	0.0668	0.0450	0.0279	0.6342	0.0227	0.0229
		$\hat{\lambda}_1$	0.0087	0.1217	1.3675	0.1212	0.1201	0.0271	0.0966	1.2200	0.0436	0.0429
		$\hat{\lambda}_2$	0.0525	0.1256	1.3745	0.1152	0.1145	0.0692	0.0867	1.1272	0.0433	0.0423
		$\hat{\lambda}_3$	0.0485	0.1143	1.3125	0.1046	0.1042	0.0451	0.0748	1.0629	0.0405	0.0406
50	I	$\hat{\alpha}$	0.0455	0.0341	0.7023	0.0664	0.0673	0.0742	0.0340	0.6651	0.0213	0.0215
		$\hat{\lambda}_1$	0.0380	0.1237	1.3713	0.1157	0.1126	0.0251	0.0866	1.1524	0.0428	0.0453
		$\hat{\lambda}_2$	0.0613	0.0920	1.1653	0.1162	0.1222	0.0514	0.0905	1.1352	0.0365	0.0363
		$\hat{\lambda}_3$	0.0182	0.0966	1.2166	0.1041	0.1052	0.0622	0.0773	1.0676	0.0397	0.0413
50	II	$\hat{\alpha}$	0.0482	0.0363	0.7229	0.0696	0.0694	0.0665	0.0352	0.6916	0.0228	0.0230
		$\hat{\lambda}_1$	0.0457	0.1116	1.2977	0.1275	0.1282	0.0960	0.0865	1.2903	0.0413	0.0431
		$\hat{\lambda}_2$	0.0653	0.1366	1.4265	0.1225	0.1249	0.0653	0.0985	1.2093	0.0447	0.0456
		$\hat{\lambda}_3$	0.0250	0.0896	1.1701	0.1289	0.1302	0.0590	0.0615	1.2081	0.0365	0.0383
50	III	$\hat{\alpha}$	0.0476	0.0341	0.6999	0.0738	0.0750	0.0459	0.0347	0.7112	0.0225	0.0233
		$\hat{\lambda}_1$	0.0490	0.1018	1.2366	0.1004	0.1005	0.1074	0.0807	1.0362	0.0389	0.0389

Table 3 continued

n = 100			MLE					Bayes				
m	Scheme		Bias	MSE	ACI	B-TCI	B-PCI	Bias	MSE	ACI	B-TCI	B-PCI
		$\hat{\lambda}_2$	0.0386	0.0821	1.1135	0.1161	0.1158	0.0352	0.0744	1.1046	0.0354	0.0363
		$\hat{\lambda}_3$	− 0.0063	0.0787	1.0998	0.1083	0.1072	0.0129	0.0812	1.1215	0.0344	0.0340
70	I	$\hat{\alpha}$	0.0337	0.0231	0.5808	0.0591	0.0594	0.0453	0.0206	0.6126	0.0185	0.0178
		$\hat{\lambda}_1$	0.0360	0.0737	1.0554	0.1156	0.1190	0.0368	0.0629	1.1033	0.0327	0.0323
		$\hat{\lambda}_2$	0.0344	0.0678	1.0121	0.0956	0.0947	0.0638	0.0651	0.9730	0.0316	0.0309
		$\hat{\lambda}_3$	0.0148	0.0608	0.9656	0.0968	0.0953	− 0.0130	0.0573	0.9416	0.0304	0.0304
70	II	$\hat{\alpha}$	0.0239	0.0227	0.5834	0.0582	0.0592	0.0450	0.0234	0.5756	0.0188	0.0184
		$\hat{\lambda}_1$	0.0318	0.0773	1.0829	0.0971	0.0990	0.0597	0.0647	0.9743	0.0348	0.0350
		$\hat{\lambda}_2$	0.0456	0.0872	1.1440	0.1285	0.1330	0.0476	0.0906	1.1873	0.0353	0.0363
		$\hat{\lambda}_3$	0.0139	0.0600	0.9594	0.0917	0.0944	0.0487	0.0459	0.9336	0.0321	0.0305

Table 4 Estimation of the parameters of BIW distribution under different schemes when $n = 30$: Case 2

	Scheme	n = 30	MLE					Bayes				
			Bias	mse	ACI	B-TCI	B-PCI	Bias	mse	ACI	B-TCI	B-PCI
10	I	$\hat{\alpha}$	0.0950	0.0454	0.7476	0.0236	0.0239	0.0809	0.0364	0.6776	0.0216	0.0218
		$\hat{\lambda}_1$	0.1806	0.7299	3.2749	0.1054	0.1095	0.0891	0.3060	2.1411	0.0693	0.0699
		$\hat{\lambda}_2$	0.1300	0.2583	1.9268	0.0599	0.0620	0.1979	0.1993	1.5696	0.0476	0.0477
		$\hat{\lambda}_3$	0.0882	0.5441	2.8721	0.0931	0.0934	0.1308	0.3438	2.2417	0.0693	0.0724
10	II	$\hat{\alpha}$	0.0723	0.0344	0.6701	0.0206	0.0207	0.0788	0.0307	0.6667	0.0202	0.0202
		$\hat{\lambda}_1$	0.1133	0.5304	2.8214	0.0900	0.0893	0.1317	0.3645	2.3108	0.0742	0.0723
		$\hat{\lambda}_2$	0.2237	0.5256	2.7046	0.0875	0.0865	0.2155	0.2679	1.8457	0.0569	0.0557
		$\hat{\lambda}_3$	0.0935	0.4801	2.6926	0.0859	0.0859	0.1580	0.3824	2.3449	0.0707	0.0711
10	III	$\hat{\alpha}$	0.0996	0.0433	0.7163	0.0235	0.0235	0.0808	0.0369	0.6839	0.0217	0.0216
		$\hat{\lambda}_1$	0.1467	0.6771	3.1756	0.1004	0.1018	0.1341	0.3649	2.3100	0.0750	0.0750
		$\hat{\lambda}_2$	0.1309	0.2823	2.0195	0.0635	0.0641	0.1622	0.1926	1.5993	0.0530	0.0513
		$\hat{\lambda}_3$	0.1835	0.6464	3.0700	0.0919	0.0930	0.1463	0.3123	2.1151	0.0677	0.0669
10	IV	$\hat{\alpha}$	0.0843	0.0366	0.6735	0.0210	0.0209	0.0742	0.0333	0.6534	0.0210	0.0219
		$\hat{\lambda}_1$	0.1270	0.5468	2.8572	0.0925	0.0914	0.0586	0.2952	2.1184	0.0665	0.0668
		$\hat{\lambda}_2$	0.1619	0.3830	2.3425	0.0748	0.0765	0.2790	0.3090	1.8856	0.0583	0.0584
		$\hat{\lambda}_3$	0.1579	0.5491	2.8396	0.0906	0.0927	0.0887	0.3037	2.1332	0.0677	0.0678
15	I	$\hat{\alpha}$	0.0613	0.0236	0.5528	0.0172	0.0175	0.0688	0.0231	0.5321	0.0174	0.0180
		$\hat{\lambda}_1$	0.1425	0.4248	2.4945	0.0809	0.0797	0.1159	0.2717	1.9931	0.0593	0.0592
		$\hat{\lambda}_2$	0.0974	0.1954	1.6912	0.0546	0.0550	0.1806	0.1870	1.5408	0.0491	0.0509
		$\hat{\lambda}_3$	0.1089	0.3588	2.3101	0.0723	0.0727	0.1283	0.2730	1.9864	0.0648	0.0658

Table 4 continued

	Scheme	n = 30	MLE					Bayes				
			Bias	mse	ACI	B-TCI	B-PCI	Bias	mse	ACI	B-TCI	B-PCI
15	II	$\hat{\alpha}$	0.0568	0.0202	0.5116	0.0160	0.0164	0.0567	0.0203	0.5127	0.0158	0.0156
		$\hat{\lambda}_1$	0.0702	0.3461	2.2909	0.0724	0.0735	0.1362	0.2977	2.0721	0.0658	0.0657
		$\hat{\lambda}_2$	0.0996	0.3093	2.1461	0.0677	0.0677	0.2135	0.2125	1.6024	0.0526	0.0536
		$\hat{\lambda}_3$	0.1333	0.3441	2.2403	0.0708	0.0733	0.1203	0.3002	2.0966	0.0674	0.0668
15	III	$\hat{\alpha}$	0.0620	0.0226	0.5364	0.0172	0.0173	0.0672	0.0218	0.5151	0.0160	0.0165
		$\hat{\lambda}_1$	0.0947	0.3306	2.2243	0.0691	0.0686	0.1184	0.2873	2.0503	0.0651	0.0653
		$\hat{\lambda}_2$	0.0838	0.2105	1.7693	0.0568	0.0572	0.1736	0.1852	1.5442	0.0483	0.0487
		$\hat{\lambda}_3$	0.1113	0.3252	2.1934	0.0684	0.0677	0.1626	0.3058	2.0727	0.0670	0.0684
25	I	$\hat{\alpha}$	0.0333	0.0116	0.4013	0.0125	0.0128	0.0249	0.0103	0.4006	0.0123	0.0124
		$\hat{\lambda}_1$	0.1213	0.2364	1.8467	0.0604	0.0610	0.1146	0.1926	1.6615	0.0541	0.0552
		$\hat{\lambda}_2$	0.0666	0.1335	1.4092	0.0448	0.0473	0.1539	0.1369	1.3195	0.0404	0.0400
		$\hat{\lambda}_3$	0.0663	0.1963	1.7179	0.0561	0.0552	0.0910	0.1805	1.6275	0.0536	0.0524
25	II	$\hat{\alpha}$	0.0350	0.0118	0.4033	0.0131	0.0128	0.0342	0.0103	0.4013	0.0122	0.0121
		$\hat{\lambda}_1$	0.0948	0.2053	1.7377	0.0533	0.0531	0.0928	0.2015	1.7019	0.0515	0.0527
		$\hat{\lambda}_2$	0.0750	0.1496	1.4883	0.0461	0.0464	0.0625	0.1460	1.4326	0.0461	0.0457
		$\hat{\lambda}_3$	0.0925	0.2134	1.7750	0.0564	0.0578	0.1208	0.2043	1.7082	0.0543	0.0570

Table 5 Estimation of the parameters of BIW distribution under different schemes when $n = 50$: Case 2

n = 50			MLE					Bayes				
m	Scheme		Bias	mse	ACI	B-TCI	B-PCI	Bias	mse	ACI	B-TCI	B-PCI
15	I	$\hat{\alpha}$	0.0655	0.0235	0.5442	0.0172	0.0166	0.0573	0.0212	0.5252	0.0168	0.0173
		$\hat{\lambda}_1$	0.0590	0.3380	2.2683	0.0709	0.0733	0.0690	0.2490	1.9383	0.0642	0.0662
		$\hat{\lambda}_2$	0.0712	0.1366	1.4221	0.0451	0.0444	0.1293	0.1361	1.4178	0.0406	0.0404
		$\hat{\lambda}_3$	0.0823	0.3052	2.1427	0.0684	0.0672	0.0568	0.2164	1.8108	0.0559	0.0575
15	II	$\hat{\alpha}$	0.0551	0.0208	0.5234	0.0163	0.0163	0.0651	0.0204	0.4986	0.0155	0.0156
		$\hat{\lambda}_1$	0.0519	0.2626	1.9996	0.0617	0.0621	0.0686	0.2223	1.8294	0.0561	0.0570
		$\hat{\lambda}_2$	0.0999	0.2512	1.9260	0.0618	0.0635	0.2101	0.2342	1.7096	0.0526	0.0539
		$\hat{\lambda}_3$	0.1037	0.3162	2.1674	0.0708	0.0704	0.0631	0.2372	1.8942	0.0580	0.0586
15	III	$\hat{\alpha}$	0.0688	0.0234	0.5355	0.0168	0.0167	0.0602	0.0211	0.5188	0.0160	0.0161
		$\hat{\lambda}_1$	0.1072	0.3505	2.2834	0.0690	0.0670	0.0616	0.2270	1.8530	0.0598	0.0601
		$\hat{\lambda}_2$	0.0867	0.1517	1.4893	0.0461	0.0457	0.0870	0.1517	1.4327	0.0461	0.0456
		$\hat{\lambda}_3$	0.0624	0.3413	2.2783	0.0677	0.0659	0.0751	0.2311	1.8621	0.0595	0.0581
15	IV	$\hat{\alpha}$	0.0547	0.0189	0.4943	0.0161	0.0165	0.0565	0.0172	0.4538	0.0167	0.0167
		$\hat{\lambda}_1$	0.0603	0.3113	2.1754	0.0719	0.0704	0.0589	0.2310	1.8709	0.0594	0.0572
		$\hat{\lambda}_2$	0.1111	0.2098	1.7428	0.0551	0.0566	0.1888	0.1831	1.5058	0.0466	0.0477
		$\hat{\lambda}_3$	0.0724	0.2803	2.0570	0.0665	0.0658	0.0699	0.2259	1.8439	0.0568	0.0563
25	I	$\hat{\alpha}$	0.0393	0.0111	0.3826	0.0121	0.0117	0.0378	0.0106	0.3761	0.0115	0.0115
		$\hat{\lambda}_1$	0.0533	0.1948	1.7185	0.0527	0.0546	0.0822	0.1789	1.6273	0.0510	0.0503
		$\hat{\lambda}_2$	0.0593	0.0976	1.2031	0.0392	0.0400	0.0486	0.0918	1.2030	0.0382	0.0375
		$\hat{\lambda}_3$	0.0871	0.1847	1.6507	0.0525	0.0522	0.0572	0.1712	1.6071	0.0488	0.0474

Table 5 continued

n = 50			MLE					Bayes				
m	Scheme		Bias	mse	ACI	B-TCI	B-PCI	Bias	mse	ACI	B-TCI	B-PCI
25	II	$\hat{\alpha}$	0.0290	0.0100	0.3747	0.0120	0.0124	0.0274	0.0099	0.3759	0.0118	0.0118
		$\hat{\lambda}_1$	0.0565	0.1830	1.6632	0.0523	0.0515	0.0746	0.1779	1.6282	0.0537	0.0545
		$\hat{\lambda}_2$	0.0607	0.1402	1.4493	0.0467	0.0470	0.1615	0.1464	1.4394	0.0460	0.0463
		$\hat{\lambda}_3$	0.0626	0.1790	1.6412	0.0526	0.0527	0.0918	0.1624	1.5390	0.0470	0.0474
25	III	$\hat{\alpha}$	0.0411	0.0119	0.3960	0.0125	0.0130	0.0323	0.0102	0.3750	0.0122	0.0122
		$\hat{\lambda}_1$	0.0774	0.1935	1.6982	0.0522	0.0503	0.0520	0.1772	1.6383	0.0539	0.0559
		$\hat{\lambda}_2$	0.1624	0.1227	1.2173	0.0375	0.0386	0.0527	0.1020	1.2352	0.0382	0.0382
		$\hat{\lambda}_3$	0.0720	0.1620	1.5533	0.0487	0.0494	0.0688	0.1791	1.6377	0.0549	0.0539
35	I	$\hat{\alpha}$	0.0380	0.0082	0.3222	0.0102	0.0102	0.0241	0.0070	0.3139	0.0097	0.0100
		$\hat{\lambda}_1$	0.0675	0.1396	1.4415	0.0465	0.0471	0.0717	0.1342	1.4507	0.0453	0.0446
		$\hat{\lambda}_2$	0.1165	0.0875	1.0662	0.0347	0.0341	0.0517	0.0782	1.0781	0.0337	0.0330
		$\hat{\lambda}_3$	0.0701	0.1451	1.4683	0.0473	0.0485	0.0547	0.1247	1.3681	0.0446	0.0431
35	II	$\hat{\alpha}$	0.0315	0.0087	0.3434	0.0106	0.0105	0.0253	0.0076	0.3266	0.0098	0.0100
		$\hat{\lambda}_1$	0.0743	0.1476	1.4784	0.0473	0.0483	0.0570	0.1422	1.4620	0.0465	0.0469
		$\hat{\lambda}_2$	0.1459	0.1170	1.2135	0.0388	0.0376	0.0475	0.1016	1.2363	0.0377	0.0379
		$\hat{\lambda}_3$	0.0553	0.1238	1.3625	0.0441	0.0434	0.0408	0.1247	1.3759	0.0434	0.0429

Table 6 Estimation of the parameters of BIW distribution under different schemes when $n = 100$: Case 2

n = 100			MLE					Bayes				
m	Scheme		Bias	mse	ACI	B-TCI	B-PCI	Bias	mse	ACI	B-TCI	B-PCI
30	I	$\hat{\alpha}$	0.0316	0.0094	0.3592	0.0109	0.0105	0.0178	0.0079	0.3423	0.0109	0.0111
		$\hat{\lambda}_1$	0.0409	0.1468	1.4939	0.0478	0.0472	− 0.0033	0.0768	1.0865	0.0341	0.0354
		$\hat{\lambda}_2$	0.0315	0.0621	0.9698	0.0307	0.0313	0.0397	0.0502	0.8651	0.0275	0.0279
		$\hat{\lambda}_3$	0.0228	0.1327	1.4258	0.0458	0.0448	0.0156	0.0721	1.0514	0.0328	0.0324
30	II	$\hat{\alpha}$	0.0281	0.0083	0.3399	0.0104	0.0107	0.0170	0.0077	0.3370	0.0106	0.0106
		$\hat{\lambda}_1$	0.0350	0.1386	1.4535	0.0448	0.0438	0.0037	0.0781	1.0962	0.0351	0.0362
		$\hat{\lambda}_2$	0.0601	0.1111	1.2859	0.0400	0.0415	0.0397	0.0625	0.9677	0.0299	0.0292
		$\hat{\lambda}_3$	0.0480	0.1325	1.4153	0.0442	0.0439	− 0.0123	0.0830	1.1291	0.0366	0.0358
30	III	$\hat{\alpha}$	0.0281	0.0083	0.3399	0.0104	0.0107	0.0185	0.0086	0.3573	0.0116	0.0119
		$\hat{\lambda}_1$	0.0350	0.1386	1.4535	0.0448	0.0438	0.0030	0.0741	1.0676	0.0337	0.0328
		$\hat{\lambda}_2$	0.0601	0.1111	1.2859	0.0400	0.0415	0.0421	0.0479	0.8423	0.0268	0.0275
		$\hat{\lambda}_3$	0.0480	0.1325	1.4153	0.0442	0.0439	0.0047	0.0725	1.0558	0.0324	0.0330
30	IV	$\hat{\alpha}$	0.0217	0.0079	0.3386	0.0107	0.0111	0.0213	0.0079	0.3245	0.0111	0.0110
		$\hat{\lambda}_1$	0.0217	0.1244	1.3806	0.0438	0.0437	0.0621	0.1304	1.3949	0.0439	0.0441
		$\hat{\lambda}_2$	0.0410	0.0757	1.0673	0.0337	0.0330	0.1278	0.0996	1.1319	0.0347	0.0339
		$\hat{\lambda}_3$	0.0571	0.1312	1.4026	0.0433	0.0431	0.0487	0.1195	1.3420	0.0433	0.0434
40	I	$\hat{\alpha}$	0.0200	0.0061	0.2959	0.0090	0.0088	0.0136	0.0058	0.2939	0.0093	0.0093
		$\hat{\lambda}_1$	0.0336	0.1107	1.2982	0.0415	0.0409	0.0018	0.0669	1.0146	0.0329	0.0332
		$\hat{\lambda}_2$	0.0215	0.0456	0.8332	0.0261	0.0269	0.0328	0.0453	0.8246	0.0264	0.0262
		$\hat{\lambda}_3$	0.0273	0.0968	1.2157	0.0400	0.0394	0.0210	0.0682	1.0208	0.0300	0.0302

Table 6 continued

n = 100			MLE					Bayes				
m	Scheme		Bias	mse	ACI	B-TCI	B-PCI	Bias	mse	ACI	B-TCI	B-PCI
40	II	$\hat{\alpha}$	0.0190	0.0064	0.3039	0.0095	0.0094	0.0152	0.0059	0.2966	0.0096	0.0100
		$\hat{\lambda}_1$	0.0373	0.1053	1.2645	0.0393	0.0393	− 0.0179	0.0649	0.9964	0.0319	0.0312
		$\hat{\lambda}_2$	0.0539	0.0782	1.0764	0.0343	0.0333	0.0250	0.0607	0.9614	0.0308	0.0309
		$\hat{\lambda}_3$	0.0273	0.1025	1.2510	0.0396	0.0383	0.0243	0.0737	1.0605	0.0315	0.0311
40	III	$\hat{\alpha}$	0.0190	0.0064	0.3039	0.0095	0.0094	0.0168	0.0061	0.2980	0.0093	0.0091
		$\hat{\lambda}_1$	0.0373	0.1053	1.2645	0.0393	0.0393	− 0.0030	0.0663	1.0096	0.0320	0.0329
		$\hat{\lambda}_2$	0.0539	0.0782	1.0764	0.0343	0.0333	0.0270	0.0440	0.8157	0.0262	0.0268
		$\hat{\lambda}_3$	0.0273	0.1025	1.2510	0.0396	0.0383	0.0105	0.0716	1.0488	0.0327	0.0330
40	IV	$\hat{\alpha}$	0.0172	0.0062	0.3001	0.0090	0.0089	0.0245	0.0061	0.2896	0.0089	0.0092
		$\hat{\lambda}_1$	0.0257	0.1062	1.2739	0.0399	0.0410	0.0491	0.1045	1.2530	0.0409	0.0410
		$\hat{\lambda}_2$	0.0322	0.0544	0.9063	0.0294	0.0294	0.0319	0.0536	0.9051	0.0280	0.0272
		$\hat{\lambda}_3$	0.0416	0.1032	1.2493	0.0387	0.0387	0.0210	0.0943	1.2012	0.0375	0.0369
50	I	$\hat{\alpha}$	0.0158	0.0050	0.2708	0.0083	0.0083	0.0166	0.0050	0.2705	0.0088	0.0086
		$\hat{\lambda}_1$	0.0388	0.0923	1.1815	0.0370	0.0378	0.0137	0.0677	1.0189	0.0334	0.0335
		$\hat{\lambda}_2$	0.0301	0.0432	0.8064	0.0259	0.0266	0.0361	0.0395	0.7669	0.0250	0.0258
		$\hat{\lambda}_3$	0.0212	0.0772	1.0869	0.0345	0.0342	0.0074	0.0606	0.9650	0.0305	0.0315
50	II	$\hat{\alpha}$	0.0164	0.0050	0.2706	0.0087	0.0090	0.0143	0.0051	0.2751	0.0087	0.0088
		$\hat{\lambda}_1$	0.0386	0.0957	1.2039	0.0387	0.0394	0.0045	0.0674	1.0177	0.0319	0.0326
		$\hat{\lambda}_2$	0.0407	0.0713	1.0354	0.0328	0.0335	0.0403	0.0562	0.9165	0.0293	0.0298
		$\hat{\lambda}_3$	0.0530	0.0841	1.1183	0.0351	0.0357	− 0.0070	0.0575	0.9402	0.0296	0.0301
50	III	$\hat{\alpha}$	0.0164	0.0050	0.2706	0.0087	0.0090	0.0144	0.0048	0.2652	0.0083	0.0081
		$\hat{\lambda}_1$	0.0386	0.0957	1.2039	0.0387	0.0394	0.0114	0.0641	0.9922	0.0314	0.0316

Table 6 continued

n = 100			MLE					Bayes				
m	Scheme		Bias	mse	ACI	B-TCI	B-PCI	Bias	mse	ACI	B-TCI	B-PCI
		$\hat{\lambda}_2$	0.0407	0.0713	1.0354	0.0328	0.0335	0.0364	0.0449	0.8184	0.0270	0.0271
		$\hat{\lambda}_3$	0.0530	0.0841	1.1183	0.0351	0.0357	− 0.0008	0.0592	0.9544	0.0298	0.0288
70	I	$\hat{\alpha}$	0.0133	0.0035	0.2246	0.0071	0.0071	0.0114	0.0034	0.2245	0.0070	0.0069
		$\hat{\lambda}_1$	0.0429	0.0636	0.9747	0.0314	0.0312	0.0095	0.0518	0.8922	0.0281	0.0275
		$\hat{\lambda}_2$	0.0144	0.0331	0.7117	0.0216	0.0223	0.0316	0.0357	0.7305	0.0232	0.0237
		$\hat{\lambda}_3$	0.0307	0.0621	0.9703	0.0309	0.0322	− 0.0019	0.0510	0.8860	0.0274	0.0266
70	II	$\hat{\alpha}$	0.0096	0.0036	0.2325	0.0074	0.0073	0.0117	0.0036	0.2297	0.0074	0.0073
		$\hat{\lambda}_1$	0.0220	0.0677	1.0171	0.0314	0.0321	0.0056	0.0576	0.9413	0.0299	0.0301
		$\hat{\lambda}_2$	0.0248	0.0421	0.7989	0.0267	0.0272	0.0452	0.0425	0.7888	0.0247	0.0257
		$\hat{\lambda}_3$	0.0247	0.0586	0.9441	0.0303	0.0311	0.0156	0.0550	0.9179	0.0292	0.0292

Table 7 Parameter estimation of BIW distribution for complete football data

BIW						
	MLE				Bayes	
	Estimate	Std	-LL	321.4475	Mean	SD
$\hat{\alpha}$	0.9221	0.0810	AIC	650.895	0.8057	0.0814
$\hat{\lambda}_1$	10.7162	3.3425	BIC	657.3387	6.5064	2.2993
$\hat{\lambda}_2$	3.5617	1.4871	HQIC	653.1667	2.8541	1.3009
$\hat{\lambda}_3$	8.7181	2.3988	CAIC	652.145	5.8251	1.9842

Table 8 The sample observation of progressive Type-II censoring schemes for real data when m = 10

m	Scheme		1	2	3	4	5	6	7	8	9	10
10	1	$x_{1i:m:n}$	2	8	16	16	18	19	22	24	25	25
		$x_{2[i:m:n]}$	2	8	75	16	18	19	14	24	9	14
	2	$x_{1i:m:n}$	2	8	22	25	28	40	49	63	64	76
		$x_{2[i:m:n]}$	2	8	14	14	28	40	49	18	15	64
	3	$x_{1i:m:n}$	2	8	16	18	19	22	24	25	26	26
		$x_{2[i:m:n]}$	2	8	16	18	19	14	24	14	20	48
	4	$x_{1i:m:n}$	2	16	19	24	26	26	27	42	44	49
		$x_{2[i:m:n]}$	2	16	19	24	20	48	47	3	30	49

6 Real Data Analysis

For illustrative purposes we have analyzed one data set to see how the proposed model works in practice. The football data set has been obtained from Meintanis [22]. Muhammed [11] discussed this data to analyze this model under complete sample. Now we estimate the parameters of BIW model by using MLE and Bayes methods. The Akaike information criterion (AIC), Bayesian information criterion (BIC), the consistent Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) are obtained (Tables 7, 8, 9, 10, 11).

The convergence of MCMC estimation of parameters can be showed as in Fig. 1 in cases of progressive Type-II censoring for the Real Data.

Table 9 The sample observation of progressive Type-II censoring schemes for real data when m = 20

m	Scheme	Count	1	2	3	4	5	6	7	8	9	10
20	1	$x_{1i:m:n}$	2	8	16	16	18	19	22	24	25	25
		$x_{2[i:m:n]}$	2	8	75	16	18	19	14	24	9	14
		count	11	12	13	14	15	16	17	18	19	20
		$x_{1i:m:n}$	26	26	27	28	34	36	39	40	41	42
		$x_{2[i:m:n]}$	20	48	47	28	34	52	39	40	3	42
		count	1	2	3	4	5	6	7	8	9	10
	2	$x_{1i:m:n}$	2	16	18	19	24	26	26	27	42	42
		$x_{2[i:m:n]}$	2	16	18	19	24	20	48	47	42	3
		count	11	12	13	14	15	16	17	18	19	20
		$x_{1i:m:n}$	44	44	49	55	63	64	66	66	72	76
		$x_{2[i:m:n]}$	13	30	49	11	18	15	85	62	72	64

Table 10 The sample observation of progressive Type-II censoring schemes for real data when m = 30

m	Scheme	Count	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
30	1	$x_{1i:m:n}$	2	8	16	16	18	19	22	24	25	25	26	26	27	28	34
		$x_{2[i:m:n]}$	2	8	75	16	18	19	14	24	9	14	20	48	47	28	34
		count	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
		$x_{1i:m:n}$	36	39	40	41	42	42	44	44	49	49	51	53	54	55	63
		$x_{2[i:m:n]}$	52	39	40	3	42	3	13	30	49	49	28	39	7	11	18
		count	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	2	$x_{1i:m:n}$	2	16	18	19	22	24	25	26	26	27	34	36	40	41	42
		$x_{2[i:m:n]}$	2	16	18	19	14	24	14	20	48	47	34	52	40	3	42
		count	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
		$x_{1i:m:n}$	42	44	44	49	51	53	54	55	63	64	66	66	72	76	82
		$x_{2[i:m:n]}$	3	13	30	49	28	39	7	11	18	15	85	62	72	64	48

7 Concluded Remarks

In this paper, the likelihood function was derived for the Marshall-Olkin family under progressive Type-II censoring. The derived likelihood function was applied on the Marshall–Olkin bivariate inverse Weibull distribution. Both Maximum likelihood and Bayesian estimation approaches are considered to estimate the unknown parameters of BIW distribution under progressive Type-II censored sample. The Bayesian estimates have more relative efficiency than MLE for most parameters of BIW distribution under progressive Type-II censored sample. Moreover, asymptotic and bootstrap confidence intervals for the unknown parameters are evaluated in both MLE and Bayesian Estimation. With regard to the interval estimation for parameters of BIW distribution, the Bootstrap CIs is more efficient than the traditional method ACI. The B-TCI has more

Table 11 Estimation for BIW Model under Progressive Type-II Censoring Schemes of the Real Data

m	Scheme		MLE		Bayes	
			Estimate	Std	Mean	SD
10	1	$\hat{\alpha}$	0.6241	0.1212	0.5704	0.1182
		$\hat{\lambda}_1$	3.1957	1.9186	3.6157	1.7989
		$\hat{\lambda}_2$	0.5269	0.8402	1.2008	0.3529
		$\hat{\lambda}_3$	7.5624	2.8980	5.1983	2.1703
		-LL	AIC	BIC	HQIC	CAIC
		79.0643	166.1286	172.5722	168.4003	167.3786
	2	$\hat{\alpha}$	0.6605	0.1491	0.8004	0.1573
		$\hat{\lambda}_1$	3.3102	1.8685	4.2495	1.3090
		$\hat{\lambda}_2$	5.5370	4.1922	2.8387	1.9395
		$\hat{\lambda}_3$	8.7172	3.7503	3.8541	1.3075
		-LL	AIC	BIC	HQIC	CAIC
		87.50663	183.0133	189.4569	185.2849	184.2633
	3	$\hat{\alpha}$	0.6502	0.1220	0.5901	0.1054
		$\hat{\lambda}_1$	3.2337	1.9487	3.7715	1.3961
		$\hat{\lambda}_2$	0.5763	0.5917	1.0055	0.5591
		$\hat{\lambda}_3$	8.3687	3.2286	5.2376	1.4343
		-LL	AIC	BIC	HQIC	CAIC
		78.5679	165.1357	171.5794	167.4074	166.3857
	4	$\hat{\alpha}$	0.66803	0.11909	0.6080141	0.1089202
		$\hat{\lambda}_1$	3.60042	2.09943	2.6300722	1.1712423
		$\hat{\lambda}_2$	1.27459	0.93147	1.7052147	0.8044416
		$\hat{\lambda}_3$	5.85135	2.29101	4.3734948	1.6010278
		-LL	AIC	BIC	HQIC	CAIC
		83.93255	175.8651	182.3088	178.1368	177.1151
20	1	$\hat{\alpha}$	0.7307	0.0960	0.6687	0.0643
		$\hat{\lambda}_1$	3.9927	1.8441	3.8146	1.2695
		$\hat{\lambda}_2$	1.6564	0.8876	1.5367	0.7599
		$\hat{\lambda}_3$	8.1930	2.5436	5.9138	1.9993
		-LL	AIC	BIC	HQIC	CAIC
		160.8097	329.6194	336.0631	331.8911	330.8694
	2	$\hat{\alpha}$	0.8385	0.1021	0.7409	0.0961
		$\hat{\lambda}_1$	6.9245	2.8117	5.1895	1.6411
		$\hat{\lambda}_2$	2.4243	1.4244	1.9029	1.0808
		$\hat{\lambda}_3$	7.4058	2.5720	4.3486	1.9854
		-LL	AIC	BIC	HQIC	CAIC
		173.4932	354.9864	361.43	357.2581	356.2364
30	1	$\hat{\alpha}$	0.8645	0.0880	0.7961	0.0523
		$\hat{\lambda}_1$	8.2886	1.9074	6.3885	1.6758
		$\hat{\lambda}_2$	1.9650	1.0123	2.3385	1.0029

Table 11 continued

m	Scheme		MLE		Bayes	
			Estimate	Std	Mean	SD
		$\hat{\lambda}_3$	8.2139	2.2812	6.0597	2.1638
		-LL	AIC	BIC	HQIC	CAIC
		246.8935	501.787	508.2307	504.0587	503.037
	2	$\hat{\alpha}$	0.8946	0.0860	0.7332	0.0855
		$\hat{\lambda}_1$	10.7332	3.5513	4.5916	1.7277
		$\hat{\lambda}_2$	2.8449	1.4192	2.0795	1.1038
		$\hat{\lambda}_3$	7.9793	2.3670	4.9554	1.6538
		-LL	AIC	BIC	HQIC	CAIC
		263.5551	535.1102	541.5539	537.3819	536.3602

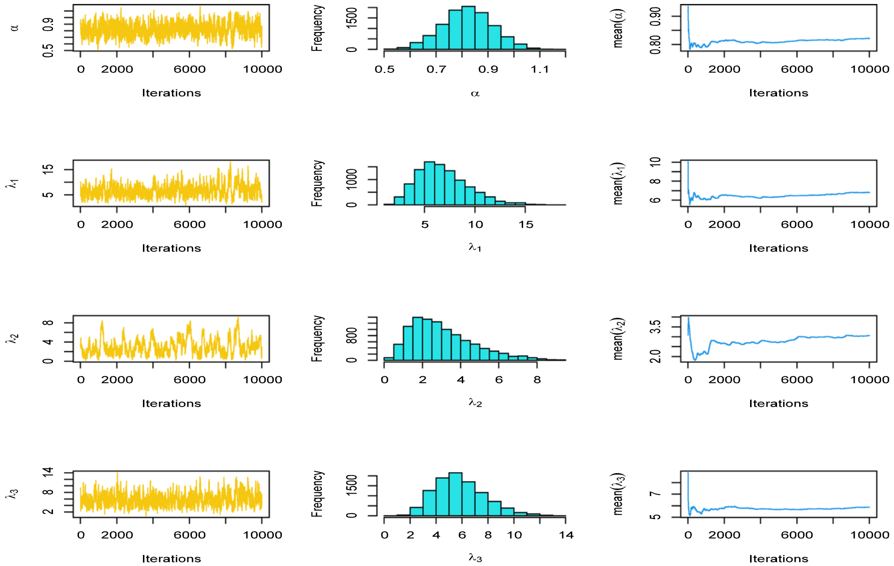

Fig. 1 Convergence of MCMC estimation of parameters for real data

efficiency than B-PCI for most parameters. Moreover, a simulation studies and a real data set are carried out for illustrative purpose.

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Availability of data Data is attached to this article.

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