Sankhyā : The Indian Journal of Statistics
2023, Volume 85-A, Part 2, pp. 1205-1242

Bayesian and Non-Bayesian Estimation for the Parameter of Bivariate Generalized Rayleigh Distribution Based on Clayton Copula under Progressive Type-II Censoring with Random Removal

El-Sayed A. El-Sherpieny, Ehab M. Almetwally and Hiba Z. Muhammed
Delta University of Science and Technology, Al-Mansoura, Egypt

Abstract

In this paper, the bivariate generalized Rayleigh distribution is introduced based on Clayton copula and denoted as (Clayton-BGR). The likelihood function for progressive Type-II censoring scheme with random removal is derived and applied on the Clayton-BGR distribution. Bayesian and non -Bayesian estimation methods based on progressive Type-II censoring have been discussed. Asymptotic confidence intervals and bootstrap confidence intervals for the unknown parameters are obtained. Also, a simulation study has been conducted to compare the performances between censoring schemes. Also, two real data sets are analyzed to investigate the models and useful results are obtained for illustrative purposes.

Keywords. Bivariate generalized Rayleigh, Clayton **copula**, **Maximum likelihood estimation**, Bayesian estimation, Progressive type-II censoring, Bootstrap confidence interval

1 Introduction

There have been several attempts made in the last few years to introduce bivariate Rayleigh distribution. For example, Abdel-Hady (2013) introduced Marshall and Olkin s bivariate model to the generalized bivariate Rayleigh distribution. Sarhan (2019) introduced bivariate generalized Rayleigh (BGR) distribution based on Marshall-Olkin (Shock model), the maximum likelihood and Bayesian estimation methods were applied to estimate the unknown parameters of this distribution. El-Sherpieny and Almetwally (2019) discussed two different estimation methods (maximum likelihood estimation and inference function for margins) for the unknown parameters of bivariate generalized Rayleigh distribution.

Bivariate Rayleigh distribution has more applications, where it applied to censored data, soccer champion s league data, stress-strength component, missing data, economic data to investigate investments in heterogeneous assets, in clinical researches to study the mortality risk of lung cancer for men versus women, in industrial studies to calculate the lifetime of different types of steels, etc. Many authors studied the bivariate Rayleigh distribution as Pak et al. (2014), Elaal et al. (2016), Elaala and Baharith (2018), El-Sherpieny et al. (2018), Shubhashree (2019), Bahari et al. (2020), and El-Morshedy et al. (2020). For more information see Kotz et al. (2004).

The copula is a convenient approach to describe a bivariate distribution with a dependence structure. Nelsen (2006) showed the definition of copulas as following; copula is a function that joins bivariate distribution functions with uniform [0, 1] margins. Sklar (1973) introduced the joint probability density function (pdf) and joint cumulative distribution function (cdf) for the two-dimension copula as follows, He considered the two random variables X_1 and X_2, with distribution functions $F_1(x_1)$ and $F_2(x_2)$ respectively, then the joint cdf and joint pdf for bivariate copula are respectively given as $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$, where C is denoted the cdf of copula and c is denoted the pdf of the copula.

$$f(x_1, x_2) = f_1(x_1)f_2(x_2)c(F_1(x_1), F_2(x_2)).$$

In Archimedian copulas, there are three copulas in common use: The Clayton, Frank, and Gumbel. Nadarajah et al. (2018) discussed Clayton in a general form as

$$C(u_1, u_2, ..., u_p) = \left[\sum_{i=1}^{p} u_i^{-\alpha} - p + 1\right]^{-1/\alpha}, \alpha > 0, p \text{ is a number of variable}$$

In this paper, we will discuss the 2-diminutions of Clayton copula, which is an asymmetric Archimedean copula, exhibiting greater dependence in the negative tail than in the positive, that have been introduced by Clayton (1978). The joint cdf of Clayton copula is given by:

$$C(u, v) = \left(\mathrm{u}^{-\theta} + \mathrm{v}^{-\theta} - 1\right)^{-1/\theta}, \tag{1.1}$$

where $u, v \in [0, 1]$ and the joint pdf of Clayton copula is given by:

$$c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v} = (\theta + 1)(uv)^{-\theta-1}\left(\mathrm{u}^{-\theta} + \mathrm{v}^{-\theta} - 1\right)^{\frac{-2\theta-1}{\theta}}. \tag{1.2}$$

If a sample is drawn from a complete-time, but either the last observations or the first are unknown, this case is called the single censored observation. The classical Type-I and Type-II censored samples do not have the flexibility of allowing removal of units at points other than the terminal points of the experiment. A more general censoring scheme is called a progressive censoring scheme. For more information see Balakrishnan and Aggarwala (2000), Balakrishnan and Cramer (2014). In the progressively censored sample, the items are removed from the life test throughout the test and during the various stages of the test, where some of the survivors are withdrawn from the above observations. Sample specimens, which continue after each stage from the censoring stages, can be observed until failure or until a subsequent stage of censoring.

In the Bivariate censoring sample, the likelihood function for bivariate distribution based on Type-II censored has been introduced by Balakrishnan and Kim (2004). The likelihood function based on a Progressive Type-II censoring scheme for bivariate distribution has been introduced by Balakrishnan and Kim (2005a, 2005b). The Type-I progressive interval censoring schemes of the Marshall-Olkin bivariate Weibull (MOBW) distribution has been introduced by Bai et al. (2018) and they discussed the confidence intervals depend on percentile bootstrap of the unknown parameters. Aly et al. (2020) derived the likelihood function for the progressive Type-I censoring and applied it on the Marshall-Olkin bivariate Kumaraswamy distribution.

Many authors have discussed applications under censored samples using different bivariate distributions. Maximum likelihood and Bayesian estimation of the unknown parameters for bivariate generalized exponential distribution under Type-II censoring scheme have been discussed by Kim et al. (2016). Parameter estimation of bivariate models under some censoring scheme has been discussed by Almetwally (2019). Dependent competing risk models from Marshall-Olkin bivariate Weibull distribution under Type-I progressive interval censoring scheme have been considered by Bai et al. (2019). Parameter estimation for the bivariate Weibull distribution based on Farlie–Gumbel–Morgenstern copula under progressive Type-II censoring sample has been introduced by El-Sherpieny et al. (2019), and Almetwally et al. (2020). Parameter estimation of bivariate Fréchet distribution based on Farlie-Gumbel-Morgenstern and Ali-Mikhail-Haq copulas has been introduced by Almetwally and Muhammed (2020). The maximum likelihood and Bayesian estimation of the unknown parameters for the Marshall-Olkin bivariate exponential distribution under generalized progressive hybrid censoring scheme with partially observed failure causes have been established by Liang et al. (2019). The maximum likelihood and Bayesian methods for estimating the bivariate Burr X-G family parameters with different distributions based on Type-II censored data have been discussed by El-

Morshedy et al. (2020). Inference of the unknown parameters for Marshall-Olkin bivariate Kumaraswamy distribution under a generalized progressive hybrid censoring has been studied by Wang et al. (2020).

Study for progressive Type-II censored with random removal is of considerable interest and practical significance in many practical situations, we develop the statistical inference of the unknown parameters in this paper when the failure times follow the Clayton-BGR distribution. Under maximum likelihood estimates (MLE) and approximate confidence intervals (ACI) are proposed for the unknown parameters, Bayesian estimates and the highest posterior density (HPD) credible intervals are also constructed by using the Monte-Carlo Markov Chain (MCMC) sampling method, and Bootstrap confidence intervals for both methods are obtained.

The rest of this paper is organized as follows. Model description and notation are given in Section 2. The maximum likelihood estimation and approximate confidence intervals are presented in Section 3. Bayesian estimates and the highest posterior density (HPD) credible intervals are presented in Section 4. Section 5 uses the Bootstrap methods to construct the confidence intervals of unknown parameters. Section 6 provides the Monte-Carlo simulation results and data analysis. In Section 7 two real data sets are introduced and analyzed to investigate the model, and some concluding remarks are given in Section 8.

2 Model Description and Notation

2.1 Clayton-BGR Distribution A random variable X has the two-parameter GR distribution with shape and scale parameters α, λ respectively, the cdf of GR distribution is given by $F(x;\alpha,\lambda)=\left(1-e^{-(\lambda x)^2}\right)^{\alpha}; x>0$, and the pdf of GR distribution is given by $f(x;\alpha,\lambda)=2\alpha\lambda^2 x e^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\alpha-1}; \alpha,\lambda>0$. The cdf, pdf, and hazard function for Clayton-BGR distribution is obtained as follows:

$$F(x_1,x_2)=\left[\left(\left(1-e^{-(\lambda_1 x_1)^2}\right)^{-\theta\alpha_1}+\left(1-e^{-(\lambda_2 x_2)^2}\right)^{-\theta\alpha_2}-1\right)^{-1/\theta}\right], \tag{2.1}$$

$$f(x_1,x_2) = 4\alpha_1\lambda_1^2 x_1 e^{-(\lambda_1 x_1)^2}\left(1-e^{-(\lambda_1 x_1)^2}\right)^{\alpha_1-1}\alpha_2\lambda_2^2 x_2 e^{-(\lambda_2 x_2)^2}\left(1-e^{-(\lambda_2 x_2)^2}\right)^{\alpha_2-1},$$
$$(\theta+1)\left(\left(1-e^{-(\lambda_1 x_1)^2}\right)^{\alpha_1}\left(1-e^{-(\lambda_2 x_2)^2}\right)^{\alpha_2}\right)^{-\theta-1}$$
$$\left(\left(1-e^{-(\lambda_1 x_1)^2}\right)^{-\theta\alpha_1}+\left(1-e^{-(\lambda_2 x_2)^2}\right)^{-\theta\alpha_2}-1\right)^{\frac{-2\theta-1}{\theta}}, \tag{2.2}$$

and

$$h(x_1,x_2) = \frac{f(x_1,x_2)}{F(x_1)+F(x_2)-1+C(1-F(x_1),1-F(x_2))}, \tag{2.3}$$

respectively, where α_1, λ_1, α_2, λ_2, $\theta > 0$ and x_1, $x_2 > 0$.

Figure 1 shows the 3-dimensions plot for the pdf and hazard function of Clayton-BGR distribution with different parameters values. Further, the HRF of the Clayton-BGR distribution has some important shapes, increasing, decreasing, upside-down-upside and bathtub.

2.2 Progressive Type-II Censoring with Random Removal Suppose we set $(x_{11:n}, x_{21:n})$, $(x_{12:n}, x_{22:n})$, ..., $(x_{1n:n}, x_{2n:n})$ as a random sample from a bivariate distribution with cdf $F(x_1, x_2)$ and pdf $f(x_1, x_2)$. Let x_{1i}, $i = 1, \ldots, n$ be arranged in ascending order of magnitude, $x_{11:n} < x_{12:n} < \cdots < x_{1n:n}$ and let $x_{2[i:n]}$ is termed the concomitant of the i^{th} order statistic. These concomitants are of interest in the selection of bivariate and prediction problems based on the ranks of the X_1. For more information about concomitants of order statistics see David and Nagaraja (1998).

This censoring scheme can be described as follows: Set n independent observations placed on a life testing and the progressive censoring scheme R_i, $i = 1, 2, \ldots, m$. Then, we shall denote the m completely observed failure times by $X_{1\,i:m:n}^{(R_i)}$, $i = 1, 2, \ldots, m$. At the time of the first failure, $(x_{11:\mathrm{m:n}}, x_{2[1:\mathrm{m:n}]})$, $R_1 \sim binomal(n-m, p)$ units are randomly removed from the remaining $(n-1)$ surviving items. In the time of the second failure, $(x_{12:\mathrm{m:n}}, x_{2[2:\mathrm{m:n}]})$, $R_2 \sim binomal(n-m-R_1, p)$ units of the remaining $n-2-R_1$ units are randomly removed and so on the test continues until the m^{th} failure at any time, all the remaining $n-m-R_1-R_2-\ldots-R_{m-1}$ units are removed. Suppose that an individual unit from X_1 being removed from the test is independent of the others but with the same removal probability p. Then, the number of units removed at each

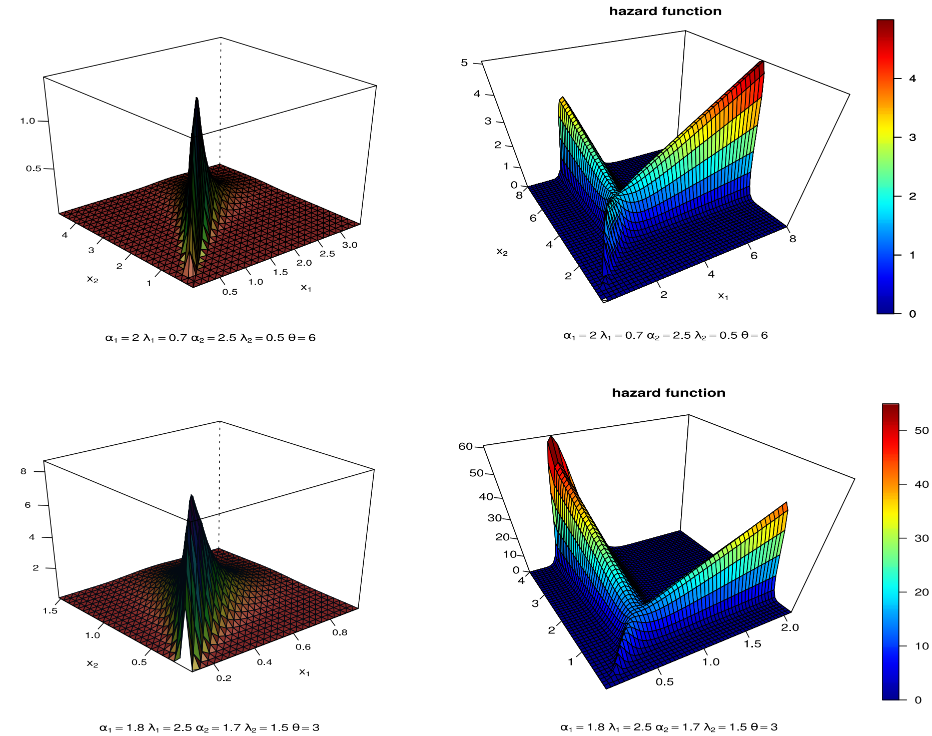

Fig. 1: The pdf and hazard function of Clayton-BGR Distribution

failure time follows a binomial distribution which is, for =2, 3, ..., $m-1$ $R_i \sim binomal\left(n-m-\sum_{j=1}^{i-1} R_j, p\right)$, and $R_m = n-m-\sum_{j=1}^{m-1} R_j$.

The number of units removed at each failure time assumed to be from a binomial distribution with the following probability mass function

$$P(R_1 = r_1) = \binom{n-m}{r_1} p^{r_1}(1-p)^{n-m-r_1} \tag{2.4}$$

While, for $i=2, 3, ..., m-1$

$$P(R_i = r_i | R_{i-1} = r_{i-1}, ..., R_1 = r_1) = \binom{n-m-\sum_{j=1}^{i-1} r_j}{r_i} p^{r_i}(1-p)^{n-m-\sum_{j=1}^{i-1} r_i} \tag{2.5}$$

where $0 \le r_i \le n-m-\sum_{j=1}^{i-1} r_i$. Furthermore, suppose that R_i is independent of $X_{1i:m:n}$ for all i. Then the full likelihood function can be found as

$$L\left(x_{1\,i:m:n}, x_{2[i:m:n]}, \Psi\right) = L_1\left(x_{1\,i:m:n}, x_{2[i:m:n]}, \Psi\right) P(\boldsymbol{R} = \boldsymbol{r})$$

According to Balakrishnan and Kim (2005a) the likelihood function $L_1(x_{1i:m:n}, x_{2[i:m:n]}, \Psi)$ in case of R removal is defined as:

$$L_1\left(x_{1\,i:m:n}, x_{2\,[i:m:n]}, \Psi\right) = c\prod_{i=1}^{m} f_{Y_1,Y_2}\left(x_{1\,i:m:n}, x_{2\,[i:m:n]}\right)\left(1-F_{Y_1}(x_{1\,i:m:n})\right)^{R_i} \quad (2.6)$$

where c is a constant which does not depend on vector parameter Ψ, n is fixed, while $m \le n$ is fixed as the number of complete failures to be observed and removal probability p is a fixed constant which depends on the experience of the experimenter. The likelihood function of the binomial distribution is

$P(\boldsymbol{R}=\boldsymbol{r}) = P(R_1=r_1, R_2=r_2, \ldots, R_{m-1}=r_{m-1})$, that is,

$$P(\boldsymbol{R}=\boldsymbol{r}) = \frac{(n-m)!}{\left(n-m-\sum_{j=1}^{m-1} r_j\right)!\prod_{j=1}^{m-1} r_j} p^{\sum_{j=1}^{m-1} r_j}(1-p)^{(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j} \quad (2.7)$$

We note that, since $L_1(x_{1i:m:n}, x_{2[i:m:n]}, \Psi)$ does not involve the binomial parameter p the MLE of p can be derived by maximizing Eq. (2.7) directly. Hence the MLE of $p$is obtained by solving the following equation

$$\frac{\partial \ln L}{\partial p} = \frac{\sum_{i=1}^{m-1} r_i}{p} - \frac{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}{1-p}. \text{Hence, } \widehat{p} = \frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i-1)r_i}.$$

3 MLE of the Clayton-BGR Distribution under Progressive Type-II Censoring

The MLE method is the most frequently used method of parameter estimation under a progressive Type-II censoring scheme. Let$\chi_{1i} = x_{1_{i:m:n}}$, $\chi_{2[i]} = x_{2_{[i:m:n]}}$ and $\Psi = (\alpha_1, \lambda_1, \alpha_2, \lambda_2, \theta)$. The likelihood function of the Clayton-BGR distribution under the progressive Type-II censoring scheme with random removal is given by using Eq. (2.2) and Eq. (2.6) as follows

$$L_1\left(\chi_{1i},\chi_{2[i]},\Psi\right)=\left(4\alpha_1\lambda_1^2\alpha_2\lambda_2^2(\theta+1)\right)^m e^{-\lambda_1^2\sum_{i=1}^{m}\chi_{1i}^2-\lambda_2^2\sum_{i=1}^{m}\chi_{2[i]}^2}\prod_{i=1}^{m}\chi_{1i}\chi_{2[i]}(\varphi(\chi_{1i},\lambda_1))^{-(1+\alpha_1\theta)}$$
$$\prod_{i=1}^{m}\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{-(1+\alpha_2\theta)}\left((\varphi(\chi_{1i},\lambda_1))^{\alpha_1\theta}+\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{\alpha_2\theta}-1\right)^{\frac{-2\theta-1}{\theta}} \tag{3.1}$$
$$\prod_{i=1}^{m}\left[1-(\varphi(\chi_{1i},\lambda_1))^{\alpha_1}\right]^{R_i}$$

Where $\varphi(\chi_{1i}\ ,\lambda_1)=1-e^{-(\lambda_1\ \chi_{1i})\ ^2}and\Im\left(\chi_{2[i]}\ ,\lambda_2\right)=1-e^{-(\lambda_2\ \chi_{2[i]})\ ^2}$.

The log-likelihood function of Clayton-BGR distribution under progressive Type-II censoring scheme can be written as

$$l(\Psi)=2m(\ln(\lambda_1)+ln(\lambda_2))+m(\ln(4)+\ln(\alpha_1)+\ln(\alpha_2)+\ln(\theta+1)) \tag{3.2}$$
$$+\sum_{i=1}^{m}\left(\ln(\chi_{1i})+\ln\left(\chi_{2[i]}\right)\right)-\sum_{i=1}^{m}(\lambda_1\chi_{1i})^2-\sum_{i=1}^{m}(\lambda_2\chi_{2i})^2-(1+\theta\alpha_1)\sum_{i=1}^{m}\ln(\varphi(\chi_{1i},\lambda_1))$$
$$-(1+\theta\alpha_2)\sum_{i=1}^{m}\ln\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)-\frac{2\theta+1}{\theta}\sum_{i=1}^{m}\ln\left((\varphi(\chi_{1i},\lambda_1))^{-\theta\alpha_1}+\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{-\theta\alpha_2}-1\right)$$
$$+\sum_{i=1}^{m}R_i\ln(1-(\varphi(\chi_{1i},\lambda_1))^{\alpha_1})$$

where Ψ is a vector of parameters.

The partial derivatives of Eq. (3.2) for the unknown parameters can be obtained as follows

$$\frac{\partial l(\Psi)}{\partial\alpha_1}=\frac{m}{\alpha_1}-\theta\sum_{i=1}^{m}\ln(\varphi(\chi_{1i},\lambda_1))-\sum_{i=1}^{m}R_i\frac{(\varphi(\chi_{1i},\lambda_1))^{\alpha_1}\ln(\varphi(\chi_{1i},\lambda_1))}{1-(\varphi(\chi_{1i},\lambda_1))^{\alpha_1}}$$
$$-\frac{2\theta+1}{\theta}\sum_{i=1}^{m}\frac{-\theta(\varphi(\chi_{1i},\lambda_1))^{-\theta\alpha_1}\ln(\varphi(\chi_{1i},\lambda_1))}{(\varphi(\chi_{1i},\lambda_1))^{-\theta\alpha_1}+\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{-\theta\alpha_2}-1},\ \frac{\partial l(\Psi)}{\partial\lambda_1}=\frac{2m}{\lambda_1}+2\lambda_1\sum_{i=1}^{m}\chi_{1i}^2-(1+\theta\alpha_1)\sum_{i=1}^{m}\frac{2\lambda_1\chi_{1i}^2e^{-(\lambda_1\chi_{1i})^2}}{\varphi(\chi_{1i},\lambda_1)}$$
$$-\sum_{i=1}^{m}R_i\frac{2\alpha_1\lambda_1\chi_{1i}^2e^{-(\lambda_1\chi_{1i})^2}(\varphi(\chi_{1i},\lambda_1))^{\alpha_1-1}}{1-(\varphi(\chi_{1i},\lambda_1))^{\alpha_1}}-\frac{2\theta+1}{\theta}\sum_{i=1}^{m}\frac{-\theta\alpha_1 2\lambda_1\chi_{1i}^2(\varphi(\chi_{1i},\lambda_1))^{-\theta\alpha_1-1}}{(\varphi(\chi_{1i},\lambda_1))^{-\theta\alpha_1}+\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{-\theta\alpha_2}-1},$$

$$\frac{\partial l(\Psi)}{\partial\alpha_2}=\frac{m}{\alpha_2}-\theta\sum_{i=1}^{m}\ln\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)-\frac{2\theta+1}{\theta}\sum_{i=1}^{m}\frac{-\theta\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{-\theta\alpha_2}\ln\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)}{(\varphi(\chi_{1i},\lambda_1))^{-\theta\alpha_1}+\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{-\theta\alpha_2}-1},$$

$$\frac{\partial l(\Psi)}{\partial\lambda_2}=\frac{2m}{\lambda_2}+2\lambda_2\sum_{i=1}^{m}\chi_{2[i]}^2-(1+\theta\alpha_2)\sum_{i=1}^{m}\frac{2\lambda_2\chi_{2[i]}^2e^{-\left(\lambda_2\chi_{2[i]}\right)^2}}{\Im\left(\chi_{2[i]},\lambda_2\right)}-\frac{2\theta+1}{\theta}\sum_{i=1}^{m}\frac{-\theta\alpha_2 2\lambda_2\chi_{2[i]}^2\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{-\theta\alpha_2-1}}{(\varphi(\chi_{1i},\lambda_1))^{-\theta\alpha_1}+\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{-\theta\alpha_2}-1},$$

and

$$\frac{\partial l(\Psi)}{\partial\theta}=\frac{m}{\theta+1}-\alpha_1\sum_{i=1}^{m}\ln(\varphi(\chi_{1i},\lambda_1))-\alpha_2\sum_{i=1}^{m}\ln\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)$$
$$+\frac{1}{\theta^2}\sum_{i=1}^{m}\ln\left((\varphi(\chi_{1i},\lambda_1))^{-\theta\alpha_1}+\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{-\theta\alpha_2}-1\right)$$
$$-\frac{2\theta+1}{\theta}\sum_{i=1}^{m}\frac{-\alpha_1(\varphi(\chi_{1i},\lambda_1))^{-\theta\alpha_1}\ln(\varphi(\chi_{1i},\lambda_1))-\alpha_2\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{-\theta\alpha_2}\ln\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)}{(\varphi(\chi_{1i},\lambda_1))^{-\theta\alpha_1}+\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{-\theta\alpha_2}-1},$$

The MLE of $\widehat{\Psi}$ of the Clayton-BGR parameters under the progressive Type-II censoring scheme is the solution of the previous non-linear equations after setting them equal to zero. These equations are very difficult to be solved, so a nonlinear optimization algorithm as Newton–Raphson method is used. Furthermore, the asymptotic confidence interval (ACI) of parameters for Clayton-BGR under the progressive Type-II censoring scheme can be approximated by numerically inverting Fisher s information matrix. Thus, the approximate confidence interval $100(1-\gamma)\%$ for Ψ can be, respectively, easily obtained by

$$\widehat{\alpha}_1\pm Z_{\gamma/2}\sqrt{Var\left(\widehat{\alpha}_1^{1:L}\right)},\widehat{\lambda}_1\pm Z_{\gamma/2}\sqrt{Var\left(\widehat{\lambda}_1^{1:L}\right)},\widehat{\alpha}_2\pm Z_{\gamma/2}\sqrt{Var\left(\widehat{\alpha}_2^{1:L}\right)},\widehat{\lambda}_2$$
$$\pm Z_{\gamma/2}\sqrt{Var\left(\widehat{\lambda}_2^{1:L}\right)}$$

and $\widehat{\theta}\pm Z_{\gamma/2}\sqrt{Var\left(\widehat{\theta}^{1:L}\right)}$, where L is the length of simulation generated.

4 Bayesian Estimation

In this section, we consider the Bayesian estimation for the unknown parameters for Clayton-BGR distribution under progressive Type-II censoring scheme under the assumption that the random variables $\Psi=(\alpha_1,\lambda_1,\alpha_2,\lambda_2,\theta)$ have an independent gamma prior distribution. Assumed that $\alpha_1\sim Gamma(a_1,b_1)$, $\lambda_1\sim Gamma(a_2,b_2)$, $\alpha_2\sim Gamma(a_3,b_3)$, $\lambda_2\sim Gamma(a_4,b_4)$, and $\theta\sim Gamma(a_5,b_5)$ then, the joint prior density of Ψ can be written as

$$\pi(\Psi)\propto\alpha_1^{a_1-1}e^{-\alpha_1 b_1}\lambda_1^{a_2-1}e^{-\lambda_1 b_2}\alpha_2^{a_3-1}e^{-\alpha_2 b_3}\lambda_2^{a_4-1}e^{-\lambda_2 b_4}\theta^{a_5-1}e^{-\theta b_5} \tag{4.1}$$

The posterior likelihood can be represented to be proportional to the product of the likelihood function given in Eq. (3.1) and the joint prior s densities given in Eq. (4.1). That is,

$$\Pi\left(\Psi|\chi_{1i},\chi_{2[i]}\right)\propto L_1\left(\chi_{1i},\chi_{2[i]},\Psi\right)\pi(\Psi)$$

Then, the joint posterior density of Ψ is

$$\begin{aligned}\Pi\left(\Psi|\chi_{1i},\chi_{2[i]}\right)\propto&\left(\alpha_1\lambda_1^2\alpha_2\lambda_2^2(\theta+1)\right)^m e^{-\lambda_1^2\sum_{i=1}^m\chi_{1i}^2-\lambda_2^2\sum_{i=1}^m\chi_{2[i]}^2-\lambda_1 b_2-\lambda_2 b_4}\prod_{i=1}^m\left(\varphi(\chi_{1i},\lambda_1)\right)^{-(1+\alpha_1\theta)}\\&\prod_{i=1}^m\chi_{1i}\chi_{2[i]}\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{-(1+\alpha_2\theta)}\left(\left(\varphi(\chi_{1i},\lambda_1)\right)^{\alpha_1\theta}+\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{\alpha_2\theta}-1\right)^{\frac{-2\theta-1}{\theta}}\\&\theta^{a_5-1}e^{-b_5\theta}\alpha_1^{a_1-1}\alpha_2^{a_3-1}\lambda_1^{a_2-1}\lambda_2^{a_4-1}e^{-b_1\alpha_1}e^{-b_3\alpha_2}\prod_{i=1}^m\left[1-\left(\varphi(\chi_{1i},\lambda_1)\right)^{\alpha_1}\right]^{R_i}\end{aligned}\tag{4.2}$$

For the Clayton-BGR distribution based on progressive Type-II censoring, the full conditional posterior distributions of the parameters are given by

$$\begin{aligned}\Pi\left(\alpha_1|\lambda_1,\alpha_2,\lambda_2,\theta,\chi_{1i},\chi_{2[i]}\right)\propto&\,\alpha_1^{a_1+m-1}e^{-b_1\alpha_1}\prod_{i=1}^m\left(\varphi(\chi_{1i},\lambda_1)\right)^{-(1+\alpha_1\theta)}\prod_{i=1}^m\left[1-\left(\varphi(\chi_{1i},\lambda_1)\right)^{\alpha_1}\right]^{R_i}\\&\prod_{i=1}^m\left(\left(\varphi(\chi_{1i},\lambda_1)\right)^{\alpha_1\theta}+\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{\alpha_2\theta}-1\right)^{\frac{-2\theta-1}{\theta}},\end{aligned}$$

$$\begin{aligned}\Pi\left(\lambda_1|\alpha_1,\alpha_2,\lambda_2,\theta,\chi_{1i},\chi_{2[i]}\right)\propto&\,\lambda_1^{2m+a_2-1}e^{-\lambda_1^2\sum_{i=1}^m\chi_{1i}^2-\lambda_1 b_2}\prod_{i=1}^m\left(\varphi(\chi_{1i},\lambda_1)\right)^{-(1+\alpha_1\theta)}\\&\prod_{i=1}^m\left(\left(\varphi(\chi_{1i},\lambda_1)\right)^{\alpha_1\theta}+\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{\alpha_2\theta}-1\right)^{\frac{-2\theta-1}{\theta}}\prod_{i=1}^m\left[1-\left(\varphi(\chi_{1i},\lambda_1)\right)^{\alpha_1}\right]^{R_i},\end{aligned}$$

$$\begin{aligned}\Pi\left(\alpha_2|\alpha_1,\lambda_1,\lambda_2,\theta,\chi_{1i},\chi_{2[i]}\right)\propto&\,\alpha_2^{m+a_3-1}e^{-b_3\alpha_2}\prod_{i=1}^m\left(\left(\varphi(\chi_{1i},\lambda_1)\right)^{\alpha_1\theta}+\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{\alpha_2\theta}-1\right)^{\frac{-2\theta-1}{\theta}}\\&\prod_{i=1}^m\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{-(1+\alpha_2\theta)}\end{aligned}$$

$$\begin{aligned}\Pi\left(\lambda_2|\alpha_1,\lambda_1,\alpha_2,\theta,\chi_{1i},\chi_{2[i]}\right)\propto&\,e^{-\lambda_2^2\sum_{i=1}^m\chi_{2[i]}^2-\lambda_2 b_4}\prod_{i=1}^m\left(\left(\varphi(\chi_{1i},\lambda_1)\right)^{\alpha_1\theta}+\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{\alpha_2\theta}-1\right)^{\frac{-2\theta-1}{\theta}}\\&\lambda_2^{2m+a_4-1}\prod_{i=1}^m\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{-(1+\alpha_2\theta)}\end{aligned}$$

and

$$\begin{aligned}\Pi\left(\theta|\alpha_1,\lambda_1,\alpha_2,\lambda_2,\chi_{1i},\chi_{2[i]}\right)\propto&\left((\theta+1)\right)^m\theta^{a_5-1}e^{-b_5\theta}\prod_{i=1}^m\left(\varphi(\chi_{1i},\lambda_1)\right)^{-(1+\alpha_1\theta)}\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{-(1+\alpha_2\theta)}\\&\prod_{i=1}^m\left(\left(\varphi(\chi_{1i},\lambda_1)\right)^{\alpha_1\theta}+\left(\Im\left(\chi_{2[i]},\lambda_2\right)\right)^{\alpha_2\theta}-1\right)^{\frac{-2\theta-1}{\theta}}.\end{aligned}$$

We can use the squared error loss function (SELF) to obtain Bayesian estimators of the parameters Ψ, that is defined by $\ell\left(\widetilde{\Psi},\Psi\right)=\left(\widetilde{\Psi}-\Psi\right)^2$. The usual estimator of the parameters under the SELF is the posterior mean. Therefore, the Bayesian estimators of the parameters Ψ under SELF *are* $\widetilde{\Psi}$ as follows:

$$\widetilde{\alpha}_1 = \int_0^\infty \alpha_1 \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \Pi\left(\Psi|\, \chi_{1i}, \chi_{2[i]}\right) d\lambda_1\, d\alpha_2\, d\lambda_2\, d\theta d\alpha_1,$$
$$\widetilde{\lambda}_1 = \int_0^\infty \lambda_1 \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \Pi\left(\Psi|\, \chi_{1i}, \chi_{2[i]}\right) d\alpha_1\, d\alpha_2\, d\lambda_2\, d\theta d\lambda_1,$$
$$\widetilde{\alpha}_2 = \int_0^\infty \alpha_2 \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \Pi\left(\Psi|\, \chi_{1i}, \chi_{2[i]}\right) d\alpha_1\, d\lambda_1\, d\lambda_2\, d\theta d\alpha_2,$$
$$\widetilde{\lambda}_2 = \int_0^\infty \lambda_2 \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \Pi\left(\Psi|\, \chi_{1i}, \chi_{2[i]}\right) d\alpha_1\, d\lambda_1\, d\alpha_2\, d\theta d\lambda_2,$$

and

$$\widetilde{\theta} = \int_0^\infty \theta \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \Pi\left(\Psi|\, \chi_{1i}, \chi_{2[i]}\right) d\alpha_1\, d\lambda_1\, d\alpha_2\, d\lambda_2\, d\theta.$$

These integrals are very hard to be solved analytically, so the MCMC approach will be used. An important sub-class of the MCMC techniques is Gibbs sampling and more general Metropolis within Gibbs samplers. Metropolis et al. (1953) and Hastings (1970) were first introduced this algorithm. In our simulation study presented in Section 6, the Markov chain Monte Carlo (MCMC) procedure is used to generate the fully conditional posterior distributions of Ψ. We set the number of periods in the MCMC process to be $N = 10{,}000$. The Metropolis-Hastings (MH) algorithm generates a sequence of draws from Clayton-BGR distribution based on progressive Type-II censoring as follows:

1. Start with any initial values $\left(\Psi_l^{\langle 0\rangle}\right)$; $l = 1, \ldots, 5$satisfying$\pi\left(\Psi_l^{\langle 0\rangle}\right) > 0$.
2. Using the initial value, sample a candidate point $(\Psi^*)$from the proposal$q(\Psi^*)$.
3. For $t = 0$ to N, given the candidate point (Θ^*), calculate the acceptance probability

$$A_l = \min\left(1, \frac{L_1\left(\Psi|\chi_{1i}, \chi_{2[i]}\right)\pi\left(\Psi_l^*\right)}{L_1\left(\Psi|\chi_{1i}, \chi_{2[i]}\right)\pi(\Psi_l)} \frac{q(\Psi_l)}{q\left(\Psi_l^*\right)}\right)$$

4. Draw a value of u from the uniform (0,1) distribution, $\Psi_l^{\langle t+1\rangle} = \begin{cases} \Psi_l^* & if \quad u \le A_l \\ \Psi_l^{\langle t\rangle} & if \quad u \le A_l \end{cases}$.
5. Repeat steps 2–5 $(t+1)$ times until we get$N$ draws.
6. The Bayes estimate ofΨ_l, for squared error loss function is $\sum_{t=1}^{N} \frac{\left(\Psi_l^{\langle t-1\rangle}\right)_t}{N}$.
7. Repeat these steps$l$ to get a Bayesian estimateΨ_l.

Hyper-parameter elicitation: The elicitation of the hyper-parameters will rely on the informative priors. These informative priors will be obtained from the maximum likelihood estimates for $(\alpha_1, \lambda_1, \alpha_2, \lambda_2, \theta)$ by equating the mean and variance of $\left(\widehat{\alpha}_1{}^j, \widehat{\lambda}_1{}^j, \widehat{\alpha}_2{}^j, \widehat{\lambda}_2{}^j, \widehat{\theta}^j\right)$ with the mean and variance of the considered gamma priors, where $j = 1, 2, \ldots L$ and L is the number of samples available from the Clayton-BGR distribution based on progressive Type-II censoring. For more information see Kundu and Dey (2009) and, Kundu and Gupta (2013).

According to Chen and Shao (1999), we obtain Bayes credible intervals of the parametersΨ as follows:

1- Arrange $\Psi_l^{\langle j \rangle}$as $\alpha_1{}^{[1]}$, $\alpha_1{}^{[2]}$, ..., $\alpha_1{}^{[L]}$, $\lambda_1{}^{[1]}$, $\lambda_1{}^{[2]}$, ..., $\lambda_1{}^{[L]}$, $\alpha_2{}^{[1]}$, $\alpha_2{}^{[2]}$, ..., $\alpha_2{}^{[L]}$, $\lambda_2{}^{[1]}$, $\lambda_2{}^{[2]}$, ..., $\lambda_2{}^{[L]}$ and $\theta^{[1]}$, $\theta^{[2]}$, ..., $\theta^{[L]}$where L is the length of simulation generated.
2- The $100(1-\gamma)\%$ symmetric credible intervals Ψbecome

$$\left(\widetilde{\alpha}_1{}^{\left[L\frac{\gamma}{2}\right]}, \widetilde{\alpha}_1{}^{\left[L\left(1-\frac{\gamma}{2}\right)\right]}\right), \left(\widetilde{\lambda}_1{}^{\left[L\frac{\gamma}{2}\right]}, \widetilde{\lambda}_1{}^{\left[L\left(1-\frac{\gamma}{2}\right)\right]}\right), \left(\widetilde{\alpha}_2{}^{\left[L\frac{\gamma}{2}\right]}, \widetilde{\alpha}_2{}^{\left[L\left(1-\frac{\gamma}{2}\right)\right]}\right), \left(\widetilde{\lambda}_2{}^{\left[L\frac{\gamma}{2}\right]}, \widetilde{\lambda}_2{}^{\left[L\left(1-\frac{\gamma}{2}\right)\right]}\right) \text{ and } \left(\widetilde{\theta}^{\left[L\frac{\gamma}{2}\right]}, \widetilde{\theta}^{\left[L\left(1-\frac{\gamma}{2}\right)\right]}\right).$$

5 Bootstrap Confidence Intervals

In this section, we propose two different methods to construct bootstrap confidence intervals for the parameters of Clayton-BGR under a progressive Type-II censoring scheme, which are percentile bootstrap and bootstrap-t confidence intervals. For more information about bootstrap Confidence Intervals see Efron (1992).

1) **Percentile Bootstrap Confidence Intervals (BP)**

i. Compute the MLE of $\Psi = (\alpha_1, \lambda_1, \alpha_2, \lambda_2, \theta)$ for Clayton-BGR under progressive Type-II censoring.
ii. Generated bootstrap samples using α_k, λ_k; $k = 1, 2$ and θ to obtain the bootstrap estimate of α_k say $\widehat{\alpha}_k^b$, λ_k say $\widehat{\lambda}_k^b$ and θ say $\widehat{\theta}^b$ using the bootstrap sample.
iii. Repeat step (ii) B times to have $\left(\alpha_k^{b(1)}, \alpha_k^{b(2)}, \ldots, \alpha_k^{b(B)}\right)$, $\left(\lambda_k^{b(1)}, \lambda_k^{b(2)}, \ldots, \lambda_k^{b(B)}\right)$ and $(\theta^{b(1)}, \theta^{b(2)}, \ldots, \theta^{b(B)})$.
iv. Arrange $\left(\alpha_k^{b(1)}, \alpha_k^{b(2)}, \ldots, \alpha_k^{b(B)}\right)$, $\left(\lambda_k^{b(1)}, \lambda_k^{b(2)}, \ldots, \lambda_k^{b(B)}\right)$ and $(\theta^{b(1)}, \theta^{b(2)}, \ldots, \theta^{b(B)})$ in ascending order as $\left(\alpha_k^{b[1]}, \alpha_k^{b[2]}, \ldots, \alpha_k^{b[B]}\right)$, $\left(\lambda_k^{b[1]}, \lambda_k^{b[2]}, \ldots, \lambda_k^{b[B]}\right)$ and $(\theta^{b[1]}, \theta^{b[2]}, \ldots, \theta^{b[\mathrm{B}]})$.
v. A two-side 100 $(1-\gamma)\%$ percentile bootstrap confidence intervals for the unknown parameters α_k, λ_k and θ are given by $\left\{\widehat{\alpha}_k^b[B\gamma/2], \widehat{\alpha}_k^b[B(1-\gamma/2)]\right\}$, $\left\{\widehat{\lambda}_k^b[B\gamma/2], \widehat{\lambda}_k^b[B(1-\gamma/2)]\right\}$ and $\left\{\widehat{\theta}^b[B\gamma/2], \widehat{\theta}^b[B(1-\gamma/2)]\right\}$.

2) **Bootstrap-t Confidence Intervals (Bt)**

i. Same as the steps (i-ii) in Boot-p

ii. Compute the t-statistic of $\Psi = (\alpha_k, \lambda_k, \theta)$ and $k = 1, 2$ as $T = \left(\widehat{\Psi}_k^b - \widehat{\Psi}_k\right) / \sqrt{V\left(\widehat{\Psi}_k^b\right)}$ where $V\left(\widehat{\Psi}_k^b\right)$ is asymptotic variances of $\widehat{\delta}_k^b$ and it can be obtained using the Fisher information matrix.

iii. Repeat step (ii) B times and obtain $T^{(1)}, T^{(2)}, \ldots, T^{(B)}$.

iv. Arrange $T^{(1)}, T^{(2)}, \ldots, T^{(B)}$ in ascending order as $T^{[1]}, T^{[2]}, \ldots, T^{[B]}$.

v. A two-sided 100 $(1-\gamma)\%$ percentile bootstrap-t confidence intervals for the unknown parameters α_k, β_k and θ are given by

$$\left\{\widehat{\alpha}_k + T_k[B\gamma/2]\sqrt{V(\widehat{\alpha}_k)},\ \widehat{\alpha}_k + T_k[B(1-\gamma/2)]\sqrt{V(\widehat{\alpha}_k)}\right\},$$

$$\left\{\widehat{\lambda}_k + T_k[B\gamma/2]\sqrt{V\left(\widehat{\lambda}_k\right)},\ \widehat{\lambda}_k + T_k[B(1-\gamma/2)]\sqrt{V\left(\widehat{\lambda}_k\right)}\right\} \quad \text{and}$$

$$\left\{\widehat{\theta} + T[B\gamma/2]\sqrt{V\left(\widehat{\theta}\right)},\ \widehat{\theta} + T[B(1-\gamma/2)]\sqrt{V\left(\widehat{\theta}\right)}\right\}$$

6 Simulation Study

In this section; A Monte Carlo simulation is done to compare the performance of the different progressive Type-II schemes for Clayton-BGR distribution. In our study, we consider binomial distribution as a random removal of censoring schemes with a different probability.

To generate random variables: Nelsen (2006) discussed generating a sample from a specified joint distribution. By conditional distribution method, the joint distribution function is as follows

$$f(x_1, x_2) = f(x_1) f(x_2 | x_1)$$

We used the R packages in the copula package to generate the random variables of the Clayton-BGR distribution.

Simulation Algorithm: A simulation experiments were carried out based on the following data-generated from Clayton-BGR distribution based on progressive Type-II censoring, the values of the parameters $\alpha_1, \lambda_1, \alpha_2, \lambda_2 and\ \theta$ are chosen as the following cases for the generated random variables:

Case 1: $(\alpha_1 = 2, \lambda_1 = 0.7, \alpha_2 = 2.5, \lambda_2 = 0.5, \theta = 6)$,

Case 2: $(\alpha_1 = 1.8, \lambda_1 = 2.5, \alpha_2 = 1.7, \lambda_2 = 1.5, \theta = 3)$,

different sample sizes $(n = 50$ and $150)$, different number of failures m, and removal probability $p = 0.25$, 0.5, and 0.75 are used to generate the number of units removed at each failure time R_i by a binomial distribution. The simulation methods are compared using the criteria of estimating parameters, the comparison is performed by calculating the Bias, the mean square error (MSE),

Table 1: MLE, Bayesian, and CI of Clayton-BGR based on progressive Type-II censoring; in case I when $n = 50$

$n = 50$			MLE					Bayesian				
p	m		Bias	MSE	L.CI	Bt	BP	Bias	MSE	L.CI	Bt	BP
0.25	25	α_1	0.2067	0.3753	2.2618	0.5103	0.5057	0.1967	0.2892	1.9629	0.3340	0.3310
		λ_1	0.0241	0.0057	0.2810	0.0585	0.0590	0.0264	0.0055	0.2709	0.0452	0.0454
		α_2	0.2764	0.6345	2.9300	0.6260	0.6258	0.2659	0.4902	2.5402	0.5194	0.5174
		λ_2	0.0158	0.0025	0.1841	0.0380	0.0380	0.0176	0.0023	0.1769	0.0440	0.0443
		θ	0.5675	3.8786	7.3963	1.8460	1.8359	0.1599	2.5574	6.2405	1.1168	1.1302
	35	α_1	0.1695	0.2984	2.0368	0.2893	0.2872	0.1524	0.2220	1.7484	0.3021	0.3015
		λ_1	0.0152	0.0040	0.2415	0.0514	0.0514	0.0172	0.0037	0.2302	0.0400	0.0394
		α_2	0.2328	0.5377	2.7270	0.3663	0.3651	0.2103	0.4102	2.3725	0.4005	0.3991
		λ_2	0.0107	0.0019	0.1640	0.0221	0.0223	0.0118	0.0017	0.1572	0.0290	0.0289
		θ	0.2686	2.3634	5.9366	1.0026	1.0049	0.0108	1.8487	5.3325	1.0243	1.0285
	45	α_1	0.1514	0.2272	1.7726	0.3372	0.3355	0.1402	0.1788	1.5648	0.2952	0.2936
		λ_1	0.0161	0.0030	0.2039	0.0345	0.0349	0.0174	0.0028	0.1962	0.0299	0.0302
		α_2	0.1932	0.3832	2.3065	0.3646	0.3659	0.1793	0.3038	2.0443	0.3116	0.3094
		λ_2	0.0099	0.0012	0.1327	0.0201	0.0204	0.0106	0.0012	0.1270	0.0205	0.0204
		θ	0.2382	1.8951	5.3175	0.8046	0.8048	0.0275	1.4153	4.6646	0.6000	0.5979
0.5	25	α_1	0.2284	0.4155	2.3641	0.5565	0.5581	0.2266	0.3321	2.0780	0.3339	0.3319
		λ_1	0.0249	0.0057	0.2800	0.0525	0.0524	0.0274	0.0056	0.2744	0.0462	0.0459
		α_2	0.3104	0.7071	3.0652	0.4576	0.4569	0.3094	0.6076	2.8058	0.4990	0.4994

Table 1 (continued)

$n=50$			MLE					Bayesian				
		λ_2	0.0170	0.0026	0.1875	0.0265	0.0266	0.0185	0.0026	0.1856	0.0250	0.0246
		θ	0.5376	3.9675	7.5221	1.5826	1.5872	0.1074	2.5040	6.1918	1.0273	1.0334
	35	α_1	0.1649	0.2807	1.9746	0.4018	0.4049	0.1714	0.2515	1.8483	0.2263	0.2266
		λ_1	0.0171	0.0037	0.2300	0.0372	0.0373	0.0194	0.0037	0.2246	0.0409	0.0408
		α_2	0.2257	0.5028	2.6365	0.3577	0.3581	0.2339	0.4391	2.4317	0.4003	0.4002
		λ_2	0.0118	0.0016	0.1515	0.0264	0.0265	0.0133	0.0016	0.1472	0.0218	0.0220
		θ	0.3662	2.6403	6.2089	0.9083	0.9044	0.0522	1.8975	5.3986	0.8511	0.8445
	45	α_1	0.1864	0.2936	1.9955	0.2487	0.2501	0.1638	0.2062	1.6613	0.2458	0.2442
		λ_1	0.0173	0.0031	0.2084	0.0299	0.0297	0.0185	0.0029	0.1988	0.0284	0.0288
		α_2	0.2496	0.5092	2.6218	0.4284	0.4319	0.2200	0.3521	2.1611	0.3495	0.3487
		λ_2	0.0118	0.0013	0.1363	0.0248	0.0249	0.0123	0.0013	0.1303	0.0214	0.0216
		θ	0.1997	2.0222	5.5220	0.9802	0.9756	0.0030	1.5293	4.8501	0.6295	0.6240
0.75	25	α_1	0.2078	0.3693	2.2397	0.3470	0.3496	0.1985	0.2725	1.8936	0.3717	0.3717
		λ_1	0.0238	0.0055	0.2763	0.0651	0.0656	0.0268	0.0052	0.2628	0.0516	0.0512
		α_2	0.2991	0.7282	3.1344	1.0061	0.9961	0.2814	0.5197	2.6029	0.5559	0.5506
		λ_2	0.0168	0.0026	0.1905	0.0393	0.0390	0.0183	0.0024	0.1799	0.0366	0.0367
		θ	0.5345	3.7695	7.3202	1.8891	1.8951	0.1063	2.3327	5.9756	0.9825	0.9846
	35	α_1	0.1857	0.3152	2.0781	0.3882	0.3940	0.1718	0.2544	1.8597	0.3390	0.3364
		λ_1	0.0176	0.0038	0.2332	0.0304	0.0305	0.0190	0.0037	0.2259	0.0269	0.0269

Table 1 (continued)

$n=50$		MLE					Bayesian				
	α_2	0.2649	0.5677	2.7664	0.5738	0.5740	0.2476	0.4666	2.4969	0.4182	0.4191
	λ_2	0.0134	0.0017	0.1533	0.0244	0.0242	0.0142	0.0016	0.1482	0.0253	0.0254
	θ	0.2751	2.5426	6.1600	1.0716	1.0711	−0.0337	1.8030	5.2646	0.9660	0.9657
45	α_1	0.1371	0.2369	1.8317	0.2350	0.2363	0.1312	0.1865	1.6137	0.2405	0.2383
	λ_1	0.0116	0.0027	0.1982	0.0271	0.0269	0.0138	0.0026	0.1909	0.0240	0.0238
	α_2	0.1913	0.4219	2.4344	0.3334	0.3368	0.1816	0.3329	2.1480	0.3644	0.3619
	λ_2	0.0086	0.0012	0.1341	0.0204	0.0205	0.0097	0.0012	0.1299	0.0141	0.0141
	θ	0.2517	2.2997	5.8651	0.8520	0.8436	0.0358	1.6311	5.0070	0.6999	0.7034

Table 2: MLE, Bayesian, and CI of Clayton-BGR based on progressive Type-II censoring; in case I when $n = 150$

$n = 150$			MLE					Bayesian				
p	M		Bias	MSE	L.CI	Bt	BP	Bias	MSE	L.CI	Bt	BP
0.25	50	α_1	0.0786	0.1081	1.2520	0.1505	0.1505	0.0818	0.0983	1.1870	0.1480	0.1484
		λ_1	0.0104	0.0023	0.1822	0.0272	0.0276	0.0116	0.0022	0.1791	0.0254	0.0255
		α_2	0.1046	0.1786	1.6056	0.2178	0.2200	0.1072	0.1634	1.5286	0.2058	0.2061
		λ_2	0.0066	0.0010	0.1217	0.0144	0.0143	0.0073	0.0010	0.1195	0.0211	0.0210
		θ	0.2809	1.3181	4.3658	0.5974	0.5899	0.0709	1.0450	3.9996	0.5315	0.5326
	75	α_1	0.0760	0.0890	1.1315	0.1195	0.1200	0.0745	0.0781	1.0563	0.1110	0.1111
		λ_1	0.0092	0.0016	0.1520	0.0166	0.0168	0.0097	0.0015	0.1478	0.0162	0.0162
		α_2	0.1000	0.1551	1.4937	0.1699	0.1709	0.0990	0.1381	1.4046	0.1648	0.1656
		λ_2	0.0058	0.0007	0.1006	0.0109	0.0109	0.0062	0.0007	0.0979	0.0117	0.0117
		θ	0.1675	1.0392	3.9438	0.4619	0.4605	0.0160	0.8328	3.5786	0.4398	0.4434
	100	α_1	0.0442	0.0729	1.0448	0.0962	0.0963	0.0488	0.0611	0.9500	0.0933	0.0937
		λ_1	0.0057	0.0012	0.1324	0.0123	0.0124	0.0066	0.0011	0.1293	0.0125	0.0126
		α_2	0.0621	0.1240	1.3596	0.1422	0.1431	0.0687	0.1049	1.2412	0.1472	0.1475
		λ_2	0.0042	0.0005	0.0877	0.0090	0.0091	0.0048	0.0005	0.0855	0.0080	0.0081
		θ	0.1286	0.7973	3.4654	0.3219	0.3211	0.0284	0.6458	3.1499	0.2925	0.2925
	120	α_1	0.0609	0.0662	0.9801	0.0961	0.0953	0.0597	0.0524	0.8666	0.0800	0.0801
		λ_1	0.0062	0.0010	0.1206	0.0105	0.0104	0.0065	0.0009	0.1148	0.0102	0.0103
		α_2	0.0786	0.1122	1.2772	0.1251	0.1271	0.0766	0.0905	1.1409	0.1150	0.1150
		λ_2	0.0036	0.0004	0.0800	0.0072	0.0072	0.0038	0.0004	0.0757	0.0077	0.0077
		θ	0.0805	0.7090	3.2872	0.2821	0.2826	−0.0287	0.5374	2.8730	0.2683	0.2663
0.5	50	α_1	0.0817	0.1128	1.2777	0.1776	0.1773	0.0888	0.1064	1.2313	0.1358	0.1339
		λ_1	0.0126	0.0023	0.1811	0.0244	0.0244	0.0143	0.0022	0.1770	0.0241	0.0241
		α_2	0.1112	0.1912	1.6584	0.1893	0.1902	0.1159	0.1782	1.5920	0.2815	0.2818
		λ_2	0.0083	0.0010	0.1195	0.0175	0.0175	0.0089	0.0010	0.1173	0.0149	0.0148
		θ	0.2590	1.6121	4.8749	0.6259	0.6330	0.0415	1.2516	4.3847	0.5451	0.5430

Table 2 (continued)

$n = 150$			MLE					Bayesian				
	75	α_1	0.0625	0.0885	1.1407	0.1538	0.1553	0.0649	0.0733	1.0308	0.1202	0.1190
		λ_1	0.0066	0.0015	0.1517	0.0158	0.0158	0.0074	0.0014	0.1455	0.0133	0.0132
		α_2	0.0874	0.1478	1.4683	0.1773	0.1792	0.0883	0.1231	1.3315	0.1414	0.1411
		λ_2	0.0051	0.0007	0.1008	0.0110	0.0111	0.0053	0.0006	0.0967	0.0098	0.0098
		θ	0.1882	1.0383	3.9275	0.4475	0.4449	0.0442	0.8176	3.5420	0.4398	0.4418
	100	α_1	0.0673	0.0769	1.0548	0.1109	0.1095	0.0703	0.0651	0.9620	0.0957	0.0973
		λ_1	0.0060	0.0012	0.1334	0.0138	0.0138	0.0071	0.0011	0.1276	0.0128	0.0128
		α_2	0.0902	0.1259	1.3458	0.1432	0.1438	0.0935	0.1056	1.2209	0.1241	0.1250
		λ_2	0.0043	0.0005	0.0873	0.0086	0.0086	0.0050	0.0005	0.0833	0.0079	0.0079
		θ	0.0765	0.8044	3.5047	0.3445	0.3450	−0.0245	0.6566	3.1765	0.2961	0.3000
	120	α_1	0.0608	0.0784	1.0718	0.0936	0.0944	0.0582	0.0620	0.9497	0.0889	0.0889
		λ_1	0.0040	0.0010	0.1242	0.0109	0.0110	0.0045	0.0009	0.1195	0.0112	0.0114
		α_2	0.0845	0.1307	1.3786	0.1251	0.1240	0.0813	0.1002	1.2001	0.1051	0.1050
		λ_2	0.0034	0.0004	0.0821	0.0068	0.0068	0.0037	0.0004	0.0779	0.0074	0.0074
		θ	0.0460	0.7391	3.3670	0.2923	0.2922	−0.0286	0.5864	3.0013	0.2621	0.2596
0.75	50	α_1	0.0822	0.1212	1.3270	0.1722	0.1728	0.0895	0.1072	1.2349	0.1764	0.1792
		λ_1	0.0113	0.0023	0.1842	0.0276	0.0275	0.0129	0.0023	0.1791	0.0273	0.0274
		α_2	0.1133	0.2204	1.7868	0.2929	0.2936	0.1233	0.1964	1.6695	0.2677	0.2668
		λ_2	0.0073	0.0010	0.1221	0.0187	0.0186	0.0084	0.0010	0.1189	0.0168	0.0170
		θ	0.3160	1.6590	4.8972	0.6202	0.6187	0.1099	1.2664	4.3925	0.6268	0.6307
	75	α_1	0.0818	0.1053	1.2318	0.1466	0.1477	0.0807	0.0870	1.1124	0.1210	0.1215
		λ_1	0.0105	0.0016	0.1527	0.0203	0.0201	0.0110	0.0016	0.1491	0.0178	0.0179
		α_2	0.1044	0.1763	1.5952	0.1529	0.1536	0.1034	0.1472	1.4489	0.1749	0.1765
		λ_2	0.0065	0.0007	0.1009	0.0099	0.0100	0.0068	0.0007	0.0982	0.0106	0.0106
		θ	0.2326	1.2121	4.2203	0.4780	0.4791	0.0874	0.9242	3.7547	0.4298	0.4290

Table 2 (continued)

$n = 150$		MLE					Bayesian				
100	α_1	0.0701	0.0794	1.0702	0.1057	0.1056	0.0712	0.0638	0.9509	0.0966	0.0981
	λ_1	0.0071	0.0012	0.1341	0.0146	0.0147	0.0082	0.0012	0.1293	0.0133	0.0133
	α_2	0.0901	0.1343	1.3931	0.1281	0.1281	0.0915	0.1071	1.2323	0.1157	0.1150
	λ_2	0.0045	0.0006	0.0904	0.0080	0.0079	0.0050	0.0005	0.0860	0.0072	0.0072
	θ	0.0570	0.7230	3.3272	0.3054	0.3071	−0.0547	0.6054	3.0441	0.2610	0.2600
120	α_1	0.0444	0.0659	0.9916	0.0897	0.0888	0.0461	0.0554	0.9051	0.0760	0.0762
	λ_1	0.0048	0.0010	0.1213	0.0116	0.0118	0.0057	0.0009	0.1175	0.0118	0.0118
	α_2	0.0629	0.1129	1.2943	0.1248	0.1252	0.0650	0.0959	1.1876	0.1071	0.1072
	λ_2	0.0037	0.0004	0.0819	0.0070	0.0070	0.0042	0.0004	0.0803	0.0078	0.0079
	θ	0.0669	0.6571	3.1685	0.2797	0.2824	−0.0168	0.5164	2.8176	0.2600	0.2608

Table 3: MLE, Bayesian, and CI of Clayton-BGR based on progressive Type-II censoring; in case II when $n=50$

$n=50$			MLE					Bayesian				
p	M		Bias	MSE	L.CI	Bt	BP	Bias	MSE	L.CI	Bt	BP
0.25	25	α_1	0.1932	0.2924	1.9807	0.3078	0.3100	0.2069	0.2721	1.8780	0.4136	0.4138
		λ_1	0.0923	0.0869	1.0979	0.1938	0.1925	0.1037	0.0887	1.0949	0.1840	0.1832
		α_2	0.1874	0.2793	1.9379	0.3762	0.3737	0.1952	0.2556	1.8290	0.4627	0.4609
		λ_2	0.0539	0.0319	0.6676	0.1408	0.1406	0.0622	0.0321	0.6589	0.1459	0.1456
		θ	0.2190	1.0589	3.9434	0.7594	0.7611	0.0734	0.8516	3.6078	0.6959	0.6935
	35	α_1	0.1378	0.2175	1.7472	0.2780	0.2799	0.1509	0.2060	1.6789	0.2475	0.2467
		λ_1	0.0530	0.0523	0.8721	0.1276	0.1267	0.0631	0.0523	0.8619	0.1378	0.1375
		α_2	0.1347	0.1998	1.6717	0.1984	0.1995	0.1424	0.1777	1.5560	0.3757	0.3751
		λ_2	0.0359	0.0208	0.5478	0.0834	0.0841	0.0418	0.0205	0.5376	0.1046	0.1048
		θ	0.1333	0.7663	3.3931	0.8017	0.7942	0.0112	0.6172	3.0808	0.5737	0.5740
	45	α_1	0.1141	0.1920	1.6592	0.2668	0.2686	0.1193	0.1705	1.5505	0.2006	0.2032
		λ_1	0.0469	0.0422	0.7842	0.1298	0.1292	0.0531	0.0419	0.7751	0.1016	0.1006
		α_2	0.0988	0.1577	1.5083	0.1733	0.1702	0.1034	0.1400	1.4102	0.1715	0.1720
		λ_2	0.0260	0.0149	0.4686	0.0643	0.0640	0.0294	0.0148	0.4623	0.0763	0.0757
		θ	0.1645	0.6809	3.1712	0.6043	0.6056	0.0620	0.5352	2.8589	0.3571	0.3583
0.5	25	α_1	0.1607	0.3013	2.0585	0.3528	0.3518	0.1738	0.2696	1.9189	0.3543	0.3513
		λ_1	0.0737	0.0766	1.0462	0.2146	0.2125	0.0851	0.0753	1.0235	0.2228	0.2222
		α_2	0.1645	0.2735	1.9469	0.2586	0.2573	0.1717	0.2379	1.7905	0.3204	0.3211

Table 3 (continued)

$n=50$			MLE					Bayesian				
		λ_2	0.0510	0.0312	0.6633	0.1460	0.1456	0.0584	0.0312	0.6535	0.1307	0.1322
		θ	0.2741	1.1089	3.9875	0.8111	0.7951	0.1000	0.9034	3.7070	0.9803	0.9868
	35	α_1	0.1309	0.2088	1.7172	0.2562	0.2539	0.1402	0.2014	1.6722	0.2168	0.2176
		λ_1	0.0554	0.0498	0.8481	0.1325	0.1313	0.0643	0.0501	0.8408	0.1363	0.1359
		α_2	0.1316	0.2212	1.7709	0.5902	0.5658	0.1361	0.2009	1.6747	0.2210	0.2208
		λ_2	0.0346	0.0184	0.5146	0.0898	0.0898	0.0393	0.0185	0.5111	0.0886	0.0881
		θ	0.1965	0.8572	3.5483	0.5019	0.4984	0.0743	0.7159	3.3056	0.4349	0.4361
	45	α_1	0.1324	0.2087	1.7148	0.1964	0.1959	0.1410	0.2004	1.6665	0.2611	0.2580
		λ_1	0.0513	0.0427	0.7847	0.1088	0.1085	0.0587	0.0431	0.7806	0.1176	0.1186
		α_2	0.1144	0.1714	1.5604	0.2155	0.2151	0.1208	0.1582	1.4863	0.2204	0.2170
		λ_2	0.0299	0.0165	0.4894	0.0736	0.0737	0.0353	0.0166	0.4867	0.0749	0.0757
		θ	0.1400	0.6416	3.0932	0.4658	0.4627	0.0400	0.5270	2.8429	0.3944	0.3992
0.75	25	α_1	0.1597	0.2549	1.8783	0.3218	0.3208	0.1684	0.2360	1.7873	0.3441	0.3403
		λ_1	0.0764	0.0721	1.0094	0.1927	0.1932	0.0876	0.0719	0.9940	0.1924	0.1920
		α_2	0.1733	0.2417	1.8043	0.3336	0.3367	0.1814	0.2319	1.7496	0.3138	0.3159
		λ_2	0.0560	0.0280	0.6184	0.1030	0.1030	0.0632	0.0283	0.6116	0.1298	0.1300
		θ	0.2814	1.1948	4.1425	0.7958	0.8012	0.1322	0.9113	3.7080	0.5764	0.5743
	35	α_1	0.1240	0.2137	1.7468	0.2116	0.2122	0.1366	0.2009	1.6744	0.1951	0.1955
		λ_1	0.0603	0.0514	0.8571	0.1294	0.1314	0.0689	0.0519	0.8513	0.1272	0.1266

Table 3 (continued)

$n = 50$		MLE					Bayesian				
	α_2	0.1329	0.2289	1.8025	0.3218	0.3218	0.1412	0.2076	1.6989	0.2483	0.2482
	λ_2	0.0399	0.0211	0.5473	0.0842	0.0836	0.0456	0.0212	0.5417	0.0694	0.0699
	θ	0.2394	0.9203	3.6433	0.5613	0.5543	0.0868	0.7068	3.2796	0.5944	0.5930
45	α_1	0.1137	0.1814	1.6097	0.2220	0.2221	0.1226	0.1723	1.5550	0.2083	0.2072
	λ_1	0.0347	0.0394	0.7670	0.1199	0.1206	0.0436	0.0391	0.7559	0.0971	0.0965
	α_2	0.1135	0.1532	1.4692	0.2088	0.2104	0.1238	0.1504	1.4412	0.1970	0.1958
	λ_2	0.0254	0.0150	0.4692	0.0744	0.0742	0.0312	0.0151	0.4661	0.0789	0.0792
	θ	0.1206	0.6458	3.1161	0.4269	0.4239	0.0306	0.5690	2.9559	0.4429	0.4408

Table 4: MLE, Bayesian, and CI of Clayton-BGR based on progressive Type-II censoring; in case II when $n = 150$

$n = 150$			MLE					Bayesian				
p	M		Bias	MSE	L.CI	Bt	BP	Bias	MSE	L.CI	Bt	BP
0.25	50	α_1	0.0621	0.0866	1.1280	0.1543	0.1560	0.0728	0.0848	1.1060	0.1642	0.1630
		λ_1	0.0366	0.0354	0.7236	0.1133	0.1132	0.0441	0.0351	0.7143	0.1019	0.1014
		α_2	0.0630	0.0798	1.0797	0.1401	0.1398	0.0700	0.0770	1.0533	0.1533	0.1551
		λ_2	0.0223	0.0132	0.4425	0.0617	0.0621	0.0265	0.0132	0.4382	0.0626	0.0627
		θ	0.1529	0.4608	2.5940	0.3323	0.3318	0.0731	0.3888	2.4287	0.2988	0.2996
	75	α_1	0.0707	0.0770	1.0523	0.1119	0.1125	0.0751	0.0745	1.0289	0.1112	0.1120
		λ_1	0.0259	0.0226	0.5812	0.0714	0.0722	0.0302	0.0228	0.5800	0.0648	0.0653
		α_2	0.0699	0.0717	1.0135	0.1111	0.1117	0.0715	0.0663	0.9704	0.0996	0.1000
		λ_2	0.0173	0.0087	0.3591	0.0451	0.0447	0.0200	0.0086	0.3556	0.0397	0.0393
		θ	0.0515	0.3179	2.2019	0.2565	0.2561	0.0043	0.2921	2.1198	0.2523	0.2539
	100	α_1	0.0661	0.0652	0.9677	0.0921	0.0926	0.0675	0.0613	0.9339	0.0911	0.0912
		λ_1	0.0246	0.0163	0.4919	0.0533	0.0532	0.0276	0.0160	0.4836	0.0477	0.0477
		α_2	0.0650	0.0576	0.9064	0.0967	0.0955	0.0646	0.0543	0.8785	0.0841	0.0844
		λ_2	0.0166	0.0062	0.3008	0.0278	0.0277	0.0180	0.0061	0.2972	0.0286	0.0290
		θ	0.0303	0.2615	2.0019	0.2195	0.2196	0.0005	0.2490	1.9572	0.1864	0.1880
	120	α_1	0.0462	0.0542	0.8953	0.0928	0.0932	0.0503	0.0522	0.8743	0.0731	0.0731
		λ_1	0.0163	0.0139	0.4583	0.0418	0.0419	0.0179	0.0137	0.4542	0.0417	0.0419
		α_2	0.0414	0.0466	0.8306	0.0745	0.0744	0.0461	0.0443	0.8059	0.0746	0.0740
		λ_2	0.0085	0.0051	0.2791	0.0270	0.0270	0.0103	0.0052	0.2800	0.0280	0.0279
		θ	0.0301	0.2125	1.8039	0.1599	0.1596	−0.0028	0.1982	1.7460	0.1507	0.1510
0.5	50	α_1	0.0627	0.0910	1.1574	0.1739	0.1734	0.0710	0.0875	1.1265	0.1524	0.1528
		λ_1	0.0409	0.0321	0.6839	0.0977	0.0972	0.0476	0.0326	0.6826	0.0990	0.1000
		α_2	0.0605	0.0835	1.1079	0.1636	0.1624	0.0653	0.0790	1.0722	0.1403	0.1413
		λ_2	0.0249	0.0126	0.4288	0.0532	0.0530	0.0287	0.0127	0.4272	0.0555	0.0548
		θ	0.1639	0.5199	2.7539	0.3782	0.3785	0.0932	0.4505	2.6070	0.3217	0.3213

Table 4 (continued)

$n = 150$			MLE					Bayesian				
	75	α_1	0.0582	0.0741	1.0427	0.0994	0.0985	0.0678	0.0729	1.0252	0.1104	0.1099
		λ_1	0.0270	0.0219	0.5713	0.0635	0.0640	0.0329	0.0220	0.5672	0.0699	0.0702
		α_2	0.0583	0.0674	0.9923	0.1194	0.1197	0.0645	0.0655	0.9716	0.1062	0.1058
		λ_2	0.0154	0.0081	0.3474	0.0437	0.0434	0.0176	0.0081	0.3459	0.0420	0.0421
		θ	0.0790	0.3140	2.1758	0.2284	0.2283	0.0239	0.3003	2.1471	0.2779	0.2771
	100	α_1	0.0453	0.0665	0.9953	0.1129	0.1133	0.0515	0.0615	0.9513	0.0719	0.0719
		λ_1	0.0169	0.0165	0.4996	0.0539	0.0540	0.0218	0.0161	0.4899	0.0434	0.0434
		α_2	0.0476	0.0581	0.9269	0.0928	0.0945	0.0527	0.0545	0.8923	0.0887	0.0887
		λ_2	0.0127	0.0058	0.2935	0.0305	0.0306	0.0156	0.0058	0.2927	0.0291	0.0291
		θ	0.0844	0.2718	2.0178	0.2212	0.2232	0.0391	0.2453	1.9364	0.1928	0.1919
	120	α_1	0.0516	0.0587	0.9288	0.0823	0.0808	0.0528	0.0524	0.8735	0.0800	0.0803
		λ_1	0.0211	0.0145	0.4656	0.0446	0.0445	0.0233	0.0139	0.4541	0.0415	0.0416
		α_2	0.0500	0.0531	0.8824	0.0696	0.0700	0.0522	0.0481	0.8350	0.0818	0.0826
		λ_2	0.0129	0.0056	0.2901	0.0257	0.0255	0.0150	0.0055	0.2854	0.0239	0.0241
		θ	0.0148	0.2093	1.7935	0.1576	0.1567	−0.0150	0.1899	1.7082	0.1538	0.1535
0.75	50	α_1	0.0667	0.0894	1.1434	0.1393	0.1399	0.0761	0.0884	1.1273	0.1691	0.1680
		λ_1	0.0376	0.0341	0.7089	0.0833	0.0826	0.0440	0.0341	0.7032	0.1086	0.1089
		α_2	0.0739	0.0886	1.1308	0.1534	0.1534	0.0815	0.0864	1.1073	0.1373	0.1356
		λ_2	0.0246	0.0127	0.4314	0.0614	0.0612	0.0286	0.0127	0.4274	0.0519	0.0520
		θ	0.1578	0.5080	2.7259	0.4242	0.4264	0.0787	0.4526	2.6204	0.3394	0.3391
	75	α_1	0.0555	0.0783	1.0757	0.1290	0.1273	0.0646	0.0727	1.0263	0.1306	0.1309
		λ_1	0.0247	0.0213	0.5635	0.0683	0.0687	0.0305	0.0206	0.5498	0.0574	0.0572
		α_2	0.0599	0.0718	1.0246	0.1207	0.1207	0.0666	0.0682	0.9908	0.1139	0.1137
		λ_2	0.0180	0.0079	0.3411	0.0418	0.0419	0.0205	0.0079	0.3387	0.0364	0.0366
		θ	0.0981	0.3799	2.3865	0.2457	0.2464	0.0305	0.3233	2.2269	0.2602	0.2610

Table 4 (continued)

$n = 150$		MLE					Bayesian				
100	α_1	0.0448	0.0594	0.9397	0.0935	0.0936	0.0488	0.0555	0.9039	0.0991	0.0987
	λ_1	0.0157	0.0162	0.4951	0.0493	0.0491	0.0194	0.0156	0.4832	0.0529	0.0528
	α_2	0.0466	0.0543	0.8955	0.0875	0.0870	0.0507	0.0504	0.8578	0.0800	0.0793
	λ_2	0.0125	0.0060	0.2994	0.0305	0.0303	0.0149	0.0060	0.2973	0.0313	0.0314
	θ	0.0595	0.2566	1.9729	0.1955	0.1942	0.0143	0.2295	1.8782	0.2014	0.2013
120	α_1	0.0340	0.0515	0.8801	0.0762	0.0765	0.0383	0.0487	0.8522	0.0772	0.0765
	λ_1	0.0115	0.0143	0.4667	0.0411	0.0410	0.0149	0.0143	0.4651	0.0382	0.0385
	α_2	0.0317	0.0463	0.8347	0.0693	0.0701	0.0375	0.0446	0.8155	0.0775	0.0786
	λ_2	0.0068	0.0053	0.2834	0.0284	0.0287	0.0095	0.0053	0.2818	0.0260	0.0262
	θ	0.0394	0.2063	1.7745	0.1478	0.1466	0.0057	0.1930	1.7230	0.1466	0.1467

Table 5: Goodness of Fit test of Clayton Copula for Football Data

	statistic	$\widehat{\theta}$	p value
Anderson–Darling-type	0.40378	1.3159	0.06424

and the length of confidence interval (L.CI) for each method of estimation as following

$$Bias = \left(\widehat{\Psi} - \Psi\right). \tag{6.1}$$

$$MSE = Mean\left(\widehat{\Psi} - \Psi\right)^2. \tag{6.2}$$

and

$$L.CI = Upper.CI - Lower.CI \tag{6.3}$$

We restricted the number of repeated-samples to 10,000.

7 Concluding Remarks on the Simulation

The simulation outcomes are recorded in Tables 1, 2, 3 and 4. The following concluding remakes are noticed based on these Tables as follows

1. When m increases with fixed values of n and p, the Bias, MSE, and the L.CI associated with the parameter estimates decrease for MLE and Bayesian methods.

Table 6: MLE, S.E., KS Goodness of Fit test of univariate GR distribution for football Data

	x_1		x_2	
	$\widehat{\alpha}_1$	$\widehat{\lambda}_1$	$\widehat{\alpha}_2$	$\widehat{\lambda}_2$
estimate	2.4715	0.0062	2.1171	0.0056
S.E.	0.8850	0.0007	0.7559	0.0007
KS	0.2482		0.0707	
P value	0.2176		0.9999	

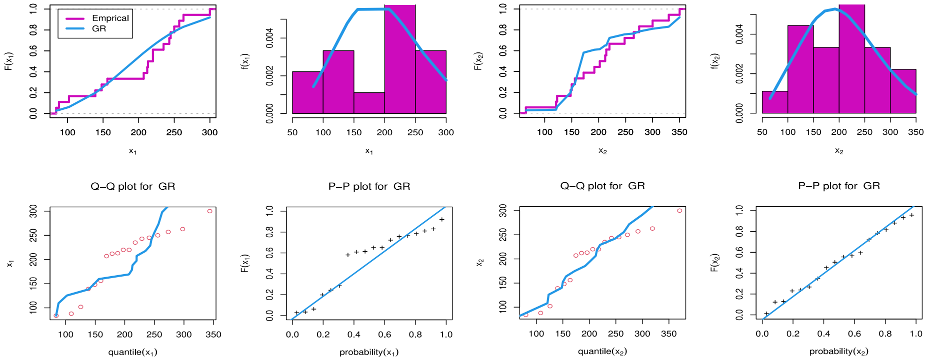

Fig. 2: Estimated PDF, CDF, PP-plot, and QQ-plot of GR for football data

2. With fixed values of m and p, then criteria for each parameter decrease for MLE and Bayesian methods of estimation as n increases.
3. For fixed n, and m, with increases p until 0.5, the criteria for each parameter increase for the MLE and Bayesian methods.
4. In approximately most situations, we notice that the measures of Bayesian estimates are more accurate than the measure of MLE estimates.
5. We notice that the Bootstrap CI is the shortest CI and the best.
6. In most situations, we notice that the Bootstrap-P CI is better than Bootstrap-t CI.
7. Also, we notice that the Bayes credible intervals are better than ACI.

Table 7: MLE and Bayesian Estimation for Clayton-BGR Model under complete sample for Football Data

	MLE		Bayesian	
	estimate	S.E.	estimate	S.E.
$\widehat{\alpha}_1$	2.4798	0.8558	2.4741	0.7240
$\widehat{\lambda}_1$	0.0061	0.0007	0.0061	0.0007
$\widehat{\alpha}_2$	2.0548	0.7046	2.0334	0.5734
$\widehat{\lambda}_2$	0.0056	0.0007	0.0056	0.0006
$\widehat{\theta}$	0.8460	0.4907	0.8674	0.3801
LL	AIC	CAIC	BIC	HQIC
200.3418	410.6836	415.6836	415.1355	411.2975

Table 8: The Sample Observation of Progressive Type-II Censoring with random sample for football Data

M	P	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
10	0.25	R	4	0	0	0	2	2	0	0	0	0					
		χ_{1i}	84	88	139	148	156	207	220	220	243	257					
		$\chi_{2[i]}$	148	202	150	123	121	214	192	275	300	330					
	0.5	R	5	0	0	1	2	0	0	0	0	0					
		χ_{1i}	84	88	139	148	156	207	220	235	243	263					
		$\chi_{2[i]}$	148	202	150	123	121	214	275	172	300	350					
	0.75	R	0	8	0	0	0	0	0	0	0	0					
		χ_{1i}	84	88	102	156	212	213	220	243	257	263					
		$\chi_{2[i]}$	148	202	65	121	220	265	192	300	330	350					
15	0.25	R	2	0	0	0	1	0	0	0	0	0	0	0	0	0	0
		χ_{1i}	84	88	102	139	148	156	207	213	220	220	235	243	245	250	263
		$\chi_{2[i]}$	148	202	65	150	123	121	214	265	192	275	172	300	156	212	350
	0.5	R	2	0	0	0	1	0	0	0	0	0	0	0	0	0	0
		χ_{1i}	84	88	102	139	148	156	207	213	220	220	235	243	245	250	263
		$\chi_{2[i]}$	148	202	65	150	123	121	214	265	192	275	172	300	156	212	350
	0.75	R	0	1	2	0	0	0	0	0	0	0	0	0	0	0	0
		χ_{1i}	84	88	102	148	156	212	213	220	220	235	243	245	257	263	300
		$\chi_{2[i]}$	148	202	65	123	121	220	265	192	275	172	300	156	330	350	248

Table 9: MLE and Bayesian Estimation for Clayton-BGR Model under Progressive Type-II Censoring with random removal of football data

m		10				15			
p		MLE		Bayesian		MLE		Bayesian	
		estimate	S.E.	estimate	S.E.	estimate	S.E.	estimate	S.E.
0.25	$\widehat{\alpha}_1$	2.2127	0.9226	2.3294	0.9112	2.4795	0.9052	2.5600	0.7744
	$\widehat{\lambda}_1$	0.0059	0.0010	0.0059	0.0009	0.0064	0.0008	0.0065	0.0007
	$\widehat{\alpha}_2$	2.8846	1.3486	3.3116	1.5258	1.9936	0.7335	1.8990	0.5769
	$\widehat{\lambda}_2$	0.0059	0.0010	0.0061	0.0009	0.0058	0.0008	0.0058	0.0007
	$\widehat{\theta}$	0.7789	0.5578	0.8414	0.5197	0.7648	0.4920	0.9363	0.4637
	LL	AIC	CAIC	BIC	HQIC	AIC	CAIC	BIC	HQIC
	113.065	236.130	251.130	237.643	234.471	343.404	350.071	346.944	343.366
0.5	$\widehat{\alpha}_1$	2.3051	0.9740	2.3729	0.8491	2.4795	0.9052	2.5600	0.7744
	$\widehat{\lambda}_1$	0.0062	0.0011	0.0063	0.0009	0.0064	0.0008	0.0065	0.0007
	$\widehat{\alpha}_2$	2.6081	1.2205	2.5228	0.9621	1.9936	0.7335	1.8990	0.5769
	$\widehat{\lambda}_2$	0.0058	0.0010	0.0058	0.0009	0.0058	0.0008	0.0058	0.0007
	$\widehat{\theta}$	0.7223	0.5552	0.7751	0.4537	0.7648	0.4920	0.9363	0.4637
	LL	AIC	CAIC	BIC	HQIC	AIC	CAIC	BIC	HQIC
	112.452	234.904	249.904	236.417	233.2441	343.404	350.071	346.944	343.366
0.75	$\widehat{\alpha}_1$	2.2030	0.8479	2.2723	0.7784	2.3444	0.8320	2.3785	0.7275
	$\widehat{\lambda}_1$	0.0062	0.0010	0.0062	0.0009	0.0059	0.0008	0.0060	0.0007
	$\widehat{\alpha}_2$	1.6004	0.6349	1.7147	0.6122	1.8460	0.6568	1.8585	0.5657
	$\widehat{\lambda}_2$	0.0047	0.0008	0.0048	0.0008	0.0053	0.0007	0.0053	0.0007
	$\widehat{\theta}$	0.8314	0.5449	0.8173	0.4533	0.7893	0.4852	0.7752	0.3746
	LL	AIC	CAIC	BIC	HQIC	AIC	CAIC	BIC	HQIC
	113.669	237.338	252.338	238.851	235.678	347.865	354.531	351.405	347.827

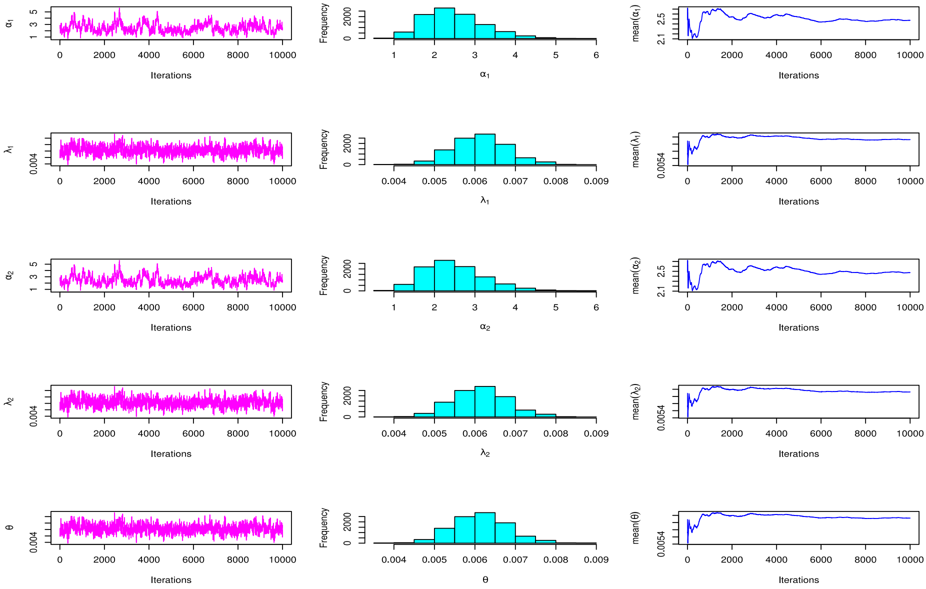

Fig. 3: Convergence of MCMC estimation of Clayton-BGR Model under complete sample for football Data

8 Applications

In this section, the Clayton-BGR distribution is fitted to two real data sets.

a. Football Data

The football data set has been obtained from Meintanis (2007). More authors discussed this data to analyze bivariate models under a complete sample. In Table 5, the p values of the goodness of fit test, it is clear that the Clayton-BGR model fits the data, for more information on this test see Genest et al. (Genest et al. 2013).

In Table 6, we estimate the parameters of univariate GR distribution by using MLE, standard errors (S.E.), and using the Kolmogorov- Smirnov (KS) statistic with its P value to check whether the data fitted the univariate GR distribution or not as also shown in Fig. 2. The parameter estimation of the Clayton-BGR model under complete sample for football data can be shown in

Table 10: Goodness of Fit test of Clayton Copula for Medical Data

	statistic	$\widehat{\theta}$	p value
Anderson–Darling-type	0.14424	0.30327	0.7497

Table 11: MLE, S.E., KS Goodness of Fit test of univariate GR distribution for Medical Data

	x_1		x_2	
	$\widehat{\alpha}_1$	$\widehat{\lambda}_1$	$\widehat{\alpha}_2$	$\widehat{\lambda}_2$
estimate	0.2495	0.0030	0.3330	0.0046
S.E.	0.0498	0.0005	0.0685	0.0008
KS	0.1922		0.1782	
P value	0.2179		0.2964	

Table 7. Assuming that, the observations are independent. Suppose, the experiment was conducted for the observations according to each censoring scheme in a random manner, the sample observation of progressive Type-II censoring with random removal based on concomitant order statistics is shown in Table 8. The parameters estimation of the Clayton-BGR model by using MLE and Bayesian estimation methods under progressive Type-II censoring with random removal of football data are obtained in Table 9. The Akaike information criterion (AIC), Bayesian information criterion (BIC), the consistent Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC) are obtained for the MLE method Table 9.

Figure 2 discusses the plot of the max distance between the empirical CDF and GR CDF curves, histogram, PP-plot, and QQ-plot for GR distribution. Therefore, it indicates that GR distribution is fitted to the football data set.

The convergence of MCMC estimation for Clayton-BGR model under complete sample for football data can be shown in Figs. 3. Histogram plot, approximate marginal posterior density, and MCMC convergence for parameters of Clayton-BGR model are represented in Fig. 3.

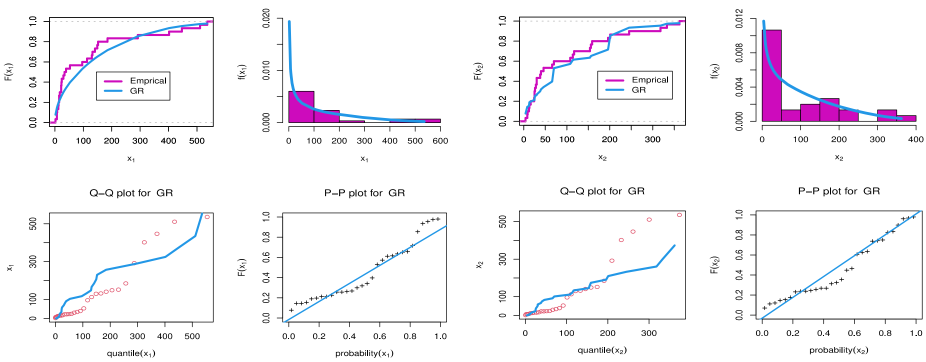

Fig. 4: Estimated PDF, CDF, PP-plot, and QQ-plot of GR for Medical data

Table 12: MLE and Bayesian Estimation for Clayton-BGR Model under complete sample for medical Data

	MLE		Bayesian	
	estimate	S.E.	estimate	S.E.
$\widehat{\alpha}_1$	0.2439	0.0492	0.2404	0.0480
$\widehat{\lambda}_1$	0.0029	0.0006	0.0030	0.0005
$\widehat{\alpha}_2$	0.3208	0.0673	0.3160	0.0634
$\widehat{\lambda}_2$	0.0045	0.0008	0.0046	0.0008
$\widehat{\theta}$	0.6128	0.4497	0.8180	0.4022
LL	AIC	CAIC	BIC	HQIC
341.0361	692.0721	694.5721	699.0781	694.3134

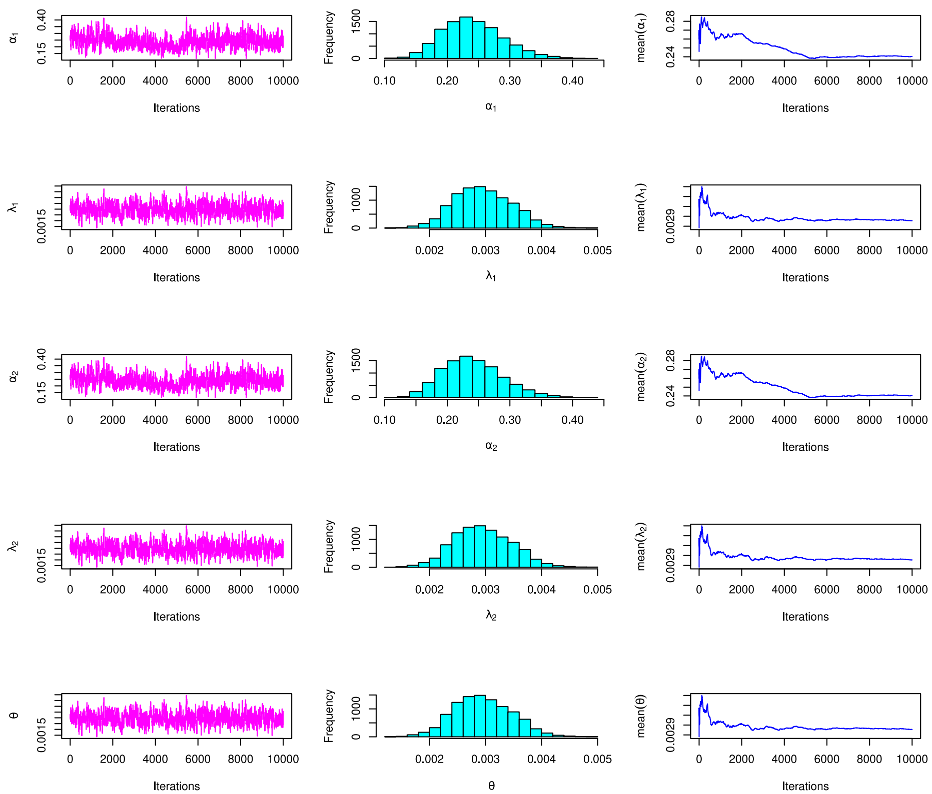

Fig. 5: Convergence of MCMC estimation of Clayton-BGR Model under complete sample for Medical Data

Table 13: The Sample Observation of Progressive Type-II Censoring with random sample for medical Data

P	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0.25	R	2	2	1	1	1	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0
	χ_{1i}	2	7	7	8	12	13	15	15	17	22	23	24	39	53	96	132	152	185	292	511
	$\chi_{2[i]}$	25	9	333	16	40	66	154	108	4	28	13	245	46	196	38	156	362	117	114	30
0.5	R	3	3	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	χ_{1i}	2	7	8	12	13	15	17	22	23	30	39	53	130	132	149	152	185	292	402	536
	$\chi_{2[i]}$	25	9	16	40	66	108	4	159	13	12	46	196	26	156	70	362	117	114	24	25
0.75	R	2	6	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	χ_{1i}	2	7	7	12	13	15	17	22	22	23	30	39	53	132	141	149	185	292	511	536
	$\chi_{2[i]}$	25	9	333	40	66	154	4	28	159	13	12	46	196	156	8	70	117	114	30	25
0.25	R	1	1	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
	χ_{1i}	2	7	7	8	12	13	15	15	17	22	22	23	30	39	53	113	130	132	141	152
	$\chi_{2[i]}$	25	9	333	16	40	66	154	108	4	28	159	13	12	46	196	201	26	156	8	362
0.5	R	1	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	χ_{1i}	2	7	7	12	13	15	15	17	22	23	30	34	39	53	96	113	141	149	152	185
	$\chi_{2[i]}$	25	9	333	40	66	154	108	4	159	13	12	30	46	196	38	201	8	70	362	117
0.75	R	1	3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	χ_{1i}	2	7	7	12	13	15	15	17	22	22	23	24	34	39	53	130	141	149	152	185
	$\chi_{2[i]}$	25	9	333	40	66	154	108	4	28	159	13	245	30	46	196	26	8	70	362	117
	I	21	22	23	24	25		i	21	22	23	24	25		i	21	22	23	24	25	
0.25	R	0	0	0	0	0		R	0	0	0	0	0	0.75	R	0	0	0	0	0	
	χ_{1i}	185	292	402	447	536		χ_{1i}	292	402	447	511	536		χ_{1i}	292	402	447	511	536	
	$\chi_{2[i]}$	117	114	24	318	25		$\chi_{2[i]}$	114	24	318	30	25		$\chi_{2[i]}$	114	24	318	30	25	

Table 14: MLE and Bayesian Estimation for Clayton-BGR Model under Progressive Type-II Censoring with random removal of medical data

m		20				25			
P		MLE		Bayesian		MLE		Bayesian	
		estimate	S.E.	estimate	S.E.	estimate	S.E.	estimate	S.E.
0.25	$\widehat{\alpha}_1$	0.2634	0.0555	0.2664	0.0552	0.2513	0.0518	0.2538	0.0496
	$\widehat{\lambda}_1$	0.0038	0.0009	0.0038	0.0009	0.0031	0.0007	0.0031	0.0006
	$\widehat{\alpha}_2$	0.3553	0.0842	0.3658	0.0779	0.3169	0.0697	0.3239	0.0693
	$\widehat{\lambda}_2$	0.0043	0.0010	0.0045	0.0009	0.0043	0.0009	0.0043	0.0008
	$\widehat{\theta}$	0.8993	0.5089	0.9084	0.4787	0.7194	0.4613	0.6039	0.3317
		AIC	CAIC	BIC	HQIC	AIC	CAIC	BIC	HQIC
		453.8108	458.0966	458.7895	454.7827	575.5951	578.7530	581.6894	577.2854
0.5	$\widehat{\alpha}_1$	0.2749	0.0591	0.2780	0.0533	0.2581	0.0537	0.2655	0.0556
	$\widehat{\lambda}_1$	0.0033	0.0008	0.0033	0.0007	0.0028	0.0006	0.0028	0.0006
	$\widehat{\alpha}_2$	0.3500	0.0827	0.3558	0.0811	0.3257	0.0725	0.3283	0.0724
	$\widehat{\lambda}_2$	0.0054	0.0012	0.0054	0.0011	0.0045	0.0009	0.0045	0.0008
	$\widehat{\theta}$	0.9209	0.4991	0.9730	0.4428	0.4764	0.4460	0.3358	0.2433
		AIC	CAIC	BIC	HQIC	AIC	CAIC	BIC	HQIC
		453.295	457.581	458.274	454.267	583.842	587.000	589.936	585.532
0.75	$\widehat{\alpha}_1$	0.2579	0.0550	0.2588	0.0524	0.2502	0.0520	0.2594	0.0521
	$\widehat{\lambda}_1$	0.0030	0.0007	0.0030	0.0007	0.0027	0.0006	0.0028	0.0005
	$\widehat{\alpha}_2$	0.3492	0.0856	0.3532	0.0818	0.3292	0.0738	0.3354	0.0716
	$\widehat{\lambda}_2$	0.0057	0.0012	0.0058	0.0012	0.0044	0.0008	0.0044	0.0008
	$\widehat{\theta}$	0.4137	0.4800	0.6308	0.4528	0.3353	0.4663	0.2445	0.2225
		AIC	CAIC	BIC	HQIC	AIC	CAIC	BIC	HQIC
		455.5928	459.8785	460.5715	456.5647	583.8717	587.0296	589.9661	585.5620

b. **Medical Data**

The data for 30 patients from McGilchrist and Aisbett (1991). Let x_1 refers to the first recurrence time and x_2 to the second recurrence time. In Table 10, the p values of the goodness of fit test, it is clear that the Clayton-BGR model fits the medical data.

In Table 11, we estimate the parameters of univariate GR distribution by using MLE, standard errors (S.E.), and using the Kolmogorov-Smirnov (KS) statistic with its P value to check these data are distributed for univariate GR distribution for football data as shown in Fig. 4.

Figure 4 discusses the plot of the max distance between the empirical CDF and GR CDF curves, histogram, PP-plot, and QQ-plot for GR distribution. Therefore, it indicates that GR distribution is fitted to the medical data set.

The parameter estimation of the Clayton-BGR model under complete sample for medical data can be shown in Table 12.

The convergence of MCMC estimation for the Clayton-BGR model under complete sample for medical data can be shown in Figs. 5. Histogram plot, approximate marginal posterior density, and MCMC convergence for parameters of Clayton-BGR model are represented in Fig. 5.

Assuming that, the observations are independent. Suppose, the experiment was conducted for the observations according to each censoring scheme in a random manner, these sample observations of progressive Type-II censoring with random removal based on concomitant order statistics are shown in Table 13.

The parameters estimation of the Clayton-BGR model by using MLE and Bayesian estimation methods under progressive Type-II censoring with random removal of medical data are obtained in Table 14.

From Tables 7, 9, 12, 14, parameter estimation of Clayton-BGR based on progressive Type-II censoring with random removal for medical and football data have been obtained. Bayesian estimation is a better fit than MLE under this model for S.E. When increase sample size, the S.E. is decreased for different causes, but the different measurement criteria are increasing.

9 Conclusion

In this paper, we discussed Bayesian and non–Bayesian estimation for the parameter of the Clayton-BGR distribution under progressive Type-II censoring schemes with random removal, which is of considerable interest and practical significance in many practical situations. Besides, we have obtained the asymptotic confidence interval and bootstrap confidence intervals by using percentile and bootstrap-t. The simulation results show that the Bayesian estimation method based on MCMC is a good performer for parameter estimation of the Clayton-BGR distribution in terms of bias, MSE, and length of CI. We notice that the Bootstrap CI is the shortest CI and the best. In most situations, we notice that the Bootstrap-P CI is better than Bootstrap-t CI. Also, we notice that the Bayes credible intervals are better than ACI. To illustrate the application of these schemes of this model, we provided two real data examples where the parameters of a Clayton-BGR distribution have been obtained.

References

ABDEL-HADY, D. H. (2013). Bivariate Generalized Rayleigh Distribution. Journal of Applied Sciences Research, 9(9), 5403-5411.

ALMETWALLY, E. M. & MUHAMMED, H. Z.. (2020). On a Bivariate Fréchet Distribution. Journal of Statistics Applications & Probability, 9(1), 1-21.

ALMETWALLY, E. M. (2019). Parameter Estimation of Bivariate Models under Some Censoring Schemes. Master Thesis, Faculty of Graduate Studies for Statistical Research, Cairo University.

ALMETWALLY, E. M., MUHAMMED, H. Z., & EL-SHERPIENY, E. S. A. (2020). Bivariate Weibull Distribution: Properties and Different Methods of Estimation. *Annals of Data Science*, *7*(1), 163-193.

ALY, H. M., MUHAMMED H. Z. & ABUELAMAYEM O. A.. (2020). Estimation of the Bivariate Kumaraswamy Lifetime Distribution under Progressive Type I Censoring. **To appear in Journal of data sciences**

BAHARI, F., PARSI, S., & GANJALI, M. (2020). Reliability of a soccer player based on the bivariate Rayleigh distribution with right-censored and ignorable missing data. Journal of Applied Statistics, 1-16.

BAI, X., SHI, Y., LIU, B., & FU, Q. (2018). Statistical inference of Marshall-Olkin bivariate Weibull distribution with three shocks based on progressive interval-censored data. Communications in Statistics-Simulation and Computation, 1-18.

BAI, X., SHI, Y., LIU, B., & FU, Q. (2019). Statistical inference of Marshall-Olkin bivariate Weibull distribution with three shocks based on progressive interval-censored data. Communications in Statistics-Simulation and Computation, 48(3), 637-654.

BALAKRISHNAN, N., & AGGARWALA, R. (2000). Progressive censoring: theory, methods, and applications. **Springer Science & Business Media**.

BALAKRISHNAN, N., & CRAMER, E. (2014). The art of progressive censoring. **Statistics for Industry and Technology**.

BALAKRISHNAN, N., & KIM, J. A. (2004). EM algorithm for Type-II right censored bivariate normal data. In Parametric and Semiparametric Models with Applications to Reliability, Survival Analysis, and Quality of Life (pp. 177-210). Birkhäuser, Boston, MA.

BALAKRISHNAN, N., & KIM, J. A. (2005_A). THE EM ALGORITHM AND OPTIMAL CENSORING SCHEMES FOR PROGRESSIVELY TYPE-II CENSORED BIVARIATE NORMAL DATA. IN ADVANCES IN RANKING AND SELECTION, MULTIPLE COMPARISONS, AND RELIABILITY (PP. 21-45). BIRKHÄUSER BOSTON.

BALAKRISHNAN, N., & KIM, J. A. (2005B). POINT AND INTERVAL ESTIMATION FOR BIVARIATE NORMAL DISTRIBUTION BASED ON PROGRESSIVELY TYPE-II CENSORED DATA. COMMUNICATIONS IN STATISTICS—THEORY AND METHODS, 34(6), 1297-1347.

CHEN, M. H., & SHAO, Q. M. (1999). Monte Carlo estimation of Bayesian credible and HPD intervals. Journal of Computational and Graphical Statistics, 8(1), 69-92.

CLAYTON, D. G. (1978). A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. Biometrika, 65(1), 141-151.

DAVID, H. A., & NAGARAJA, H. N. (1998). Concomitants of order statistics. Handbook of statistics, 16, 487-513.

EFRON, B. (1992). Bootstrap methods: another look at the jackknife. In Breakthroughs in statistics. Springer, New York, NY.

ELAAL, M. K. A., & JARWAN, R. S. (2017). Inference of Bivariate Generalized Exponential Distribution Based on Copula Functions. Applied Mathematical Sciences, 11(24), 1155-1186.

ELAAL, M. K. A., MAHMOUD, M. R., EL-GOHARY, M. M., & BAHARITH, L. A. (2016). Univariate and bivariate Burr Type X distributions based on mixtures and copula. Int. J. Math. Stat, 17(1), 113-127.

ELAALA, M. K. A., & BAHARITH, L. A. (2018). Univariate and bivariate Burr x-type distributions. International journal of advanced and applied sciences, 5(6), 64-69.

EL-MORSHEDY, M., ALHUSSAIN, Z. A., ATTA, D., ALMETWALLY, E. M., & ELIWA, M. S. (2020). Bivariate Burr X generator of distributions: properties and estimation methods with applications to complete and type-II censored samples. Mathematics, 8(2), 264.

EL-SHERPIENY, E. A., & ALMETWALLY, E. M. (2019) BIVARIATE GENERALIZED RAYLEIGH DISTRIBUTION BASED ON CLAYTON COPULA. IN PROCEEDINGS OF THE ANNUAL CONFERENCE ON STATISTICS (54RD), COMPUTER SCIENCE AND OPERATION RESEARCH , FACULTY OF GRADUATE STUDIES FOR STATISTICAL RESEARCH, CAIRO UNIVERSITY, 1-19.

EL-SHERPIENY, E. A., MUHAMMED, H. Z., & ALMETWALLY, E. M. (2018). FGM Bivariate Weibull Distribution. In Proceedings of the Annual Conference in Statistics (53rd), Computer Science, and Operations Research, Institute of Statistical Studies and Research, Cairo University.

EL-SHERPIENY, E. S. A., MUHAMMED, H. Z., & ALMETWALLY, E. M. (2019). Parameter Estimation of the Bivariate Weibull Distribution under Progressive Type-II Censored Samples. Journal of Modern Applied Statistical Methods. Accepted, To appear.

GENEST, C., HUANG, W., AND DUFOUR, J.-M. (2013). A regularized goodness-of-fit test for copulas. Journal de la Société française de statistique 154, 64–77.

HASTINGS, W. K. (1970). Monte Carlo sampling methods using Markov chains and their applications.

KIM, S. W., NG, H. K. T., & JANG, H. (2016). Estimation of parameters in a bivariate generalized exponential distribution based on Type-II censored samples. Communications in Statistics-Simulation and Computation, 45(10), 3776-3797.

KOTZ, S., BALAKRISHNAN, N., & JOHNSON, N. L. (2004). *Continuous multivariate distributions*, Volume 1: Models and applications (Vol. 1). John Wiley & Sons.

KUNDU, D., & DEY, A. K. (2009). Estimating the parameters of the Marshall–Olkin bivariate Weibull distribution by EM algorithm. *Computational Statistics & Data Analysis*, *53*(4), 956-965.

KUNDU, D., & GUPTA, A. K. (2013). Bayes estimation for the Marshall–Olkin bivariate Weibull distribution. *Computational Statistics & Data Analysis*, *57*(1), 271-281.

LIANG, W. A. N. G., HUANYU, L. I., & JIN'GE, M. (2019). Inference for dependence competing risks from bivariate exponential model under generalized progressive hybrid censoring with partially observed failure causes. Journal of Systems Engineering and Electronics, 30(1), 201-208.

MCGILCHRIST, C. A., & AISBETT, C. W. (1991). Regression with frailty in survival analysis. Biometrics, 47(2), 461-466.

MEINTANIS, S. G. (2007). Test of fit for Marshall–Olkin distributions with applications. Journal of Statistical Planning and inference, 137(12), 3954-3963.

METROPOLIS, N., ROSENBLUTH, A. W., ROSENBLUTH, M. N., TELLER, A. H., AND TELLER, E. (1953). Equation of state calculations by fast computing machines. J. Chem Phys., 21, 1087 – 1091.

NADARAJAH, S., AFUECHETA, E., & CHAN, S. (2018). A compendium of copulas. Statistica, 77(4), 279-328.

NELSEN, R.B.. (2006). An Introduction to Copulas. Springer, New York.

PAK, A., KHOOLENJANI, N. B., & JAFARI, A. A. (2014). Inference on P (Y¡ X) in bivariate Rayleigh distribution. *Communications in Statistics-Theory and Methods*, *43*(22), 4881-4892.

SARHAN, A. (2019). The Bivariate Generalized Rayleigh Distribution. Journal of Mathematical Sciences and Modelling, 2(2), 99-111.

SHUBHASHREE, J. (2019). Estimation of Multicomponent System Reliability for a Bivariate Generalized Rayleigh Distribution. International Journal of Computational and Theoretical Statistics, 6(1), 51-63.

SKLAR A (1973) RANDOM VARIABLES, JOINT DISTRIBUTIONS, AND COPULAS. KYBERNETICA 9:449-460.

WANG, L., LI, M., & TRIPATHI, Y. M. (2020). Inference for dependent competing risks from bivariate Kumaraswamy distribution under generalized progressive hybrid censoring. Communications in Statistics-Simulation and Computation, 1-24.

Publisher´s Note. Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

El-Sayed A. El-Sherpieny
Hiba Z. Muhammed
Faculty of Graduate Studies for Statistical Research, Cairo University, Cairo, Egypt

Ehab M. Almetwally
Faculty of Business Administration, Delta University of Science and Technology, Al-Mansoura, Egypt
E-mail: ehabxp_2009@hotmail.com