# UNIVERSAL INTERVALS

## *Towards a Dependency-Aware Interval Algebra*

CHAPTER ABSTRACT • JOHN WILEY & SONS, 2020

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**ABSTRACT**

Interval computations are most fundamental in addressing uncertainty and imprecision. The intended status of this chapter is to be both an introduction and a treatise on some theoretical and practical aspects of interval mathematics. In the body of the work, there is room for novelties which may not be devoid of interest to researchers and specialists. The theories of classical intervals and parametric intervals are formally constructed and their mathematical structures are uncovered. By means of the logical concepts of Skolemization and quantification dependence, the notion of interval dependency is formalized by putting on a systematic basis its meaning, and thus gaining the advantage of indicating formally the criteria by which it is to be characterized and, accordingly, deducing its fundamental properties in a merely logical manner. Moreover, with a view to treating some problems of the present interval theories, a new alternate theory of intervals, namely the "theory of universal intervals", is presented and proved to have a nice S-field algebra, which extends the ordinary field of the reals. Our approach is formal by the pursuit of formulating the mathematical concepts in a strictly accurate manner, our perspective is systematic by taking the passage from the informal treatments to the formal technicalities of mathematical logic, and our concern is to take one small step towards paving the way for developing dependency-aware interval methods.

***Keywords*** • Interval mathematics, Classical interval arithmetic, Parametric interval arithmetic, Universal interval arithmetic, Interval dependency, Functional dependence, Guaranteed enclosures, S-Semiring, S-Field, Skolemization.

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*The most reliable way of carrying out a proof, obviously, is to follow pure logic, a way that, disregarding the particular characteristics of objects, depends solely on those laws upon which all knowledge rests.*

Gottlob Frege (1848–1925)

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