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Parametric Intervals: More Reliable or Foundationally Problematic?

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Abstract

Interval arithmetic has been proved to be very subtle, reliable, and most fundamental in addressing uncertainty and imprecision. However, the theory of classical interval arithmetic and all its alternates suffer from algebraic anomalies, and all have difficulties with interval dependency. A theory of interval arithmetic that seems promising is the theory of parametric intervals. The theory of parametric intervals is presented in the literature with the zealous claim that it provides a radical solution to the long-standing dependency problem in the classical interval theory, along with the claim that parametric interval arithmetic, unlike Moore's classical interval arithmetic, has additive and multiplicative inverse elements, and satisfies the distributive law. So, does the theory of parametric intervals accomplish these very desirable objectives? Here it is argued that it does not.

Keywords: Interval mathematics; Classical interval arithmetic; Parametric interval arithmetic; Constrained interval arithmetic; Overestimation-free interval arithmetic; Interval dependency; Functional dependence; Dependency predicate; Interval enclosures; S-semiring; Uncertainty; Reliability

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> If you have to prove a theorem, do not rush. First of all, understand fully what the theorem says, try to see clearly what it means. Then check the theorem; it could be false. Examine the consequences, verify as many particular instances as are needed to convince yourself of the truth. When you have satisfied yourself that the theorem is true, you can start proving it.
>
> – George Polya (1887–1985)

1 Introduction

Does the way, in which we learn about the physical world, meet our persistent seeking for deterministic scientific knowledge? Our knowledge of real-world systems is acquired through *observing*, *experimenting*, *measuring identifiable features*, and *formulating hypotheses* with the aid of formal reasoning ([12] and [21]). Through our encounters with the physical world, it reveals itself to us as systems of uncertain quantifiable properties. Knowledge, then, is not absolute certainty. Knowledge, per contra, is the theoretical and practical tools we develop to purposely get better and better outcomes through our learning about the world [12]. In the effort to deal with quantifiable uncertainties, various theoretical approaches have been developed. One approach that proved to be subtle, reliable, and most fundamental in all of mathematics of uncertainty is interval mathematics. The field of interval mathematics has its roots from the Greek mathematician Archimedes of Syracuse (287–212 BC), who used guaranteed lower and upper bounds to compute his constant π [32], to the American mathematician Ramon Edgar Moore (1929–2015), who was the first to define interval analysis in its modern sense and recognize its power as a viable computational tool for intervalizing uncertainty [45]. In between, historically speaking, several distinguished constructions of interval arithmetic by John Charles Burkill, Rosalind Cecily Young, Paul S. Dwyer, Teruo Sunaga, and others (see, e.g., [4], [61], [24], and [56]) have emphasized the very idea of reasoning about uncertain values through calculating with intervals. In the fifties and sixties of the twentieth century, research in interval mathematics and its applications has started to blossom with the works of Paul S. Dwyer, Ramon Edgar Moore, Raymond E. Boche, Sidney Shayer, and others who made the term "interval arithmetic" popular (see, [24], [45], [3], and [54]). By integrating the complementary powers of rigorous mathematics and scientific

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computing, interval arithmetic is able to offer highly reliable accounts of uncertainty. It should therefore come as no surprise that the interval theory has been profitably applied in many areas that deal intensely with uncertain quantitative data (see, e.g., [12], [14], [23], [22], [27], [29], [38], [39], and [46]).

As a result of the computational power against error provided by the interval methods, machine realizations of interval arithmetic are of utmost importance. As a matter of course, there are various software implementations of interval arithmetic. As instances, one may mention INTLAB, Sollya, InCLosure and others (see, e.g., [53], [5], [13], and [44]). Fortunately, with the tremendous computing power of parallel and distributed processing, computers are getting faster and faster. That being so, it is very possible to make interval computations as fast as floating point computations (For further reading about machine arithmetizations and hardware circuitries for interval arithmetic, see, e.g., [10], [14], [37], [36], [41], [48], [34], and [35]).

Nevertheless, despite all these advantages of interval mathematics, it has its pitfalls as well. The algebraic system of classical interval arithmetic is a *commutative* S*-semiring* [22], which is a primitive algebraic structure, if compared to the totally ordered field of real numbers. Two useful properties of ordinary real arithmetic fail to hold in classical interval arithmetic: additive and multiplicative inverses do not always exist for interval numbers, and there is no distributivity between addition and multiplication except for certain special cases. Another main drawback of the classical interval theory emerges from a peculiarity of interval arithmetic: interval arithmetic considers *all instances* of variables as *independent*. Accordingly, when computing interval enclosure of real functions some of whose variables are *functionally dependent*, we usually get overestimations, and in many situations, the interval enclosure might be too wide to be useful. This phenomenon is known as the *interval dependency problem* [21].

A considerable scientific effort is put into developing special methods and algorithms that try to overcome the difficulties imposed by the algebraic system of classical intervals. Also, various proposals for possible alternate theories of interval arithmetic were introduced to reduce the dependency effect or to enrich the algebraic structure of interval numbers (For further details, the reader may consult, e.g., [11], [28], [30], [31], [40], [42], and [43]). One theory of interval arithmetic that seems promising is the theory of *parametric intervals*. The term “parametric interval arithmetic” is reasonably recent, but the idea perhaps has an earlier root in Cleary’s “logical arithmetic” which is a logical technique, for real arithmetic in Prolog, that uses constraints over real intervals (see [7] and [8]). In the course of history, parametric interval arithmetic has been invented and re-invented several times, under different names: “logical intervals”, “constrained intervals”, “instantiation intervals”, “RDM intervals”, and “Overestimation-free intervals” (see, e.g., [7], [10], [11], [22], [25], [42], and [49]).

But, what exactly is the algebraic structure of parametric interval arithmetic? The interval literature does not provide an answer for this question which looks simple and answerable, but when subject to formal investigation, develops into a problematic situation. What is noteworthy here also is that the parametric approach to interval analysis is usually introduced in the interval literature with the very zealous claims that parametric interval arithmetic, unlike classical interval arithmetic, has additive and multiplicative inverse elements, satisfies the distributive law, and explicitly provides a solution to the long-standing dependency problem. So, *does the theory of parametric intervals accomplish these very desirable objectives?* In this article, *it is argued that it does not*.

The underlying idea of parametric interval arithmetic seems elegant and simple, but it is *too simple* to fully account for the notion of interval dependency or to achieve a richer algebraic structure for interval arithmetic. It is therefore imperative both to supply the defect in the parametric approach and to present an alternative theory with a mathematical construction that avoids the defect. The former is attempted in this article, and the latter is attempted in [22].

Our main purpose in this article is then to introduce the theory of parametric intervals and to mathematically examine to what extent it can accomplish these very desirable algebraic properties. For this goal to be met, we have to set the stage by formalizing a number of purely logical and algebraic notions of particular importance for our purpose. This formalization is mainly done in section 2. Section 3 is devoted to recasting “classical interval arithmetic” in a formalized theory $\mathsf{Th}_{\mathcal{I}}$ of interval algebra over the real field, along with providing a logical systematization of interval dependency as presented in [21]. With the aid of this systematization, we provide, in sections 4 and 5, an in-depth investigation of the theory $\mathsf{Th}_{\mathcal{P}}$ of parametric intervals and examine to what extent its claims are accomplishable. The investigation of parametric intervals conducted in this article is *metatheoretical* in nature and based indispensably on the systematization of interval dependency presented in [21] and the metamathematical apparatus fixed in section 2. That is, most of the results deduced in this article are *metatheorems about* the theory $\mathsf{Th}_{\mathcal{P}}$ of parametric intervals.

2 On Some Fundamental Notions of Logic and Algebra

The adaptation of a particular formal approach, other than that of natural language, is of the utmost importance and has been forced on us by the pursuit of a strictly accurate formulation of the mathematical concepts. Therefore, we commence the work by specifying some notational conventions and formalizing some purely logical and algebraic ingredients we shall need throughout this text (For further details about the notions prescribed here, the reader may consult, e.g., [2], [16], [17], [20], [14], [21], [22], and [57]).

A set-theoretical relation is a particular type of sets. Let $\mathcal{S}^2$ be the binary Cartesian power of a set $\mathcal{S}$. A binary relation on $\mathcal{S}$ is a subset of $\mathcal{S}^2$. That is, a set $\mathfrak{R}$ is a binary relation on a set $\mathcal{S}$ iff $(\forall \mathbf{r} \in \mathfrak{R})\,((\exists x, y \in \mathcal{S})\,(\mathbf{r} = (x,y))$. We will follow Suppes [57] and Tarski [58] in defining, within a set-theoretical framework, the notion of a *finitary relation* and some related concepts. Let $\mathcal{U}^n$ be the n-th Cartesian power of a set $\mathcal{U}$. A set $\mathfrak{R} \subseteq \mathcal{U}^n$ is an n-ary relation on $\mathcal{U}$ iff $\mathfrak{R}$ is a binary relation from $\mathcal{U}^{n-1}$ to $\mathcal{U}$. That is, for $\mathbf{v} = (x_1, \dots, x_{n-1}) \in \mathcal{U}^{n-1}$ and $y \in \mathcal{U}$, an n-ary relation $\mathfrak{R}$ is defined to be $\mathfrak{R} \subseteq \mathcal{U}^n = \{(\mathbf{v}, y)\,|\,\mathbf{v} \in \mathcal{U}^{n-1} \wedge y \in \mathcal{U}\}$. In this sense, an n-ary relation is a binary relation (or simply a relation); then its *domain*, *range*, *field*, and *converse* are defined to be, respectively $\mathrm{dom}(\mathfrak{R}) = \{\mathbf{v} \in \mathcal{U}^{n-1}|\,(\exists y \in \mathcal{U})\,(\mathbf{v}\mathfrak{R}y)\}$, $\mathrm{ran}(\mathfrak{R}) = \{y \in \mathcal{U}|\,\left(\exists \mathbf{v} \in \mathcal{U}^{n-1}\right)(\mathbf{v}\mathfrak{R}y)\}$, $\mathrm{fld}(\mathfrak{R}) = \mathrm{dom}(\mathfrak{R}) \cup \mathrm{ran}(\mathfrak{R})$, and $\widehat{\mathfrak{R}} = \{(y, \mathbf{v}) \in \mathcal{U}^n|\mathbf{v}\mathfrak{R}y\}$. It is thus obvious that $y\widehat{\mathfrak{R}}\mathbf{v} \Leftrightarrow \mathbf{v}\mathfrak{R}y$ and $\widehat{\widehat{\mathfrak{R}}} = \mathfrak{R}$.

The notions of the *image* and *preimage* of a set, with respect to an n-ary relation, are characterized as follows ([21] and [22]).

Definition 2.1 (Image and Preimage of a Relation). *Let $\mathfrak{R}$ be an n-ary relation on a set $\mathcal{U}$, and for $(\mathbf{v}, y) \in \mathfrak{R}$, let $\mathbf{v} = (x_1, \dots, x_{n-1})$, with each x_k is restricted to vary on a set $X_k \subset \mathcal{U}$, that is, $\mathbf{v}$ is restricted to vary on a set $\mathbf{V} \subset \mathcal{U}^{n-1}$. Then, the image of $\mathbf{V}$ (or the image of the sets X_k) with respect to $\mathfrak{R}$, denoted $\mathrm{I}_{\mathfrak{R}}$, is defined to be*

$$\begin{aligned} Y = \mathrm{I}_{\mathfrak{R}}(\mathbf{V}) &= \mathrm{I}_{\mathfrak{R}}(X_1, \dots, X_{n-1}) \\ &= \{y \in \mathcal{U}|\,(\exists \mathbf{v} \in \mathbf{V})\,(\mathbf{v}\mathfrak{R}y)\} \\ &= \{y \in \mathcal{U}|\,\left(\exists_{k=1}^{n-1} x_k \in X_k\right)((x_1, \dots, x_{n-1})\,\mathfrak{R}y)\}, \end{aligned}$$

where the set $\mathbf{V}$, called the preimage of Y, is defined to be the image of Y with respect to the converse relation $\widehat{\mathfrak{R}}$, that is

$$\mathbf{V} = \mathrm{I}_{\widehat{\mathfrak{R}}}(Y) = \{\mathbf{v} \in \mathcal{U}^{n-1}|\,(\exists y \in Y)\left(y\widehat{\mathfrak{R}}\mathbf{v}\right)\}.$$

In accordance with this definition and the fact that $y\widehat{\mathfrak{R}}\mathbf{v} \Leftrightarrow \mathbf{v}\mathfrak{R}y$, we obviously have $Y = \mathrm{I}_{\mathfrak{R}}(\mathbf{V}) \Leftrightarrow \mathbf{V} = \mathrm{I}_{\widehat{\mathfrak{R}}}(Y)$.

Now for a mathematically satisfactory characterization of a finitary function. Within this set-theoretical framework, a completely general definition of the notion of an *n-ary function* can be formulated. A set f is an n-ary function on a set $\mathcal{U}$ iff f is an $(n+1)$-ary relation on $\mathcal{U}$, and $(\forall \mathbf{v} \in \mathcal{U}^n)\,(\forall y, z \in \mathcal{U})\,(\mathbf{v} f y \wedge \mathbf{v} f z \Rightarrow y = z)$. Thus, an n-ary function is a *many-one* $(n+1)$-ary relation; that is, a relation, with respect to which, any element in its domain is related exactly to one element in its range. Getting down from relations to the particular case of functions, we have at hand the standard notation: $y = f(\mathbf{v})$ in place of $\mathbf{v} f y$. From the fact that an n-ary function is a special kind of relation, then all the preceding definitions and results, concerning the *domain*, *range*, *field*, and *converse* of a relation, apply to functions as well.

With some criteria satisfied, a function is called *invertible*. A function f has an inverse, denoted f^{-1}, iff its converse relation $\widehat{f}$ is a function, in which case $f^{-1} = \widehat{f}$. In other words, f is invertible if, and only if, it is an *injection* from its domain to its range, and obviously the inverse f^{-1} is *unique*, from the fact that the converse relation is always *definable* and *unique*.

With the aid of the above concepts, we next define the notions of *partial* and *total operations*.

Definition 2.2 (Partial and Total Operations). *Let $\mathcal{S}^n$ be the n-th Cartesian power of a set $\mathcal{S}$. An n-ary (total) operation on $\mathcal{S}$ is a total function $t_n : \mathcal{S}^n \longmapsto \mathcal{S}$. An n-ary partial operation in $\mathcal{S}$ is a partial function $p_n : \mathcal{U} \longmapsto \mathcal{S}$, where $\mathcal{U} \subset \mathcal{S}^n$. The ordinal n is called the arity of t_n or p_n.*

A binary operation is an n-ary operation for $n = 2$. Addition and multiplication on the set $\mathbb{R}$ of real numbers are best-known examples of binary *total* operations, while division is a *partial* operation in $\mathbb{R}$.

A formalized theory is characterized by two things; an object language in which the theory is formalized (the symbolism of the theory), and a set of axioms. Let $\mathfrak{L}$ be an object formal language. A formalized theory in $\mathfrak{L}$ (or an $\mathfrak{L}$-theory) is a set of $\mathfrak{L}$-sentences which is closed under its associated deductive apparatus. Let $\Lambda_{\mathfrak{T}}$ denote a finite set of $\mathfrak{L}$-sentences, and let φ denote an $\mathfrak{L}$-sentence. The formalized $\mathfrak{L}$-theory $\mathfrak{T}$ of the set $\Lambda_{\mathfrak{T}}$ is the deductive closure of $\Lambda_{\mathfrak{T}}$ under logical consequence, that is

$$\mathfrak{T} = \{\varphi \in \mathfrak{L}|\varphi \text{ is a consequence of } \Lambda_{\mathfrak{T}}\}.$$

The set $\Lambda_{\mathfrak{T}}$ is called the set of axioms (or postulates) of $\mathfrak{T}$.

A model (or an interpretation) of a theory $\mathfrak{T}$ is some particular (algebraic or relational) structure that satisfies every formula of $\mathfrak{T}$. The notion of a structure of a formalized language $\mathfrak{L}$ (or an $\mathfrak{L}$-structure) is characterized in the following definition (see [21] and [22]).

Definition 2.3 (Structures). *Let $\mathfrak{L}$ be a formalized language (possibly with no individual constants). By an $\mathfrak{L}$-structure we understand a system $\mathfrak{M} = \langle \mathcal{A}; F^{\mathfrak{M}}; R^{\mathfrak{M}} \rangle$, where*

- *$\mathcal{A}$ is a (possibly empty) set called the* individuals universe *of $\mathfrak{M}$. The elements of $\mathcal{A}$ are called the* individual elements *of $\mathfrak{M}$;*
- *$F^{\mathfrak{M}}$ is a (possibly empty) set of finitary* total *operations on $\mathcal{A}$. The elements of $F^{\mathfrak{M}}$ are called the $\mathfrak{M}$-operations;*
- *$R^{\mathfrak{M}}$ is a (possibly empty) set of finitary relations on $\mathcal{A}$. The elements of $R^{\mathfrak{M}}$ are called the $\mathfrak{M}$-relations.*

An $\mathfrak{L}$-structure with an empty individuals universe is called an *empty $\mathfrak{L}$-structure* [1]. By a *many-sorted $\mathfrak{L}$-structure* we understand an $\mathfrak{L}$-structure with more than one universe set. By a *relational $\mathfrak{L}$-structure* we understand an $\mathfrak{L}$-structure with a non-empty set of relations and an empty set of functions. By an *algebraic $\mathfrak{L}$-structure* (or an *$\mathfrak{L}$-algebra*) we understand an $\mathfrak{L}$-structure with a non-empty set of functions. An $\mathfrak{L}$-algebra endowed with a compatible ordering relation $\mathfrak{R}$ is called an $\mathfrak{R}$-ordered $\mathfrak{L}$-algebra.

The notion of *inverse elements* can be precisely formulated in the following definition ([22]).

Definition 2.4 (Inverse Elements). *Let $\langle \mathcal{A}; \bullet; \mathbf{e}_\bullet \rangle$ be an algebra with $\bullet$ is a binary operation on $\mathcal{A}$ and $\mathbf{e}_\bullet$ is the identity element for $\bullet$. We say that every element of $\mathcal{A}$ has an inverse element with respect to the operation $\bullet$ iff*

$$(\forall x \in \mathcal{A})\,(\exists y \in \mathcal{A})\,(x \bullet y = \mathbf{e}_\bullet).$$

Next, we characterize some algebraic structures of particular importance to our purpose (see [21] and [22]).

Definition 2.5 (Ringoid). *A ringoid (or a ring-like structure) is a structure $\mathfrak{R} = \langle \mathcal{R}; +_\mathcal{R}, \times_\mathcal{R} \rangle$ with $+_\mathcal{R}$ and $\times_\mathcal{R}$ are total binary operations on the universe set $\mathcal{R}$. The operations $+_\mathcal{R}$ and $\times_\mathcal{R}$ are called respectively the addition and multiplication operations of the ringoid $\mathfrak{R}$.*

Definition 2.6 (S-Ringoid). *An S-ringoid (or a subdistributive ringoid) is a ringoid that satisfies at least one of the following subdistributive criteria.*

(i) $(\forall x,y,z \in \mathcal{R})\,(x \times_\mathcal{R} (y +_\mathcal{R} z) \subseteq x \times_\mathcal{R} y +_\mathcal{R} x \times_\mathcal{R} z)$,

(ii) $(\forall x,y,z \in \mathcal{R})\,((y +_\mathcal{R} z) \times_\mathcal{R} x \subseteq y \times_\mathcal{R} x +_\mathcal{R} z \times_\mathcal{R} x)$.

Criteria (i) and (ii) in the preceding definition are called respectively left and right subdistributivity (or S-distributivity).

Definition 2.7 (Semiring). *Let $\mathfrak{R} = \langle \mathcal{R}; +_\mathcal{R}, \times_\mathcal{R} \rangle$ be a ringoid. $\mathfrak{R}$ is said to be a semiring iff*

(i) *$\langle \mathcal{R}; +_\mathcal{R} \rangle$ is a commutative monoid with identity element $0_\mathcal{R}$,*

(ii) *$\langle \mathcal{R}; \times_\mathcal{R} \rangle$ is a monoid with identity element $1_\mathcal{R}$,*

(iii) *Multiplication, $\times_\mathcal{R}$, left and right distributes over addition, $+_\mathcal{R}$,*

(iv) *$0_\mathcal{R}$ is an absorbing element for $\times_\mathcal{R}$.*

A commutative semiring is one whose multiplication is commutative.

Definition 2.8 (S-Semiring). *An S-semiring (or a subdistributive semiring) is an S-ringoid that satisfies criteria (i), (ii), and (iv) in definition 2.7. A commutative S-semiring is one whose multiplication is commutative.*

Here, let us make a note: the notion of S-semiring is a generalization of the notion of a *near*-semiring; a near-semiring is a ringoid that satisfies the criteria of a semiring except that it is *either* left or right distributive (For detailed discussions of near-semirings and related concepts, the interested reader may consult, e.g., [60], [50], and [6]).

Two further definitions we shall need are those of a *number system* and an S-*number system* ([22]).

Definition 2.9 (Number System). *A number system is a ringoid $\mathfrak{N} = \langle \mathcal{N}; +_\mathcal{N}, \times_\mathcal{N} \rangle$ with $+_\mathcal{N}$ and $\times_\mathcal{N}$ are each both commutative and associative, and $\times_\mathcal{N}$ distributes over $+_\mathcal{N}$.*

Definition 2.10 (S-Number System). *An S-number system (or a subdistributive number system) is an S-ringoid $\mathfrak{N} = \langle \mathcal{N}; +_\mathcal{N}, \times_\mathcal{N} \rangle$ with $+_\mathcal{N}$ and $\times_\mathcal{N}$ are each both commutative and associative.*

[1] First-order logics with empty structures were first considered by Mostowski in [47], and then studied by many logicians (see, e.g., [51], [33], and [1]). Such logics are now referred to as *free logics*.

The notion of *dependency* is one of the most fundamental ingredients that underlie mathematical reasoning and scientific reasoning in general. The notion of *dependency* comes from the notion of a *function*. Not surprisingly, therefore, there is scarcely a mathematical theory which does not involve the notion of a function (See, e.g., [11], [21], and [22]). We close this section by fixing a symbolic apparatus suitable and adequate for reducing the notion of *functional dependence* to the pure logical concepts of *Skolemization* and *quantification dependence*.

Two pure logical notions we shall need are those of a quantification matrix and a prenex sentence. A *quantification matrix* $\mathcal{Q}$ is a sequence $(Q_1 x_1) \ldots (Q_n x_n)$, where $x_1, \ldots, x_n$ are variable symbols and each Q_i is $\forall$ or $\exists$. A *prenex sentence* is a sentence of the form $\mathcal{Q}\varphi$, where $\mathcal{Q}$ is a quantification matrix and φ is a quantifier-free formula.

The most fundamental part of all mathematics is formal logic. So, getting down to the most elementary fundamentals, it can be shown that in all mathematical theories, any type of dependence can be reduced to the following simple logical definition (see, e.g., [21] and [22]).

Definition 2.11 (Quantification Dependence). *Let $\mathcal{Q}$ be a quantification matrix and let $\varphi(x_1, \ldots, x_m; y_1, \ldots, y_n)$ be a quantifier-free formula. For any universal quantification $(\forall x_i)$ and any existential quantification $(\exists y_j)$ in $\mathcal{Q}$, the variable y_j is dependent on the variable x_i in the prenex sentence $\mathcal{Q}\varphi$ iff $(\exists y_j)$ is in the scope of $(\forall x_i)$ in $\mathcal{Q}$. Otherwise x_i and y_j are independent.*

That is, the order of quantifiers in a quantification matrix determines the mutual dependence between the variables in a sentence. For example, consider the prenex sentence

$$(\exists x)(\forall y)(\exists z)(y = x \circ y \wedge x = z \circ y),$$

which asserts that there exists an identity element x, for the operation $\circ$, with respect to which every element possesses an inverse z. According to the order in which quantifiers are written, the variable z depends only on y, while there is no dependency between x and y or between x and z. Also, in the prenex sentence

$$(\forall x)(\exists y)(\forall z)(\exists u)\varphi(x, y, z, u),$$

the variable y depends on x, and the variable u depends on both x and z.

By means of a *Skolem equivalent form* or a *Skolemization*[2], a quantification dependence is translated into a functional dependence. The notion of a Skolem equivalent form is characterized in the following definition (see, e.g., [21], [22], [26], and [59]).

Definition 2.12 (Skolem Equivalent Form). *Let σ be a sentence that takes the prenex form*

$$(\forall_{i=1}^{m} x_i)\left(\exists_{j=1}^{n} y_j\right)\varphi(x_1, \ldots, x_m; y_1, \ldots, y_n).$$

where φ is a quantifier-free formula.

The Skolem equivalent form of σ is defined to be

$$\left(\exists_{j=1}^{n} f_j\right)(\forall_{i=1}^{m} x_i)\varphi(x_1, \ldots, x_m; f_1, \ldots, f_n),$$

where $f_j(x_1, \ldots, x_m) = y_j$ are the dependency functions of y_j upon $x_1, \ldots, x_m$, for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Not surprisingly, therefore, that in all mathematics, any instance of a dependence is, in fact, a functional dependence. To further illustrate, it is sufficient to give an example. Let a sentence σ take the prenex form

$$(\forall x)(\exists y)(\forall z)(\exists u)\varphi(x, y, z, u).$$

The Skolem equivalent form of σ is

$$(\exists f)(\exists g)(\forall x)(\forall z)\varphi(x, f(x), z, g(x, z)).$$

3 Classical Interval Algebra and Interval Dependence: A Formalized Treatment

There are many theories of interval arithmetic (see, e.g., [30], [41], [28], [43], [42], [34], [35], [10], [19], [15], and [22]). We are here interested in *classical interval arithmetic* as introduced in, e.g., [45], [54], [46], [12], [14], and [21]. The present section therefore is devoted to recasting "classical interval arithmetic" in a formalized theory $\mathsf{Th}_{\mathcal{I}}$ of interval algebra over the real field. We shall do such a recasting by putting on a systematic basis the fundamental notions of

[2] *Skolemization* is named after the Norwegian logician Thoralf Skolem (1887–1963), who first presented the notion in [55].

the classical interval theory, with the requisite formalized apparatus of section 2. After characterizing the theory $\mathsf{Th}_{\mathcal{I}}$, we close this section by providing a logical systematization of interval dependency as presented in [21]. With the aid of this systematization, we provide, in section 5, an in-depth investigation of the theory of parametric intervals and examine to what extent its claims are accomplishable.

Hereafter and throughout this work, the machinery used, and assumed priori, is the *standard* (*classical*) predicate calculus and axiomatic set theory. Moreover, in all the proofs, elementary facts about operations and relations on the real numbers are usually used without explicit reference.

A theory $\mathsf{Th}_{\mathcal{I}}$ of a *real interval algebra* (a *classical interval algebra* or an *interval algebra over the real field*) can be characterized in the following definition (see [21] and [22]).

Definition 3.1 (Theory of Real Interval Algebra). *Take $\sigma = \{+,\times;-,^{-1};0,1\}$ as a set of non-logical constants and let $\mathsf{R} = \langle \mathbb{R};\sigma^{\mathsf{R}} \rangle$ be the totally $\leq$-ordered field of real numbers. The theory $\mathsf{Th}_{\mathcal{I}}$ of an interval algebra over the field R is the theory of a two-sorted structure $\mathfrak{I}_{\mathbb{R}} = \langle \mathcal{I}_{\mathbb{R}};\mathbb{R};\sigma^{\mathfrak{I}_{\mathbb{R}}} \rangle$ prescribed by the following set of axioms.*

(I1) $(\forall X \in \mathcal{I}_{\mathbb{R}})\,(X = \{x \in \mathbb{R}|\,(\exists \underline{x} \in \mathbb{R})\,(\exists \overline{x} \in \mathbb{R})\,(\underline{x} \leq_{\mathbb{R}} x \leq_{\mathbb{R}} \overline{x})\})$,

(I2) $(\forall X,Y \in \mathcal{I}_{\mathbb{R}})\,\big(\circ \in \{+,\times\} \Rightarrow X \circ_{\mathcal{I}_{\mathbb{R}}} Y = \{z \in \mathbb{R}|\,(\exists x \in X)\,(\exists y \in Y)\,(z = x \circ_{\mathbb{R}} y)\}\big)$,

(I3) $(\forall X \in \mathcal{I}_{\mathbb{R}})\,\big(\diamond \in \{-\} \vee \big(\diamond \in \{^{-1}\} \wedge 0_{\mathcal{I}_{\mathbb{R}}} \not\subseteq X\big) \Rightarrow \diamond_{\mathcal{I}_{\mathbb{R}}} X = \{z \in \mathbb{R}|\,(\exists x \in X)\,(z = \diamond_{\mathbb{R}} x)\}\big)$.

The sentence (I1) of definition 3.1 characterizes what an interval number (or a closed $\mathbb{R}$-interval) is. The sentences (I2) and (I3) prescribe, respectively, the binary and unary operations for $\mathbb{R}$-intervals. Hereafter, the upper-case Roman letters X, Y, and Z (with or without subscripts), or equivalently $[\underline{x},\overline{x}]$, $[\underline{y},\overline{y}]$, and $[\underline{z},\overline{z}]$, shall be employed as variable symbols to denote real interval numbers. A *point* (*singleton*) interval number $\{x\}$ shall be denoted by $[x]$. The letters A, B, and C, or equivalently $[\underline{a},\overline{a}]$, $[\underline{b},\overline{b}]$, and $[\underline{c},\overline{c}]$, shall be used to denote constants of $\mathcal{I}_{\mathbb{R}}$. Also, we shall single out the symbols $1_{\mathcal{I}}$ and $0_{\mathcal{I}}$ to denote, respectively, the singleton $\mathbb{R}$-intervals $\{1_{\mathbb{R}}\}$ and $\{0_{\mathbb{R}}\}$. For the purpose at hand, it is convenient to define two proper subsets of $\mathcal{I}_{\mathbb{R}}$: the sets of *symmetric interval numbers* and *point interval numbers*. Respectively, these are defined and denoted by

$$\mathcal{I}_{\mathsf{S}} = \{X \in \mathcal{I}_{\mathbb{R}}|(\exists x \in \mathbb{R})(0 \leq x \wedge X = [-x,x])\},$$
$$\mathcal{I}_{[x]} = \{X \in \mathcal{I}_{\mathbb{R}}|(\exists x \in \mathbb{R})(X = [x,x])\}.$$

From the fact that real intervals are totally $\leq_{\mathbb{R}}$-ordered subsets of $\mathbb{R}$, equality of $\mathbb{R}$-intervals follows immediately from the axiom of extensionality [3] of set theory. That is,

$$[\underline{x},\overline{x}] =_{\mathcal{I}} [\underline{y},\overline{y}] \Leftrightarrow \underline{x} =_{\mathbb{R}} \underline{y} \wedge \overline{x} =_{\mathbb{R}} \overline{y}.$$

Since $\mathbb{R}$-intervals are *ordered sets* of real numbers, it follows that the next theorem is derivable from definition 3.1 (see [21] and [22]).

Theorem 3.1 (Interval Operations). *For any two interval numbers $[\underline{x},\overline{x}]$ and $[\underline{y},\overline{y}]$, the binary and unary interval operations are formulated in terms of the intervals' endpoints as follows.*

(i) $[\underline{x},\overline{x}] +_{\mathcal{I}} [\underline{y},\overline{y}] = [\underline{x} +_{\mathbb{R}} \underline{y}, \overline{x} +_{\mathbb{R}} \overline{y}]$,

(ii) $[\underline{x},\overline{x}] \times_{\mathcal{I}} [\underline{y},\overline{y}] = [\min\{\underline{x} \times_{\mathbb{R}} \underline{y}, \underline{x} \times_{\mathbb{R}} \overline{y}, \overline{x} \times_{\mathbb{R}} \underline{y}, \overline{x} \times_{\mathbb{R}} \overline{y}\}, \max\{\underline{x} \times_{\mathbb{R}} \underline{y}, \underline{x} \times_{\mathbb{R}} \overline{y}, \overline{x} \times_{\mathbb{R}} \underline{y}, \overline{x} \times_{\mathbb{R}} \overline{y}\}]$,

(iii) $-_{\mathcal{I}} [\underline{x},\overline{x}] = [-_{\mathbb{R}}\overline{x}, -_{\mathbb{R}}\underline{x}]$,

(iv) $0_{\mathcal{I}} \not\subseteq [\underline{x},\overline{x}] \Rightarrow [\underline{x},\overline{x}]^{-1_{\mathcal{I}}} = \left[\overline{x}^{-1_{\mathbb{R}}}, \underline{x}^{-1_{\mathbb{R}}}\right]$,

where min *and* max *are respectively the $\leq_{\mathbb{R}}$-minimal and $\leq_{\mathbb{R}}$-maximal.*

Wherever there is no confusion, we shall drop the subscripts $\mathcal{I}$ and $\mathbb{R}$. It is obvious that all the interval operations, except interval reciprocal, are *total* operations. The additional operations of interval subtraction and division can be defined respectively as $X - Y = X + (-Y)$ and $X \div Y = X \times \left(Y^{-1}\right)$.

Classical interval arithmetic has a number of peculiar algebraic properties: The point intervals $0_{\mathcal{I}}$ and $1_{\mathcal{I}}$ are identity elements for addition and multiplication, respectively; interval addition and multiplication are both commutative and associative; interval addition is cancellative; interval multiplication is cancellative only for zeroless intervals; an

[3] The axiom of extensionality asserts that two sets are equal if, and only if they have precisely the same elements, that is, for any two sets $\mathcal{S}$ and $\mathcal{T}$, $\mathcal{S} = \mathcal{T} \Leftrightarrow (\forall z)\,(z \in \mathcal{S} \Leftrightarrow z \in \mathcal{T})$.

interval number is invertible for addition (respectively, multiplication) if and only if it is a point interval (respectively, a nonzero point interval); and interval multiplication left and right *subdistributes* over interval addition (see definition 2.6 of section 2). To sum up, by definitions 2.8 and 2.10, we have the following theorem and its corollary ([14] and [22]).

Theorem 3.2 (S-Semiring of Classical Intervals). *the structure* $\langle \mathcal{I}_{\mathbb{R}}; +_{\mathcal{I}}, \times_{\mathcal{I}}; 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle$ *of classical interval numbers is a commutative* S*-semiring.*

Corollary 3.1 (Number System of Classical Intervals). *The theory of classical intervals defines an* S*-number system on the set* $\mathcal{I}_{\mathbb{R}}$.

Throughout this text, we shall employ the following theorem and its corollary (see [11] and [12]).

Theorem 3.3 (Inclusion Monotonicity in Classical Intervals). *Let* X_1*,* X_2*,* Y_1*, and* Y_2 *be interval numbers such that* $X_1 \subseteq Y_1$ *and* $X_2 \subseteq Y_2$*. Then for any binary operation* $\circ \in \{+, \times\}$ *and any definable unary operation* $\diamond \in \{-, ^{-1}\}$*, we have*

(i) $X_1 \circ_{\mathcal{I}} X_2 \subseteq Y_1 \circ_{\mathcal{I}} Y_2$,

(ii) $\diamond_{\mathcal{I}} X_1 \subseteq \diamond_{\mathcal{I}} Y_1$.

In consequence of this theorem, from the fact that $[x, x] \subseteq X \Leftrightarrow x \in X$, we have the following important special case.

Corollary 3.2 (Membership Monotonicity for Classical Intervals). *Let* X *and* Y *be real interval numbers with* $x \in X$ *and* $y \in Y$*. Then for any binary operation* $\circ \in \{+, \times\}$ *and any definable unary operation* $\diamond \in \{-, ^{-1}\}$*, we have*

(i) $x \circ_{\mathbb{R}} y \in X \circ_{\mathrm{c}} Y$,

(ii) $\diamond_{\mathbb{R}} x \in \diamond_{\mathrm{c}} X$.

By virtue of theorem 3.3, if the algebra $\langle \mathcal{I}_{\mathbb{R}}; +_{\mathcal{I}}, \times_{\mathcal{I}}; 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle$ of classical intervals is endowed with the *compatible* (*preserving*, or *monotonic*) partial ordering $\subseteq$, then we have a *partially-ordered commutative* S*-semiring* [11]. In addition to ordering intervals by the set inclusion relation $\subseteq$, there are many orders presented in the interval literature. Some of these orders are compatible with the algebraic operations on $\mathcal{I}_{\mathbb{R}}$ and others are not (see, e.g., [11], [18], [41], and [46]).

As stated in the introduction and proved in theorem 3.2, the algebraic system of classical interval arithmetic is a commutative S-semiring, which is a primitive algebraic structure, if compared to the totally ordered field of real numbers. Two useful properties of ordinary real arithmetic fail to hold in classical interval arithmetic: additive and multiplicative inverses do not always exist for interval numbers, and there is no distributivity between addition and multiplication except for certain special cases. Another main drawback of the classical interval theory emerges from a peculiarity of interval arithmetic: the set-theoretic characterization of the interval operations (definition 3.1) implies that interval arithmetic considers *all instances* of variables as *independent*. Accordingly, for two interval variables X and Y assigned the same interval constant A, both the interval operations $X \circ_{\mathcal{I}} X$ and $X \circ_{\mathcal{I}} Y$ are equal and they are the same as the image of the *multivariate* real function $f_{\text{ind}}(x, y) = x \circ_{\mathbb{R}} y$, with $x \in A$ and $y \in A$. In fact, this is one of the strengths of interval mathematics: since images of real functions are inclusion monotonic (see, e.g., [11], [22], and [52]), it follows that the image of the function f_{ind} is an enclosure of the image of a unary real function $f_{\text{dep}}(x) = x \circ_{\mathbb{R}} x$, with $x \in A$, and therefore $X \circ_{\mathcal{I}} X = X \circ_{\mathcal{I}} Y$ is a *guaranteed enclosure* of the image of f_{dep}. However, in many situations, this enclosure might be too wide to be useful. This phenomenon is known as the *interval dependency problem*.

A logical formalization of the notion of interval dependency, as a logical predicate $\mathfrak{D}$, has been presented by Dawood in [21]. In the remaining of this section, we introduce briefly this systematization of interval dependency and its fundamental properties. With the aid of the dependency predicate $\mathfrak{D}$, in section 5, we mathematically examine the validity of the claims of the parametric interval theory.

Before we proceed, it is convenient here to introduce some notational conventions. By a *finitary* real-valued function in real arguments (in short, a *real function* or $\mathbb{R}$-function), we understand a function $f_{\mathbb{R}} : \mathcal{D}_{\mathbb{R}} \subseteq \mathbb{R}^n \mapsto \mathbb{R}$, and by an *interval function* (or $\mathcal{I}$-function) we understand a function $f_{\mathcal{I}} : \mathcal{D}_{\mathcal{I}} \subseteq \mathcal{I}^n \mapsto \mathcal{I}$. The $\mathbb{R}$-subscripted letters $f_{\mathbb{R}}$, $g_{\mathbb{R}}$, $h_{\mathbb{R}}$ shall be employed to denote real-valued functions, while the $\mathcal{I}$-subscripted letters $f_{\mathcal{I}}$, $g_{\mathcal{I}}$, $h_{\mathcal{I}}$ shall be employed to denote interval-valued functions. If the type of function is clear from its arguments, and if confusion is not likely to ensue, we shall usually drop the subscripts "$\mathbb{R}$" and "$\mathcal{I}$". Thus, we may, for instance, write $f(x_1, \ldots, x_n)$ and $f(X_1, \ldots, X_n)$ for, respectively, a real-valued function and an interval-valued function, which are both defined by the same rule.

An important notion we shall need is that of the image set of real closed intervals, under an n-ary real-valued function. This notion is a special case of that of the corresponding $(n+1)$-ary relation on $\mathbb{R}$. More precisely, we have the following definition.

Definition 3.2 (Image of Real Closed Intervals). *Let f be an n-ary function on $\mathbb{R}$, and for $(\mathbf{v}, y) \in f$, let $\mathbf{v} = (x_1, ..., x_n)$, with each x_k is restricted to vary on a real closed interval $X_k \subset \mathbb{R}$, that is, $\mathbf{v}$ is restricted to vary on a set $\mathbf{V} \subset \mathbb{R}^n$. Then, the image of the closed intervals X_k with respect to f, denoted I_f, is defined to be*

$$\begin{aligned} Y = \mathrm{I}_f(\mathbf{V}) &= \mathrm{I}_f(X_1, ..., X_n) \\ &= \{y \in \mathbb{R} | (\exists \mathbf{v} \in \mathbf{V})(\mathbf{v} f y)\} \\ &= \{y \in \mathbb{R} | (\exists_{k=1}^{n} x_k \in X_k)(y = f(x_1, ..., x_n))\} \subseteq \mathbb{R}, \end{aligned}$$

where the set $\mathbf{V}$, called the preimage[4] *of Y, is defined to be the image of Y with respect to the converse relation $\widehat{f}$, that is*

$$\mathbf{V} = \mathrm{I}_{\widehat{f}}(Y) = \{\mathbf{v} \in \mathbb{R}^n | (\exists y \in Y)\left(y \widehat{f} \mathbf{v}\right)\}.$$

An immediate consequence of definition 3.2 and the *extreme value theorem* is the following important property [21].

Theorem 3.4 (Main Theorem of Image Evaluation). *Let an n-ary real-valued function f be continuous in the real closed intervals X_k. The (accurate) image $\mathrm{I}_f(X_1, ..., X_n)$, of X_k, is in turn a real closed interval such that*

$$\mathrm{I}_f(X_1, ..., X_n) = \left[\min_{x_k \in X_k} f(x_1, ..., x_n), \max_{x_k \in X_k} f(x_1, ..., x_n)\right].$$

A cornerstone result from the above theorem, that should be stressed at once, is that the *best* way to evaluate the *accurate* image of a continuous real-valued function is to apply minimization and maximization directly to determine the exact lower and upper endpoints of the image.

By definitions 2.11 and 2.12 plus theorem 3.4, the following indispensable result is provable [21].

Theorem 3.5 (Image Inclusions in Prenex Sentences). *Let σ_1 and σ_2 be the two prenex sentences such that*

$$\begin{aligned} \sigma_1 &\Leftrightarrow (\forall_{i=1}^{m} x_i \in X_i)\left(\exists_{j=1}^{n} y_j \in Y_j\right)(\exists z)(z = f(x_1, ..., x_m; y_1, ..., y_n)), \\ \sigma_2 &\Leftrightarrow (\forall_{i=1}^{m} x_i \in X_i)\left(\forall_{j=1}^{n} y_j \in Y_j\right)(\exists z)(z = f(x_1, ..., x_m; y_1, ..., y_n)), \end{aligned}$$

where X_i and Y_j are real closed intervals, and f is a continuous real-valued function with $x_i \in X_i$ and $y_j \in Y_j$.

If $\mathrm{I}_f^{\sigma_1}$ and $\mathrm{I}_f^{\sigma_2}$ are the images of f, respectively, in σ_1 and σ_2, then $\mathrm{I}_f^{\sigma_1} \subseteq \mathrm{I}_f^{\sigma_2}$.

From the fact that existential quantification over a nonempty set $\mathcal{S}$ defines a set $\mathcal{T}$ such that $\mathcal{T} \subseteq \mathcal{S}$, the previous theorem entails, as a special case, the following important result of "real analysis" ([21] and [22]).

Corollary 3.3 (Inclusion Monotonicity of Real Images). *Let $\mathbf{V}_X = (X_1, ..., X_i, ..., X_n)$ and $\mathbf{V}_Y = (Y_1, ..., Y_i, ..., Y_n)$ be two preimages of a continuous real-valued function f. Then, the image I_f is inclusion monotonic. That is*

$$(\forall_{i=1}^{n} X_i, Y_i)\left(X_i \subseteq Y_i \Rightarrow \mathrm{I}_f(\mathbf{V}_X) \subseteq \mathrm{I}_f(\mathbf{V}_Y)\right).$$

To make the statement of theorem 3.5 clear, let us consider an example. Let σ_1 and σ_2 be the two prenex sentences such that

$$\begin{aligned} \sigma_1 &\Leftrightarrow (\forall x \in [1,2])(\exists y \in [1,2])(\exists z \in \mathbb{R})(z = f(x,y) = y - x), \\ \sigma_2 &\Leftrightarrow (\forall x \in [1,2])(\forall y \in [1,2])(\exists z \in \mathbb{R})(z = f(x,y) = y - x). \end{aligned}$$

In the sentence σ_1, the variable y depends on x, and therefore there is some function $g(x)$ such that σ_1 has the Skolem equivalent form

$$(\exists g)(\forall x \in [1,2])(\exists z \in \mathbb{R})(z = f(x, g(x)) = g(x) - x).$$

Let g be the identity function. Consequently, the image of f in σ_1 is $\mathrm{I}_f^{\sigma_1} = \{0\}$. Obviously, the image of f in σ_2 is $\mathrm{I}_f^{\sigma_2} = [-1, 1]$, and therefore $\mathrm{I}_f^{\sigma_1} \subseteq \mathrm{I}_f^{\sigma_2}$.

An *exact* (or *generalized*) interval operation can then be characterized in the following definition [21].

[4] From the fact that the converse relation $\widehat{f}$ is always definable, the preimage of a function f is always definable, regardless of the definability of the inverse function f^{-1}.

Definition 3.3 (Exact Interval Operation). *Let* $\circ_{\mathbb{R}} \in \{+,\times\}$ *be a binary real operation, and let* $\mathrm{I}_f = \mathrm{I}_f^{\sigma_{\mathrm{Dep}}} \vee \mathrm{I}_f = \mathrm{I}_f^{\sigma_{\mathrm{Ind}}}$*, where* $\mathrm{I}_f^{\sigma_{\mathrm{Dep}}}$ *and* $\mathrm{I}_f^{\sigma_{\mathrm{Ind}}}$ *are the images of a function* f *for two real closed intervals* X *and* Y *in, respectively, two prenex sentences* σ_{Dep} *and* σ_{Ind} *such that*

$$\sigma_{\mathrm{Dep}} \Leftrightarrow (\forall x \in X)(\exists y \in Y)(\exists z \in \mathbb{R})(z = f(x,y) = x \circ_{\mathbb{R}} y),$$
$$\sigma_{\mathrm{Ind}} \Leftrightarrow (\forall x \in X)(\forall y \in Y)(\exists z \in \mathbb{R})(z = f(x,y) = x \circ_{\mathbb{R}} y).$$

Then, an exact interval operation $\circ_{\mathcal{J}} \in \{+,\times\}$ *is defined by*

$$X \circ_{\mathcal{J}} Y = \mathrm{I}_f(X,Y).$$

In accordance with this definition, plus definition 3.1, inexactness of classical interval arithmetic figures in the following obvious theorem.

Theorem 3.6 (Inexactness of Classical Interval Operations). *The value of a classical interval operation* $X \circ_{\mathcal{I}} Y$ *is exact only when the real variables* $x \in X$ *and* $y \in Y$ *are independent, that is*

$$X \circ_{\mathcal{I}} Y = \mathrm{I}_f^{\sigma_{\mathrm{Ind}}}(X,Y).$$

We are now ready to pass to a logical systematization of the notion of interval dependency [21].

Definition 3.4 (Interval Dependency Relation). *Let* $S_1,\ldots,S_m$ *be some arbitrary real closed intervals. For two interval variables* X *and* Y*, we say that* Y *is dependent on* X*, in symbols* $Y\mathfrak{D}X$*, iff there is some* given *real-valued function* f *such that* Y *is the image of* $(X;S_1,\ldots,S_m)$ *with respect to* f*. That is*

$$Y\mathfrak{D}X \Leftrightarrow Y = \mathrm{I}_f(X;S_1,\ldots,S_m),$$

where f *is called the dependency function of* Y *on* X*. Otherwise* Y *is not dependent on* X*, in symbols* $Y\mathfrak{I}X$*, that is*

$$Y\mathfrak{I}X \Leftrightarrow \neg Y\mathfrak{D}X \Leftrightarrow \neg Y = \mathrm{I}_f(X;S_1,\ldots,S_m).$$

Hereon, the following notational convention shall be adopted. We write $Y\mathfrak{D}_f X$ (with the subscript f) to mean that Y is dependent on X by some given dependency function f, and we write $\mathfrak{I}(X,Y)$ to mean that X and Y are mutually independent.

So, to say that an interval variable Y is dependent on an interval variable X, we must be *given* some real-valued function f such that Y is the image of X under f. This characterization of interval dependency is completely compatible with the concept of functional dependence of real variables: for two real variables x and y, the variable y is *functionally* dependent on x if there is some *given* function f such that $y = f(x)$. If x and y are mutually dependent by an *idempotence* $y = f(x)$ and $x = g(y)$, then, to keep the dependency information, it suffices to write either $x \circ_{\mathbb{R}} f(x)$ or $g(y) \circ_{\mathbb{R}} y$. In case there is neither such a given function f nor such a given function g, then it is obvious that the real variables x and y are *not* functionally dependent. Definition 3.4 extends this concept to the set of real closed intervals.

The preceding definition, along with two deductions that we shall presently make (theorem 3.7 and corollary 3.4), touches the notion of interval dependency in a way which copes with all possible cases. Noteworthy also is that the dependency relation characterized in definition 3.4 is a *meta-relation*, not an *object-ingredient* of the numerous interval theories heretofore presented in the literature. In other words, the results of this formalization of interval dependency are not only about classical intervals, but they are meant to apply also to any possible theory of interval arithmetic.

To illustrate, let us give an example. Let X and Y be two interval variables that both are assigned the same individual constant $[0,1]$. Then, we may have one of the following cases:

(i) Y is not dependent on X (there is no given dependency function),

(ii) Y is dependent on X, by the identity function $y = f(x) = x$,

(iii) Y is dependent on X, by the square function $y = f(x) = x^2$.

To reiterate, if two interval *variables* X and Y both are assigned the same *individual constant* (both have the same *value*), it does not necessarily follow that X and Y are *identical*, unless they are dependent by the identity function [5].

[5] The notions of *identity* and *equality* are commonly confused and treated as synonyms. However, they are two *distinct* logical concepts. Despite the fact that equality implies identity in the theory of real numbers, this is not always the case. Two line halves are *equal* but not *identical* (one and the same). Every line *equals* infinitely many other lines, but no line *is* (identical to) any other line (see [9] and [58]). Identity, which is the most fundamental ingredient of any mathematical theory, is characterized by *Leibniz's principle of the identity of indiscernibles* which states that two entities x and y are identical iff any property of x is also a property of y and vice versa.

Thereupon, for an interval theory to be *dependency-aware*, it must incorporate in its symbolism the dependency relation as an *object-ingredient* in such a way that two intervals are equal iff they are "*one and the same*". This fact itself calls for a new characterization of the *equality relation for interval numbers*: Two interval variables X and Y are equal (identical) iff they are *dependent by identity*, that is $X = Y \Leftrightarrow X\mathfrak{D}_{\mathrm{Id}}Y$. A theory of intervals that incorporates the dependency relation in its symbolism, namely the "theory of universal intervals", is presented by Dawood in [22].

As a consequence of the above characterization of interval dependency, we have the next immediate theorem that establishes that the interval dependency relation is a *quasi-ordering* relation [21].

Theorem 3.7 (Quasi-Orderness of the Dependency Relation). *The interval dependency relation is a quasi-ordering relation on the set of real closed intervals. That is, for any three interval variables X, Y, and Z, the following statements are true:*

(i) $\mathfrak{D}$ *is reflexive, in symbols* $(X\mathfrak{D}X)$,

(ii) $\mathfrak{D}$ *is transitive, in symbols* $(X\mathfrak{D}Y \wedge Y\mathfrak{D}Z \Rightarrow X\mathfrak{D}Z)$.

In accordance with this theorem and definition 3.3, we also have the following corollary [21].

Corollary 3.4 (Dependency Relation Properties). *For any interval operation $\circ_{\mathcal{J}}$, and for any three interval variables X, Y, and Z, the following two assertions are true:*

(i) $(X \circ_{\mathcal{J}} Y)\mathfrak{D}X$,

(ii) $(X \circ_{\mathcal{J}} Y)\mathfrak{D}Y$.

Finally, the interval dependency problem can now be formulated in the following theorem ([21] and [22]).

Theorem 3.8 (Dependency Problem). *Let X_k be real closed intervals and let $f(x_1,...,x_n)$ be a continuous real-valued function with $x_k \in X_k$. Evaluating the accurate image of f for the interval numbers X_k, using classical interval arithmetic, is not always possible if there exist X_i and X_j such that $X_j\mathfrak{D}X_i$ for $i \neq j$. That is,*

(i) $(\exists f)\left(\mathrm{I}_f(X_1,...,X_n) \neq f(X_1,...,X_n)\right)$.

In general,

(ii) $(\forall f)\left(\mathrm{I}_f(X_1,...,X_n) \subseteq f(X_1,...,X_n)\right)$.

4 Parametric Interval Arithmetic: Elegant Idea but Too Optimistic Claims

In this section, first, we formalize the basic notions and investigate the fundamental properties of the theory $\mathsf{Th}_{\mathcal{P}}$ of parametric intervals, and then we set forth its claims of desirable mathematical properties. To what extent these claims are accomplishable is to be examined in the succeeding section. The language of the theory $\mathsf{Th}_{\mathcal{P}}$ of parametric intervals is an extension of the language of the theory $\mathsf{Th}_{\mathcal{I}}$ of classical intervals, by defining new symbols or by introducing new abbreviations to allow for the possibility of expressing new concepts of the theory $\mathsf{Th}_{\mathcal{P}}$.

Starting with idea that a real closed interval is a *convex* subset [6] of the reals, and motivated by the fact that the *best* way to evaluate the *accurate* image of a continuous real-valued function is to apply minimization and maximization directly to determine the exact lower and upper endpoints of the image; parametric interval arithmetic can be constructed as a simplified type of a min-max optimization problem, with constraints varying in the unit interval [7].

A definition of a parametric interval can then be formulated as follows.

Definition 4.1 (Parametric Interval). *Let $\underline{x},\overline{x} \in \mathbb{R}$ such that $\underline{x} \leq \overline{x}$. A parametric interval is defined to be*

$$[\underline{x},\overline{x}] = \{x \in \mathbb{R} | (\exists \lambda_x \in [0,1])(x = (\overline{x}-\underline{x})\lambda_x + \underline{x})\},$$

where $\min\limits_{\lambda_x}(x) = \underline{x}$ and $\max\limits_{\lambda_x}(x) = \overline{x}$ are, respectively, the lower and upper bounds (endpoints) of $[\underline{x},\overline{x}]$.

[6] Let $\mathcal{V}$ be a vector space over an ordered field $\langle \mathbb{F}; +_{\mathbb{F}}, \times_{\mathbb{F}}; \leq_{\mathbb{F}} \rangle$. A set $\mathcal{C}$ in $\mathcal{V}$ is said to be convex iff

$$(\forall x,y \in \mathcal{C})(\forall \lambda \in [0_{\mathbb{F}},1_{\mathbb{F}}])(((1-\lambda)x+\lambda y) \in \mathcal{C}).$$

[7] Note that various parametrizations can be employed to characterize a parametric interval. As an instance, Elishakoff used a *trigonometric* parametrization in [25]. Here we use the simple *linear* parametrization adopted in [10], [22], and [42].

Obviously, definition 4.1 is equivalent to the definition of a classical interval number, and a parametric interval is a closed and bounded nonempty real interval. However, to simplify the exposition, we shall denote[8] the set of parametric intervals by $\mathcal{P}$, and the upper-case Roman letters X, Y, and Z (with or without subscripts), or equivalently $[\underline{x},\overline{x}]$, $[\underline{y},\overline{y}]$, and $[\underline{z},\overline{z}]$, shall be still employed as variable symbols to denote elements of $\mathcal{P}$. The sets of *point* and *symmetric* parametric intervals shall be denoted by $\mathcal{P}_{[x]}$ and $\mathcal{P}_{\mathrm{S}}$, respectively.

In definition 4.1, a parametric interval is defined as the image of a continuous real-valued function x of one variable $\lambda_x \in [0,1]$ and two constants $\underline{x}$ and $\overline{x}$. The endpoints, $\underline{x}$ and $\overline{x}$, are respectively the minimum and maximum of x with the constraint

$$\begin{aligned} 0 \le \lambda_x \le 1 &\Rightarrow 0 \le (\overline{x}-\underline{x})\,\lambda_x \le (\overline{x}-\underline{x}) \\ &\Rightarrow \underline{x} \le (\overline{x}-\underline{x})\,\lambda_x + \underline{x} \le \overline{x} \\ &\Rightarrow \underline{x} \le x \le \overline{x}. \end{aligned}$$

Since the endpoints $\underline{x}$ and $\overline{x}$ are known *inputs*, they are parameters and hence the name "*parametric interval arithmetic*", whereas λ_x is a variable that is constrained between 0 and 1. The binary parametric interval operations can be guaranteed to be *continuous* by introducing two constrained variables $\lambda_x, \lambda_y \in [0,1]$. From the fact that $x \in [\underline{x},\overline{x}]$ and $y \in [\underline{y},\overline{y}]$ are continuous real-valued functions of λ_x and λ_y respectively, the result of a parametric interval operation shall be the image of the continuous function

$$x \circ_{\mathbb{R}} y = ((\overline{x}-\underline{x})\,\lambda_x + \underline{x}) \circ_{\mathbb{R}} \left((\overline{y}-\underline{y})\,\lambda_y + \underline{y}\right),$$

with $\lambda_x, \lambda_y \in [0,1]$, and $\circ_{\mathbb{R}} \in \{+,-,\times,\div\}$ such that $y \neq 0$ if $\circ_{\mathbb{R}}$ is "$\div$".

According to the functional dependence of real variables, Lodwick in [42] defines two types of parametric interval operations, namely "*dependent operations*" and "*independent operations*". The dependent and independent parametric interval operations can be characterized in the following two definitions (see [11] and [22]).

Definition 4.2 (Parametric Dependent Operations). *For any parametric interval $[\underline{x},\overline{x}]$, there exists a parametric interval $[\underline{z},\overline{z}]$ such that*

$$\begin{aligned} [\underline{z},\overline{z}] &= [\underline{x},\overline{x}] \circ_{\mathrm{dep}} [\underline{x},\overline{x}] \\ &= \{z \in \mathbb{R} \,|\, (\exists x \in [\underline{x},\overline{x}])\,(z = x \circ_{\mathbb{R}} x)\} \\ &= \{z \in \mathbb{R} \,|\, (\exists \lambda_x \in [0,1])\,(z = ((\overline{x}-\underline{x})\,\lambda_x + \underline{x}) \circ_{\mathbb{R}} ((\overline{x}-\underline{x})\,\lambda_x + \underline{x}))\}, \end{aligned}$$

where

$$\begin{aligned} \underline{z} &= \min_{\lambda_z}(z) = \min_{\lambda_x}(x \circ_{\mathbb{R}} x), \\ \overline{z} &= \max_{\lambda_z}(z) = \max_{\lambda_x}(x \circ_{\mathbb{R}} x), \end{aligned}$$

and $\circ \in \{+,-,\times,\div\}$ such that $0 \notin [\underline{x},\overline{x}]$ if $\circ$ is "$\div$".

Definition 4.3 (Parametric Independent Operations). *For any two parametric intervals $[\underline{x},\overline{x}]$ and $[\underline{y},\overline{y}]$, there exists a parametric interval $[\underline{z},\overline{z}]$ such that*

$$\begin{aligned} [\underline{z},\overline{z}] &= [\underline{x},\overline{x}] \circ_{\mathrm{ind}} [\underline{y},\overline{y}] \\ &= \{z \in \mathbb{R} \,|\, (\exists x \in [\underline{x},\overline{x}])\,(\exists y \in [\underline{y},\overline{y}])\,(z = x \circ_{\mathbb{R}} y)\} \\ &= \{z \in \mathbb{R} \,|\, (\exists \lambda_x \in [0,1])\,(\exists \lambda_y \in [0,1]) \\ &\qquad \left(z = ((\overline{x}-\underline{x})\,\lambda_x + \underline{x}) \circ_{\mathbb{R}} \left((\overline{y}-\underline{y})\,\lambda_y + \underline{y}\right)\right)\}, \end{aligned}$$

where

$$\begin{aligned} \underline{z} &= \min_{\lambda_z}(z) = \min_{\lambda_x,\lambda_y}(x \circ_{\mathbb{R}} y), \\ \overline{z} &= \max_{\lambda_z}(z) = \max_{\lambda_x,\lambda_y}(x \circ_{\mathbb{R}} y), \end{aligned}$$

and $\circ \in \{+,-,\times,\div\}$ such that $0 \notin [\underline{y},\overline{y}]$ if $\circ$ is "$\div$".

It is obvious, from definitions 4.2 and 4.3, that parametric interval arithmetic is a *mathematical programming problem*, and therefore parametric interval operations can be easily performed by any constraint solver such as GeCode[9],

[8]Throughout the text, we always deal with the *same* set of real closed intervals. However, for legibility and brevity; we employ *different notations* for the set of interval numbers, according to what theory of intervals is being discussed. So, we can write, for instance, the brief expressions "*addition on $\mathcal{I}$*" and "*addition on $\mathcal{P}$*" instead respectively of the expressions "*classical interval addition*" and "*parametric interval addition*".

[9]*http://www.gecode.org/*

HalPPC[10], and MINION[11]. The minimization and maximization are well-defined, attained, and the resultant $[\underline{z}, \overline{z}]$ is in turn a parametric interval.

The idea underlying parametric intervals seems elegant and the theory makes a promising first impression. Notably, it is presented in the literature with a number of optimistic claims, namely:

Claim 1. Parametric interval arithmetic has additive and multiplicative inverse elements.

Claim 2. Parametric interval arithmetic satisfies the distributive law.

Claim 3. The theory of parametric intervals provides a solution to the long-standing dependency problem.

So, *are these very desirable properties accomplishable by the theory of parametric intervals?* In the succeeding section, *we argue that they are not*.

5 On the Structure of Parametric Intervals: Disproving the Claims

With the aid of the systematic formalization of the notion of interval dependency as a binary predicate $\mathfrak{D}$ and the related theorems established in section 3, along with the notions prescribed in section 4, we are now ready to investigate if the theory of parametric interval arithmetic accomplishes its claims. The investigation conducted in this section is *metatheoretical* in nature and is based mainly on the formalized apparatus fixed in section 2. That is, most of the deductions of this section are *metatheorems about* the theory $\mathsf{Th}_{\mathcal{P}}$ of parametric intervals.

In the first place, we must generally ask: *what exactly is the algebraic structure of parametric interval arithmetic?* The interval literature does not provide an answer for this question. However, the parametric approach to interval analysis is usually introduced with the emphasis that parametric interval arithmetic, unlike Moore's classical interval arithmetic, has additive and multiplicative inverse elements, satisfies the distributive law, and explicitly keeps track of functional dependencies (see, e.g., [25], [42], and [49]). For example, on page 1 in [42], Lodwick says:

> "Unlike (classical) interval arithmetic, constrained interval arithmetic has an additive inverse, a multiplicative inverse and satisfies the distributive law. This means that the algebraic structure of constrained interval arithmetic is different than that of (classical) interval arithmetic.",

and then presents proofs for the following three statements:

(i) Additive inverse. $(\forall X \in \mathcal{P})\left(X -_{\text{dep}} X = [0,0]\right)$.

(ii) Multiplicative inverse. $(\forall X \in \mathcal{P})\left(0 \notin X \Rightarrow X \div_{\text{dep}} X = [1,1]\right)$.

(iii) Distributive law. $(\forall X,Y,Z \in \mathcal{P})\left(Z \times (X+Y) = Z \times X + Z \times Y\right)$.

The first two statements are derivable by simple substitution in definition 4.2 for parametric dependent operations, hence the subscript "dep". For the third statement, the matter is much more complicated, and therefore we dropped the subscripts for the operation symbols of addition and multiplication.

Getting down to particulars with the above three statements, we must turn to ask, then, the corresponding three questions:

(1) Is the statement "$A -_{\text{dep}} A = [0,0]$" equivalent to "$(-A)$ is the inverse element of A with respect to the operation $+$ on the set $\mathcal{P}$, according to the dependent operation $X +_{\text{dep}} X$"?

(2) Is the statement "$A \div_{\text{dep}} A = [1,1]$" equivalent to "$\left(A^{-1}\right)$ is the inverse element of A with respect to the operation $\times$ on the set $\mathcal{P}$, according to the dependent operation $X \times_{\text{dep}} X$"?

(3) Does the distributive law hold, according to the dependent and independent operations?

In the sequel, we prove that the answers of the above three questions are all *negative*. On the face of it, the theory of parametric intervals seems to fit squarely into its objectives, but, however the elegance of its underlying idea, we shall argue both that the fit is problematic, and that its mathematical formulation constitutes a serious algebraic defect.

A careful formal investigation of the parametric interval operations will show that the problems of the theory of parametric intervals are 'foundational' in nature. As shown in section 3, the classical interval theory considers all *numerically-equal* intervals as independent which is not necessarily true. On the other hand, definition 4.2 considers only dependency by *identity* and it implies that two numerically-equal intervals are always dependent which is obviously not true (Take, for example, the interval extension of $f(x,y) = x - y$ with each of the variables x and y

[10] *http://sourceforge.net/projects/halppc*
[11] *http://minion.sourceforge.net/*

varies *independently* on the same interval $[1,2]$). A consequence of this issue is that definition 4.3 will not consider numerically-equal intervals even if they are independent. In other words, the domain of the dependent operations is the set of pairs (X,X), and the domain of the independent operations is the set of pairs (X,Y) where X and Y cannot be assigned the same individual constant.

Recalling definition 3.4 of the interval dependency relation, it is also convenient to single out two sets of elements of $\mathcal{P}^2$, according to interval dependencies.

Definition 5.1 (Dependent Parametric Set). $\mathcal{K}_{\text{dep}} = \{(X,Y) \in \mathcal{P}^2 | \, Y \mathfrak{D} X\}$.

Definition 5.2 (Independent Parametric Set). $\mathcal{K}_{\text{ind}} = \{(X,Y) \in \mathcal{P}^2 | \, \mathfrak{I}(X,Y)\}$.

With definitions 2.2, 2.3, 2.4, and 2.9 of, respectively, *partial and total operations*, an $\mathfrak{L}$*-structure*, *inverse elements*, and a *number system* at our disposal, we are now ready to prove our statements about the theory of parametric intervals. We begin by investigating what type of algebraic operations the parametric operations are.

Theorem 5.1 (Partiality of Parametric Dependent Operations). *Parametric dependent addition and multiplication are partial operations in the set* $\mathcal{P}$.

Proof. For $\circ_{\text{dep}} \in \{+,\times\}$, from definition 4.2, we have $\circ_{\text{dep}} : \text{Id}_{\mathcal{P}} \to \mathcal{P}$, where

$$\text{Id}_{\mathcal{P}} = \{(X,X) \,|\, X \in \mathcal{P}\}.$$

Obviously the set $\text{Id}_{\mathcal{P}}$ is the *identity* relation on $\mathcal{P}$, which, by definition 5.1, is a proper subset of $\mathcal{K}_{\text{dep}}$, and hence a proper subset of $\mathcal{P}^2$. Therefore, according to definition 2.2, the operations $\circ_{\text{dep}} \in \{+,\times\}$ are partial operations in the set $\mathcal{P}$ of parametric intervals. □

One immediate result that this theorem implies is that the parametric dependent operations consider only a single special case of interval dependency, namely the *dependency by identity*, $X \mathfrak{D} X$. Other cases of interval dependency, characterized in definition 3.4, are not considered by the parametric dependent operations.

Theorem 5.2 (Partiality of Parametric Independent Operations). *Parametric independent addition and multiplication are partial operations in the set* $\mathcal{P}$.

Proof. For $\circ_{\text{ind}} \in \{+,\times\}$, definition 4.3 does not consider independent intervals if they are numerically-equal. Accordingly $\circ_{\text{ind}} : \mathcal{V} \to \mathcal{P}$, where $\mathcal{V}$, by definition 5.2, is a proper subset of $\mathcal{K}_{\text{ind}}$, and hence a proper subset of $\mathcal{P}^2$. Therefore, by definition 2.2, the operations $\circ_{\text{ind}} \in \{+,\times\}$ are partial operations in the set $\mathcal{P}$ of parametric intervals. □

In accordance with the preceding two results, we are now led to the following three theorems, which answer our questions concerning inverse elements and distributivity.

Theorem 5.3 (Unprovability of Parametric Additive Inverses). *Inverse elements for addition cannot be proved to exist in parametric interval arithmetic. In other words, the statement*

$$(\forall X \in \mathcal{P})\,(X + (-X) = [0,0]),$$

is undecidable in the theory $\mathsf{Th}_{\mathcal{P}}$ *of parametric intervals.*

Proof. For any parametric interval X, define the negation of X, in the standard way, to be

$$-X = \{z \in \mathbb{R} | \, (\exists x \in X)\,(z = -x)\}.$$

Obviously, the relation $(-X)\,\mathfrak{D} X$ is true. But, by theorem 5.1, the pair $((-X)\,,X) \notin \text{Id}_{\mathcal{P}}$ unless $X = [0,0]$, and the expression $X + (-X)$ thus is not expressible as a parametric dependent operation. On the other hand, by theorem 5.2, $((-X)\,,X) \notin \mathcal{K}_{\text{ind}}$ because the predicate $\mathfrak{I}\,((-X)\,,X)$ is not true, and the expression $X + (-X)$ thus is not expressible as a parametric independent operation.

It follows, therefore, that the existence of additive inverses is undecidable in the parametric interval theory. □

Theorem 5.4 (Unprovability of Parametric Multiplicative Inverses). *Inverse elements for multiplication cannot be proved to exist in parametric interval arithmetic. In other words, the statement*

$$(\forall X \in \mathcal{P})\,\left(0 \notin X \Rightarrow X \times \left(X^{-1}\right) = [1,1]\right),$$

is undecidable in the theory $\mathsf{Th}_{\mathcal{P}}$ *of parametric intervals.*

Proof. For any parametric interval X with $0 \notin X$, define the reciprocal of X, in the standard way, to be

$$X^{-1} = \{z \in \mathbb{R} | (\exists x \in X) \left(z = x^{-1}\right)\}.$$

Obviously, the relation $\left(X^{-1}\right) \mathfrak{D} X$ is true. But, by theorem 5.1, the pair $\left(\left(X^{-1}\right), X\right) \notin \mathrm{Id}_{\mathcal{P}}$ unless $X = [1,1]$, and the expression $X \times \left(X^{-1}\right)$ thus is not expressible as a parametric dependent operation. On the other hand, by theorem 5.2, $\left(\left(X^{-1}\right), X\right) \notin \mathcal{K}_{\text{ind}}$ because the predicate $\mathfrak{I}\left(\left(X^{-1}\right), X\right)$ is not true, and the expression $X \times \left(X^{-1}\right)$ thus is not expressible as a parametric independent operation.

It follows, therefore, that the existence of multiplicative inverses is undecidable in the parametric interval theory. □

Theorem 5.5 (Unprovability of Parametric Distributivity). *The distributive law does not hold in parametric interval arithmetic. In other words, the statement*

$$(\forall X, Y, Z \in \mathcal{P})\,(Z \times (X + Y) = Z \times X + Z \times Y),$$

is not provable in the theory $\mathsf{Th}_{\mathcal{P}}$ *of parametric intervals.*

Proof. Obviously, in the left-hand side

$$Z \times (X + Y),$$

all the variables are mutually independent. Then, applying definition 4.3 of the parametric independent operations, we obtain the same result as in classical interval arithmetic, that is

$$Z \times_{\text{ind}} (X +_{\text{ind}} Y) = Z \times_{\mathcal{I}} (X +_{\mathcal{I}} Y).$$

Let us now consider the right-hand side

$$(Z \times X) + (Z \times Y).$$

It is clear that the relations $(Z \times X)\,\mathfrak{D} Z$ and $(Z \times Y)\,\mathfrak{D} Z$ are true. However, by theorem 5.1, the pair $((Z \times X), (Z \times Y)) \notin \mathrm{Id}_{\mathcal{P}}$, and the expression $(Z \times X) + (Z \times Y)$ thus is not expressible as a parametric dependent operation. Then, applying definition 4.3 of the parametric independent operations, we again have the same result as in classical interval arithmetic, that is

$$(Z \times_{\text{ind}} X) +_{\text{ind}} (Z \times_{\text{ind}} Y) = (Z \times_{\mathcal{I}} X) +_{\mathcal{I}} (Z \times_{\mathcal{I}} Y).$$

According to the subdistributive property of the classical interval theory, we also have only the subdistributive law

$$Z \times_{\text{ind}} (X +_{\text{ind}} Y) \subseteq (Z \times_{\text{ind}} X) +_{\text{ind}} (Z \times_{\text{ind}} Y),$$

for parametric interval arithmetic.

It follows, therefore, that distributivity is not provable in the parametric interval theory. □

Thus, the preceding three theorems prove that the answers of our questions are all *negative*. The parametric dependent and independent operations do not qualify as *total* operations on $\mathcal{P}$, and in their full extent, do not suffice to cope with interval dependencies except for the special case when the operands are trivially *dependent by identity*, that is, $X \mathfrak{D} X$.

In order to make this clear, we next give an example.

Example 5.1 (Inexpressibility of Parametric Intervals). *Let* σ *be the prenex sentence such that*

$$\sigma \Leftrightarrow (\forall x \in [-1,1])\,(\exists y \in [0,1])\,(\exists z \in \mathbb{R})\,(z = y - x).$$

In the sentence σ*, the variable* y *depends on* x*, and therefore there is some function* $g(x)$ *such that* σ *has the Skolem equivalent form*

$$(\exists g)\,(\forall x \in [-1,1])\,(\exists z \in \mathbb{R})\,(z = g(x) - x).$$

The dependency function g *can be, for instance, the quadratic function, that is* $y = g(x) = x^2$*. It is clear that the relation* $[0,1]\,\mathfrak{D}\,[-1,1]$ *is true, that is, the interval number* $[0,1]$ *is dependent on* $[-1,1]$*. However, the pair* $([0,1], [-1,1]) \notin \mathrm{Id}_{\mathcal{P}}$*, and the expression* $[0,1] - [-1,1]$ *thus is not expressible as a parametric dependent operation.*

We now pass to our general question concerning the algebraic system of parametric interval arithmetic. The following theorem clarifies an answer.

Theorem 5.6 (Undefinability of Parametric Interval Algebra). *The theory* $\mathsf{Th}_{\mathcal{P}}$ *of parametric intervals does not define an algebra for addition or multiplication on the set* $\mathcal{P}$.

Proof. By theorems 5.1 and 5.2, the operations $\circ_{\text{dep}}$ and $\circ_{\text{ind}}$, in $\{+, \times\}$, are partial operations in $\mathcal{P}$, and therefore, according to definition 2.3, the algebras $\langle \mathcal{P}; \circ_{\text{dep}} \rangle$ and $\langle \mathcal{P}; \circ_{\text{ind}} \rangle$ are not definable. ☐

That is, the structures $\langle \mathcal{P}; \circ_{\text{dep}} \rangle$ and $\langle \mathcal{P}; \circ_{\text{ind}} \rangle$ are *undefinable*, for the requirement that an algebraic operation must be *total* on the universe set $\mathcal{P}$.

In consequence of the last theorem, we also have the following important result.

Theorem 5.7 (Undefinability of Parametric Number System). *The theory* $\mathsf{Th}_{\mathcal{P}}$ *of parametric intervals does not define a number system on the set* $\mathcal{P}$.

Proof. The proof immediately follows from definitions 2.9 and 2.10, plus theorem 5.6, by the fact that every number system is an algebra. ☐

Thus, parametric intervals are neither "*numbers*" nor "S-*numbers*", and therefore we cannot talk of "*parametric interval numbers*".

From the above investigation, one can conclude that the underlying idea of parametric interval arithmetic seems elegant and simple, but it is *too simple* to fully account for the notion of interval dependency or to achieve a richer algebraic structure for interval arithmetic. It is therefore imperative both to supply the defect in the parametric approach and to present an alternative theory with a mathematical construction that avoids the defect. The former was attempted in this article, and the latter is attempted in [22].

6 Conclusion

What exactly is the algebraic structure of parametric interval arithmetic? The interval literature does not provide an answer for this question which looks simple and answerable, but when subject to formal investigation, develops into a problematic situation. Notably, the parametric approach to interval analysis is usually introduced in the interval literature with the very zealous claims that parametric interval arithmetic, unlike classical interval arithmetic, has additive and multiplicative inverse elements, satisfies the distributive law, and explicitly provides a solution to the long-standing dependency problem. In this article, with the aid of a logical systematization of interval dependency, we provided an in-depth investigation of the algebraic system of parametric interval arithmetic and argued that the optimistic claims of the theory of parametric intervals are not accomplishable.

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