

RESEARCH ARTICLE

On Some Algebraic and Order-Theoretic Aspects of Machine Interval Arithmetic

Hend Dawood

Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt

CORRESPONDING AUTHOR: Hend Dawood  (hend.dawood@sci.cu.edu.eg)

Abstract

Interval arithmetic is a fundamental and reliable mathematical machinery for scientific computing and for addressing uncertainty in general. In order to apply interval mathematics to real life uncertainty problems, one needs a computerized (machine) version thereof, and so, this article is devoted to some mathematical notions concerning the algebraic system of machine interval arithmetic. After formalizing some purely mathematical ingredients of particular importance for the purpose at hand, we give formal characterizations of the algebras of real intervals and machine intervals along with describing the need for interval computations to cope with uncertainty problems. Thereupon, we prove some algebraic and order-theoretic results concerning the structure of machine intervals.

Keywords: Interval mathematics; Machine interval arithmetic; Outward rounding; Floating-point arithmetic; Machine monotonicity; Dense orders; Orderability of intervals; Symmetricity; Singletonicity; Subdistributive semiring; S-semiring

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1 Introduction

A natural and elegant idea is that of expressing uncertain real-valued quantities as real closed intervals. In this very simple and old idea, the field of interval mathematics has its roots from the Greek mathematician *Archimedes of Syracuse* to the American mathematician *Ramon Edgar Moore*, who was the first to define interval analysis in its modern sense and recognize its power as a viable computational tool for intervalizing uncertainty. In between, historically speaking, several distinguished constructions of interval arithmetic by John Charles Burkill, Rosalind Cecily Young, Paul S. Dwyer, Teruo Sunaga, and others (see, e.g., [3], [50], [20], and [46]) have emphasized the very idea of reasoning about uncertain values through calculating with intervals. By integrating the complementary powers of rigorous mathematics and scientific computing, interval arithmetic is able to offer highly reliable accounts of uncertainty. It should therefore come as no surprise that the interval theory has been fruitfully applied in diverse areas that deal intensely with uncertain quantitative data (see, e.g., [7], [9], and [27]).

In order to apply interval mathematics to real life uncertainty problems, we need first to *digitize* it so that it can be processed by a computer. Although calculating with intervals is an old idea, Moore was the first to recognize the practical power of machine interval arithmetic as a viable computational tool for bounding errors and intervalizing uncertainties (see, e.g., [35], [36], [37], and [38]).

In view of this computational power against error, machine implementations of interval arithmetic are of great importance. It should therefore come as no surprise that there are many software implementations of interval arithmetic. As instances, we may mention INTLAB, Sollya, InCLosure and others (see, e.g., [44], [5], [10], [19], and [33]). Fortunately, computers are getting faster and most existing *parallel processors* provide a tremendous computing power. So, with little extra hardware, it is very possible to make interval computations as fast as floating point computations (For further reading about machine arithmetizations and hardware circuitries for interval arithmetic, see, e.g., [7], [8], [26], [30], [28], [29], [40], [41], and [25]).

The objective of this article is then to investigate some mathematical notions concerning the algebraic system of machine interval arithmetic. We begin in section 2 by specifying some notational conventions and formalizing some purely algebraic and order-theoretical ingredients of importance to our purpose. In section 3, we give a formal characterization of an interval algebra over the real field. In section 4, we provide a discussion of the limitations and loss of precision of machine real arithmetic along with a clarification of the need for the *infinite precision* of machine

EMAIL ADDRESS, Hend Dawood: hend.dawood@sci.cu.edu.eg

interval arithmetic. Thereupon, in section 5, we give an algebraic characterization of the key concepts of machine real arithmetic and machine interval arithmetic. Finally, in section 6, we deduce some algebraic and order-theoretic results concerning the structure of machine intervals.

2 On Some Fundamental Notions of Relations and Structures

Before moving on to characterize the algebra of intervals, we begin in this section by specifying some notational conventions and formalizing some algebraic and order-theoretical ingredients we shall need throughout this article (For further details about the notions prescribed here, the reader may consult, e.g., [2], [11], [13], [14], [17], and [34]).

Most of our notions are characterized in terms of *ordinals* and *ordinal tuples*. So, we first define what an *ordinal* is.

Definition 2.1 (Ordinal). *An ordinal is the well-ordered set of all ordinals preceding it. That is, for each ordinal n , there exists an ordinal $S(n)$ called the successor of n such that*

$$(\forall n)(\forall k)(k = S(n) \Leftrightarrow (\forall m)(m \in k \Leftrightarrow m \in n \vee m = n)).$$

In other words, we have $S(n) = n \cup \{n\}$. Accordingly, the the first *infinite (transfinite) ordinal* is the set $\omega = \{0, 1, 2, \dots\}$. All ordinals preceding ω (all elements of ω) are *finite ordinals*. The idea of *transfinite counting (counting beyond the finite)* is due to Cantor (See [4]).

With the aid of ordinals, the notions of *countably finite*, *countably infinite* and *uncountably infinite* sets can be characterized as follows.

Definition 2.2 (Countably Finite and Infinite Sets). *A set S is countably finite if there is a bijective mapping from S onto some finite ordinal $n \in \omega$. A set S is countably infinite (or denumerable) if there is a bijective mapping from S onto the infinite ordinal ω .*

For example the set $\{a_0, a_1, a_2\}$ is countably finite because it can be bijectively mapped onto the finite ordinal $3 = \{0, 1, 2\}$, while the set $\{a_0, a_1, a_2, \dots\}$ is denumerable because it can be bijectively mapped onto the infinite ordinal $\omega = \{0, 1, 2, \dots\}$.

Definition 2.3 (Uncountably Infinite Sets). *An uncountably infinite set is an infinite set which is not countably infinite.*

For example the set \mathbb{R} of real numbers is uncountably infinite.

The notion of an *n-tuple* is characterized in the following definition.

Definition 2.4 (Ordinal Tuple). *For an ordinal $n = S(k)$, an n -tuple (ordinal tuple) is any mapping τ whose domain is n . A finite n -tuple is an n -tuple for some finite ordinal n . That is*

$$\begin{aligned} \tau_{S(k)} &= \langle \tau(0), \tau(1), \dots, \tau(k) \rangle \\ &= \langle (0, \tau(0)), (1, \tau(1)), \dots, (k, \tau(k)) \rangle. \end{aligned}$$

If $n = 0 = \emptyset$, then, for any set S , there is exactly one mapping (*the empty mapping*) $\tau_\emptyset = \emptyset$ from \emptyset into S .

An important definition we shall need is that of the Cartesian power of a set.

Definition 2.5 (Cartesian Power). *Let \emptyset denote the empty set. For a set S and an ordinal n , the n -th Cartesian power of S is the set S^n of all mappings from n into S , that is*

$$S^n = \begin{cases} \{\emptyset\} & n = 0, \\ \text{the set of all } n\text{-tuples of elements of } S & n = 1 \vee 1 \in n. \end{cases}$$

If S is the empty set \emptyset , then¹

$$\emptyset^n = \begin{cases} \{\emptyset\} & n = 0, \\ \emptyset & n = 1 \vee 1 \in n; \end{cases} \quad \text{and} \quad \emptyset^{\emptyset^n} = \begin{cases} \emptyset & n = 0, \\ \{\emptyset\} & n = 1 \vee 1 \in n. \end{cases}$$

The preceding definition can be further generalized by introducing the notion of the *Cartesian product* (or *cross-product*).

¹Amer in [1] used the n -th Cartesian power of \emptyset to define empty structures, and axiomatized their first-order theory.

Definition 2.6 (Cartesian Product). Let \mathcal{S}_i be sets, for an ordinal $1 \leq i \leq n$. The Cartesian product of \mathcal{S} , denoted $\mathcal{S}_1 \times \mathcal{S}_2 \times \dots \times \mathcal{S}_n$ or equivalently $\prod_{i=1}^n \mathcal{S}_i$, is defined to be

$$\prod_{i=1}^n \mathcal{S}_i = \begin{cases} \mathcal{S}^n & (\forall \mathcal{S}_i) (\forall \mathcal{S}_j) (\mathcal{S}_i = \mathcal{S}_j = \mathcal{S}), \\ \emptyset & (\exists \mathcal{S}_i) (\mathcal{S}_i = \emptyset), \\ \{\langle s_1, \dots, s_n \rangle \mid s_i \in \mathcal{S}_i\} & \text{otherwise.} \end{cases}$$

In accordance with the preceding definitions, a set-theoretical relation is a particular type of sets. Let \mathcal{S}^2 be the binary Cartesian power of a set \mathcal{S} . A binary relation on \mathcal{S} is a subset of \mathcal{S}^2 . That is, a set \mathfrak{R} is a binary relation on a set \mathcal{S} iff $(\forall \mathbf{r} \in \mathfrak{R}) ((\exists x, y \in \mathcal{S}) (\mathbf{r} = \langle x, y \rangle))$. In this set-theoretical sense, the notion of *density* of a set with respect to an ordering relation \mathfrak{R} is characterized in the following definition.

Definition 2.7 (Density). A set \mathcal{S} is dense with respect to a strict ordering relation \mathfrak{R} (or an \mathfrak{R} -dense) iff²

$$(\forall x, y \in \mathcal{S}) (x \mathfrak{R} y \Rightarrow (\exists z \in \mathcal{S}) (x \mathfrak{R} z \wedge z \mathfrak{R} y)).$$

By means of the above concepts, we next define the notions of *partial* and *total operations*.

Definition 2.8 (Partial and Total Operations). Let \mathcal{S}^n be the n -th Cartesian power of a set \mathcal{S} . An n -ary (total) operation on \mathcal{S} is a total function $t_n : \mathcal{S}^n \rightarrow \mathcal{S}$. An n -ary partial operation in \mathcal{S} is a partial function $p_n : \mathcal{U} \rightarrow \mathcal{S}$, where $\mathcal{U} \subset \mathcal{S}^n$. The ordinal n is called the *arity* of t_n or p_n .

A binary operation is an n -ary operation for $n = 2$. Addition and multiplication on the set \mathbb{R} of real numbers are best-known examples of binary *total* operations, while division is a *partial* operation in \mathbb{R} .

We finalize this section by characterizing an important notion we shall need, that of a structure of a formalized language \mathcal{L} (or an \mathcal{L} -structure).

Definition 2.9 (Structures). Let \mathcal{L} be a formalized language (possibly with no individual constants). By an \mathcal{L} -structure we understand a system $\mathfrak{M} = \langle \mathcal{A}; F^{\mathfrak{M}}; R^{\mathfrak{M}} \rangle$, where

- \mathcal{A} is a (possibly empty) set called the individuals universe of \mathfrak{M} . The elements of \mathcal{A} are called the individual elements of \mathfrak{M} ;
- $F^{\mathfrak{M}}$ is a (possibly empty) set of finitary total operations on \mathcal{A} . The elements of $F^{\mathfrak{M}}$ are called the \mathfrak{M} -operations;
- $R^{\mathfrak{M}}$ is a (possibly empty) set of finitary relations on \mathcal{A} . The elements of $R^{\mathfrak{M}}$ are called the \mathfrak{M} -relations.

An \mathcal{L} -structure with an empty individuals universe is called an *empty \mathcal{L} -structure*³. By a *many-sorted \mathcal{L} -structure* we understand an \mathcal{L} -structure with more than one universe set. By a *relational \mathcal{L} -structure* we understand an \mathcal{L} -structure with a non-empty set of relations and an empty set of functions. By an *algebraic \mathcal{L} -structure* (or an *\mathcal{L} -algebra*) we understand an \mathcal{L} -structure with a non-empty set of functions. An \mathcal{L} -algebra endowed with a compatible ordering relation \mathfrak{R} is called an \mathfrak{R} -ordered \mathcal{L} -algebra.

3 Interval Algebra over the Real Field

The main business of this section is to give a formalized characterization of the theory of real interval arithmetic. There are many theories of interval arithmetic (see, e.g., [22], [29], [21], [32], [31], [7], [16], and [12]). We are here interested in characterizing classical interval arithmetic as introduced in, e.g., [35], [45], [38], and [9]. An algebra for machine intervals over a different theory of intervals will be fundamentally the same as the one presented in this article, but it might differ in the resulting algebraic properties.

A theory $\text{Th}_{\mathcal{I}}$ of a *real interval algebra* (a *classical interval algebra* or an *interval algebra over the real field*) is characterized in the following definition (see [9] and [15]).

Definition 3.1 (Theory of Real Interval Algebra). Take $\sigma = \{+, \times; -, ^{-1}; 0, 1\}$ as a set of non-logical constants and let $\mathbb{R} = \langle \mathbb{R}; \sigma^{\mathbb{R}} \rangle$ be the totally \leq -ordered field of real numbers. The theory $\text{Th}_{\mathcal{I}}$ of an interval algebra over the field \mathbb{R} is the theory of a two-sorted structure $\mathfrak{I}_{\mathbb{R}} = \langle \mathcal{I}_{\mathbb{R}}; \mathbb{R}; \sigma^{\mathfrak{I}_{\mathbb{R}}} \rangle$ prescribed by the following set of axioms.

²The abbreviation “*iff*”, for “*if and only if*”, is attributed to the great mathematician Paul Richard Halmos (1916–2006) who preferred it for *definitions*. Despite the customary usage of “*if*”, rather than “*iff*”, in statements of definitions, it is a usual practice to prefer “*iff*” for definitions, in formalized treatments where steering clear of ambiguity is a must (see, e.g., [24], [47], and [48]). Accordingly, we shall follow the formal tradition of Hunter, Suppes and Tarski, and deploy “*iff*” as an ordinary English translation of “ \Leftrightarrow ” in all statements of our definitions.

³First-order logics with empty structures were first considered by Mostowski in [39], and then studied by many logicians (see, e.g., [43], [23], and [1]). Such logics are now referred to as *free logics*.

- (I1) $(\forall X \in \mathcal{I}_{\mathbb{R}})(X = \{x \in \mathbb{R} \mid (\exists \underline{x} \in \mathbb{R})(\exists \bar{x} \in \mathbb{R})(x \leq_{\mathbb{R}} x \leq_{\mathbb{R}} \bar{x})\})$,
- (I2) $(\forall X, Y \in \mathcal{I}_{\mathbb{R}})(\circ \in \{+, \times\} \Rightarrow X \circ_{\mathcal{I}_{\mathbb{R}}} Y = \{z \in \mathbb{R} \mid (\exists x \in X)(\exists y \in Y)(z = x \circ_{\mathbb{R}} y)\})$,
- (I3) $(\forall X \in \mathcal{I}_{\mathbb{R}})(\diamond \in \{-\} \vee (\diamond \in \{-1\} \wedge 0_{\mathcal{I}_{\mathbb{R}}} \not\subseteq X) \Rightarrow \diamond_{\mathcal{I}_{\mathbb{R}}} X = \{z \in \mathbb{R} \mid (\exists x \in X)(z = \diamond_{\mathbb{R}} x)\})$.

The sentence (I1) of definition 3.1 characterizes what an interval number (or a closed \mathbb{R} -interval) is. The sentences (I2) and (I3) prescribe, respectively, the binary and unary operations for \mathbb{R} -intervals. Hereafter, the upper-case Roman letters X, Y , and Z (with or without subscripts), or equivalently $[\underline{x}, \bar{x}]$, $[\underline{y}, \bar{y}]$, and $[\underline{z}, \bar{z}]$, shall be employed as variable symbols to denote real interval numbers. A point (singleton) interval number $\{x\}$ shall be denoted by $[x]$. The letters A, B , and C , or equivalently $[\underline{a}, \bar{a}]$, $[\underline{b}, \bar{b}]$, and $[\underline{c}, \bar{c}]$, shall be used to denote constants of $\mathcal{I}_{\mathbb{R}}$. Also, we shall single out the symbols $1_{\mathcal{I}}$ and $0_{\mathcal{I}}$ to denote, respectively, the singleton \mathbb{R} -intervals $\{1_{\mathbb{R}}\}$ and $\{0_{\mathbb{R}}\}$. For the purpose at hand, it is convenient to define two proper subsets of $\mathcal{I}_{\mathbb{R}}$: the sets of *symmetric interval numbers* and *point interval numbers*. Respectively, these are defined and denoted by

$$\mathcal{I}_S = \{X \in \mathcal{I}_{\mathbb{R}} \mid (\exists x \in \mathbb{R})(0 \leq x \wedge X = [-x, x])\},$$

$$\mathcal{I}_{[x]} = \{X \in \mathcal{I}_{\mathbb{R}} \mid (\exists x \in \mathbb{R})(X = [x, x])\}.$$

From the fact that real intervals are totally $\leq_{\mathbb{R}}$ -ordered subsets of \mathbb{R} , equality of \mathbb{R} -intervals follows immediately from the axiom of extensionality⁴ of set theory. That is,

$$[\underline{x}, \bar{x}] =_{\mathcal{I}} [\underline{y}, \bar{y}] \Leftrightarrow \underline{x} =_{\mathbb{R}} \underline{y} \wedge \bar{x} =_{\mathbb{R}} \bar{y}.$$

From the fact that \mathbb{R} -intervals are *ordered sets* of real numbers, the following theorem is derivable from definition 3.1 (see [9] and [11]).

Theorem 3.1 (Interval Operations). *For any two interval numbers $[\underline{x}, \bar{x}]$ and $[\underline{y}, \bar{y}]$, the binary and unary interval operations are formulated in terms of the intervals' endpoints as follows.*

- (i) $[\underline{x}, \bar{x}] +_{\mathcal{I}} [\underline{y}, \bar{y}] = [\underline{x} +_{\mathbb{R}} \underline{y}, \bar{x} +_{\mathbb{R}} \bar{y}]$,
- (ii) $[\underline{x}, \bar{x}] \times_{\mathcal{I}} [\underline{y}, \bar{y}] = [\min\{\underline{x} \times_{\mathbb{R}} \underline{y}, \underline{x} \times_{\mathbb{R}} \bar{y}, \bar{x} \times_{\mathbb{R}} \underline{y}, \bar{x} \times_{\mathbb{R}} \bar{y}\}, \max\{\underline{x} \times_{\mathbb{R}} \underline{y}, \underline{x} \times_{\mathbb{R}} \bar{y}, \bar{x} \times_{\mathbb{R}} \underline{y}, \bar{x} \times_{\mathbb{R}} \bar{y}\}]$,
- (iii) $-_{\mathcal{I}} [\underline{x}, \bar{x}] = [-_{\mathbb{R}} \bar{x}, -_{\mathbb{R}} \underline{x}]$,
- (iv) $0_{\mathcal{I}} \not\subseteq [\underline{x}, \bar{x}] \Rightarrow [\underline{x}, \bar{x}]^{-1_{\mathcal{I}}} = [\bar{x}^{-1_{\mathbb{R}}}, \underline{x}^{-1_{\mathbb{R}}}]$,

where *min* and *max* are respectively the $\leq_{\mathbb{R}}$ -minimal and $\leq_{\mathbb{R}}$ -maximal.

Wherever there is no confusion, we shall drop the subscripts \mathcal{I} and \mathbb{R} . By definition 2.8, it is obvious that all the interval operations, except interval reciprocal, are *total* operations. The additional operations of interval subtraction and division can be defined respectively as $X - Y = X + (-Y)$ and $X \div Y = X \times (Y^{-1})$.

Interval multiplication left and right *subdistributes* over interval addition⁵. In other words, the structure $\langle \mathcal{I}_{\mathbb{R}}; +_{\mathcal{I}}, \times_{\mathcal{I}}; 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle$ of real interval numbers is a commutative *S-semiring* (*subdistributive semiring*) [9].

Throughout this article, we shall employ the following theorem and its corollary (see, [8], and [9]).

Theorem 3.2 (Inclusion Monotonicity in Classical Intervals). *Let X_1, X_2, Y_1 , and Y_2 be interval numbers such that $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$. Then for any binary operation $\circ \in \{+, \times\}$ and any definable unary operation $\diamond \in \{-, -1\}$, we have*

- (i) $X_1 \circ_{\mathcal{I}} X_2 \subseteq Y_1 \circ_{\mathcal{I}} Y_2$,
- (ii) $\diamond_{\mathcal{I}} X_1 \subseteq \diamond_{\mathcal{I}} Y_1$.

⁴The axiom of extensionality asserts that two sets are equal if, and only if they have precisely the same elements, that is, for any two sets S and T , $S = T \Leftrightarrow (\forall z)(z \in S \Leftrightarrow z \in T)$.

⁵An S-ringoid (or a *subdistributive ringoid*) is a ring-like structure that satisfies at least one of the following subdistributive criteria (see [11] and [15]).

- (i) $(\forall x, y, z \in \mathcal{R})(x \times_{\mathcal{R}} (y +_{\mathcal{R}} z) \subseteq x \times_{\mathcal{R}} y +_{\mathcal{R}} x \times_{\mathcal{R}} z)$,
- (ii) $(\forall x, y, z \in \mathcal{R})((y +_{\mathcal{R}} z) \times_{\mathcal{R}} x \subseteq y \times_{\mathcal{R}} x +_{\mathcal{R}} z \times_{\mathcal{R}} x)$.

Noteworthy, the notion of an S-ringoid is a generalization of the notion of a *near-semiring*; a near-semiring is a ringoid that satisfies the criteria of a semiring except that it is *either* left or right distributive (For detailed discussions of near-semirings and related concepts, the reader may refer to, e.g., [49], [42], and [6]).

In consequence of this theorem, from the fact that $[x, x] \subseteq X \Leftrightarrow x \in X$, we have the following important special case.

Corollary 3.1 (Membership Monotonicity for Classical Intervals). *Let X and Y be real interval numbers with $x \in X$ and $y \in Y$. Then for any binary operation $\circ \in \{+, \times\}$ and any definable unary operation $\diamond \in \{-, ^{-1}\}$, we have*

- (i) $x \circ_{\mathbb{R}} y \in X \circ_c Y$,
- (ii) $\diamond_{\mathbb{R}} x \in \diamond_c X$.

In addition to ordering intervals by the set inclusion relation \subseteq , there are many orders presented in the interval literature. Among these is Moore's partial ordering which is defined by $[\underline{x}, \bar{x}] <_M [\underline{y}, \bar{y}] \Leftrightarrow \bar{x} <_{\mathbb{R}} \underline{y}$. In contrast to the case for \subseteq , Moore's partial ordering $<_M$ is not compatible with the algebraic operations on $\mathcal{I}_{\mathbb{R}}$ (see [9] and [15]).

4 From Approximations to Infinite Precision: The Need for Machine Intervals

In order to clarify the need for machine interval arithmetic, this section provides a brief discussion of the limitations and loss of precision of machine real arithmetic. Machine real numbers have finite decimal places of precision. The *finite* precision provided by modern computers is enough in many real life applications, and there is scarcely a physical quantity which can be measured beyond the maximum representable value of this precision.

So when is the finite precision not enough? The problem arises when doing arithmetic. The operation of "subtraction", for example, results, in many situations, in an inevitable loss of precision. Consider, for instance, the expression $x - y$, with $x = 0.963$ and $y = 0.962$. The exact result of $x - y$ is 0.001, but when evaluating this expression on a machine with 2 significant digits, the values of x and y are rounded *downward* to have the same machine value 0.96, and the machine result of $x - y$ becomes 0, which is a complete loss of precision. Instead, by enclosing the exact values of x and y in interval numbers, with outward rounding (see section 5), a *guaranteed enclosure* of the exact result of $x - y$ can be obtained easily by computing with machine interval numbers.

This problem of machine subtraction has dangerous consequences in numerical computations. To illustrate, consider the problem of calculating the derivative of a real-valued function f at a given point. The method of *finite differences* is a numerical method which can be performed by a computer to approximate the derivative. For a differentiable function f , the first derivative $f^{(1)}$ can be approximated by

$$f^{(1)} \cong \frac{f(x+dx) - f(x)}{dx},$$

for a *small* nonzero value of dx . As dx approaches zero, the derivative is better approximated. But as dx gets smaller, the rounding error increases because of the finite precision of machine real arithmetic, and we accordingly get the problematic situation of $f(x+dx) - f(x) = 0$. That is, for small enough values of dx , the derivative will be always computed as *zero*, regardless of the rule of the function f . Using interval enclosures of the function f instead, we can find a way out of this problem, by virtue the *infinite precision* of machine interval arithmetic. For further details on interval enclosures of derivatives, see, e.g., [9] and [18].

Another problem of finite precision arises when *truncating* an infinite operation by a computable finite operation. For example, The exponential function e^x may be written as a Taylor series

$$e^x = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

In order to compute this infinite series on a machine, we have to *truncate* it to the partial sum

$$S_k = \sum_{n=0}^k \frac{x^n}{n!},$$

for some finite k , and the truncation error then is $|e^x - S_k|$. Using interval bounds for this error term, machine interval arithmetic can provide a *guaranteed enclosure* of the exact value of the exponential function e^x .

Moreover, in some practical situations, the numerical approximations provided by machine real arithmetic are not beneficial. In robotics and control applications, for example, it is important to have guaranteed inclusions of the exact values in order to guarantee stability under uncertainty.

The preceding sample examples shed light on the fact that taking the passage from machine real arithmetic to machine interval arithmetic opens the way to the rich technicalities and the infinite precision of interval computations.

5 Machine Realization of the Interval Operations

The arithmetic of intervals defined in section 3 may be called an *exact interval arithmetic*, in the sense that no rounding or approximation is involved. However, when interval arithmetic is realized on a computer, we get some loss of accuracy due to round-off errors. Therefore, due to the fact that there is only a *finite* subset $\mathbb{M} \subset \mathbb{R}$ of *machine-representable numbers*, special care has to be taken to guarantee a proper hardware implementation of interval arithmetic. Thus, we need a *machine interval arithmetic* in which interval numbers have to be rounded so that the interval result computed by a machine always contains the exact interval result.

The algebraic operations of the classical theory of interval arithmetic are defined in such a way that they satisfy the property of *inclusion monotonicity* (see theorem 3.2). An important immediate consequence of the inclusion monotonicity is that given two interval numbers $[x, \bar{x}]$ and $[y, \bar{y}]$ with $x \in [x, \bar{x}]$ and $y \in [y, \bar{y}]$, then for any definable unary operation $\diamond \in \{-, ^{-1}\}$ and any binary operation $\circ \in \{+, \times\}$, the real and interval results shall satisfy

$$\begin{aligned} \diamond x &\in \diamond [x, \bar{x}], \\ x \circ y &\in [x, \bar{x}] \circ [y, \bar{y}]. \end{aligned}$$

That is, guaranteed enclosures of the real-valued results can be obtained easily by computing on interval numbers.

The preceding formulas use the arithmetic of real numbers that are not machine-representable. However, using *outward rounding* for interval numbers, we can obtain alternate formulas that use floating-point arithmetic, and still satisfy the property of inclusion monotonicity.

Let n be a finite ordinal. Henceforth, we shall understand by a set of machine-representable real numbers with n significant digits, any finite subset \mathbb{M}_n of rational numbers that can be represented by n significant decimal digits. Two definitions we shall need are those of the *downward* and *upward* rounding operators.

Definition 5.1 (Downward Rounding). *Let x be any real number and let x_m denote a machine-representable real number with n significant digits. Then there exists a machine-representable real number $\nabla_n x \in \mathbb{M}_n$ such that*

$$\nabla_n x = \sup\{x_m \in \mathbb{M}_n \mid x_m \leq x\},$$

where ∇ is called the downward rounding operator.

Definition 5.2 (Upward Rounding). *Let x be any real number and let x_m denote a machine-representable real number with n significant digits. Then there exists a machine-representable real number $\Delta_n x \in \mathbb{M}_n$ such that*

$$\Delta_n x = \inf\{x_m \in \mathbb{M}_n \mid x \leq x_m\},$$

where Δ is called the upward rounding operator.

Let, for instance, \mathbb{M}_2 be the set of machine-representable real numbers with *two* significant digits. Then, $\nabla_2(0.432) = 0.43$ and $\Delta_2(0.432) = 0.44$.

In order to be able to do useful arithmetic with machine real numbers, next we characterize the notion of a *sufficient* (or *arithmetical*) set of machine real numbers. This is made precise in the following definition.

Definition 5.3 (Arithmetical Machine Real Numbers). *For a finite ordinal n , a sufficient (or arithmetical) set \mathbb{M}_n of machine real numbers is characterized as follows.*

- (i) \mathbb{M}_n is finite,
- (ii) $x_m \in \mathbb{M}_n \Leftrightarrow (\exists x \in \mathbb{R}) (x_m = \nabla_n x \vee x_m = \Delta_n x)$,
- (iii) $0 \in \mathbb{M}_n$,
- (iv) $(\forall x_m) (x_m \in \mathbb{M}_n \Rightarrow (-x_m) \in \mathbb{M}_n)$.

Hereafter, if confusion is unlikely, we shall usually drop the subscript n . Also, unless stated explicitly otherwise, we shall understand by \mathbb{M} an *arithmetical* set of machine real numbers.

The binary and unary operations for machine real numbers can be characterized in the following definition.

Definition 5.4 (Machine Real Operations). *Let \blacklozenge be in $\{\nabla, \Delta\}$, and let \diamond and \circ be, respectively, in $\{-, ^{-1}\}$ and $\{+, \times\}$. For any two real numbers x and y , the unary and binary machine real operations are defined as*

$$\begin{aligned} \diamond_{\blacklozenge} x &= \blacklozenge(\diamond x), \\ x \circ_{\blacklozenge} y &= \blacklozenge(x \circ y). \end{aligned}$$

On the basis of these definitions, we can obtain a finite set $\mathcal{M} \subset \mathcal{I}_{\mathbb{R}}$ of machine interval numbers by rounding interval numbers outward.

Definition 5.5 (Outward Rounding). Let $[x, \bar{x}]$ be any interval number. Then there exists a machine-representable interval number $\square[x, \bar{x}]$ such that

$$\square[x, \bar{x}] = [\nabla x, \Delta \bar{x}],$$

where \square is called the outward rounding operator.

Accordingly, the set \mathcal{M} of machine interval numbers is

$$\mathcal{M} = \{X_m \mid (\exists X \in \mathcal{I}_{\mathbb{R}}) (X_m = \square X)\},$$

and obviously outward rounding is a function that maps elements of $\mathcal{I}_{\mathbb{R}}$ to the set \mathcal{M} of machine interval numbers, that is $\square_n : \mathcal{I}_{\mathbb{R}} \mapsto \mathcal{M}_n$.

With outward rounding, a machine interval arithmetic can be defined such that the result of a machine interval operation is a machine interval number which is guaranteed to contain the exact result of an interval operation. In this manner, the classical interval operations can be redefined, in the language of machine interval arithmetic, as follows.

Definition 5.6 (Machine Interval Operations). Let $[x, \bar{x}]$ and $[y, \bar{y}]$ be interval numbers. The unary and binary machine interval operations are defined as

- (i) $\square(-[x, \bar{x}]) = [\nabla(-\bar{x}), \Delta(-x)],$
- (ii) $0 \notin [x, \bar{x}] \Rightarrow \square([x, \bar{x}]^{-1}) = [\nabla(\bar{x}^{-1}), \Delta(x^{-1})],$
- (iii) $\square([x, \bar{x}] + [y, \bar{y}]) = [\nabla(x + y), \Delta(\bar{x} + \bar{y})],$
- (iv) $\square([x, \bar{x}] \times [y, \bar{y}]) = [\nabla(\min\{x\underline{y}, x\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\}), \Delta(\max\{x\underline{y}, x\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\})].$

For simplicity of the language, throughout this article, we shall deploy the following abbreviations.

$$\begin{aligned} -\square[x, \bar{x}] &= \square(-[x, \bar{x}]), \\ [x, \bar{x}]^{-1\square} &= \square([x, \bar{x}]^{-1}), \\ [x, \bar{x}] + \square[y, \bar{y}] &= \square([x, \bar{x}] + [y, \bar{y}]), \\ [x, \bar{x}] \times \square[y, \bar{y}] &= \square([x, \bar{x}] \times [y, \bar{y}]). \end{aligned}$$

Outward rounding of interval numbers involves performing computations with two rounding modes (upward and downward). This can be much costlier than performing the computations with one single rounding direction. By virtue of definition 5.3, we have

$$(\forall x_m) (x_m \in \mathbb{M} \Rightarrow (-x_m) \in \mathbb{M}),$$

and accordingly

$$(\forall x \in \mathbb{R}) (\nabla(-x) = -\Delta(x)),$$

which makes it possible to use upward rounding as one single rounding mode. In this manner, for instance, machine interval addition can be reformulated as

$$\square([x, \bar{x}] + [y, \bar{y}]) = [-\Delta((-x) - y), \Delta(\bar{x} + \bar{y})].$$

Similar optimal roundings can be applied to other interval operations so that one can get more efficient implementations of interval arithmetic.

6 On Some Algebraic and Order-Theoretic Properties of Machine Intervals

By means of the notions prescribed in the preceding sections of this article, we shall now inquire into some algebraic and order-theoretic theorems concerning machine interval arithmetic.

A first important result figures in the following theorem.

Theorem 6.1 (Symmetricity is Machine Monotonic). Let \mathcal{M}_S be the set of machine symmetric intervals. The machine representation of a real symmetric interval X is always an element of \mathcal{M}_S . That is

$$(\forall X) (X \in \mathcal{I}_S \Rightarrow \square X \in \mathcal{M}_S).$$

Proof. Let $[-x, x] \in \mathcal{I}_S$. Then, by definition 5.3, $\square[-x, x] = [\nabla(-x), \Delta(x)] = [-\Delta(x), \Delta(x)] \in \mathcal{M}_S$. □

In contrast to symmetricity, singletonicity is not machine monotonic. This is established in the following theorem.

Theorem 6.2 (Singletonicity is not Machine Monotonic). *Let $\mathcal{M}_{[x]}$ be the set of machine point (singleton) intervals. The machine representation of a real point interval X is not necessarily an element of $\mathcal{M}_{[x]}$. That is*

$$(\exists X) (X \in \mathcal{I}_{[x]} \wedge \square X \notin \mathcal{M}_{[x]}).$$

Proof. Let $[x, x]$ be an element of $\mathcal{I}_{[x]}$. According to definition 5.5, the machine representation of $[x, x]$ is

$$\square[x, x] = [\nabla x, \Delta x],$$

which is not necessarily a machine point interval. □

The following example makes this clear.

Example 6.1. *Let \mathbb{M}_2 be the set of machine-representable real numbers with two significant digits. The real point interval $[0.432, 0.432]$ then has the machine representation*

$$\begin{aligned} \square_2([0.432, 0.432]) &= [\nabla_2(0.432), \Delta_2(0.432)] \\ &= [0.43, 0.44], \end{aligned}$$

which is not a machine point interval.

This property is not a problem of machine interval arithmetic; rather, it is a guarantee that the interval result computed by a machine always contains the exact interval result.

Consequently, the following theorem is derivable.

Theorem 6.3 (Nontotality of Point Machine Operations). *The machine interval operations are not total operations on $\mathcal{M}_{[x]}$. That is for X and Y are elements of $\mathcal{M}_{[x]}$, the results of machine interval operations for X and Y are not always elements of $\mathcal{M}_{[x]}$.*

Proof. By virtue of the previously proved fact that outward rounding does not preserve singletonicity, the proof is immediate from definition 5.6 and definition 2.8. □

To illustrate this, we next give an example.

Example 6.2. *Let \mathbb{M}_2 be the set of machine-representable real numbers with two significant digits. The result of multiplying the two machine point intervals $[2.26, 2.26]$ and $[2.27, 2.27]$ is computed as*

$$\begin{aligned} \square_2([2.26, 2.26] \times [2.27, 2.27]) &= [\nabla_2(5.1302), \Delta_2(5.1302)] \\ &= [5.13, 5.14], \end{aligned}$$

which is not a machine point interval.

The questions of *definability* of the structure of machine point intervals is the subject of the following theorem.

Theorem 6.4 (Undefinability of Machine Point Algebra). *The structure $\langle \mathcal{M}_{[x]}; +_{\square}, \times_{\square} \rangle$ of machine point intervals is not definable with respect to the machine operations $+_{\square}$ and \times_{\square} .*

Proof. Since, by theorem 6.3, the machine interval operations $+_{\square}$ and \times_{\square} are not total operations on $\mathcal{M}_{[x]}$, it follows, from definition 2.9, that the structure $\langle \mathcal{M}_{[x]}; +_{\square}, \times_{\square} \rangle$ is not definable. □

The following theorem asserts that the set \mathbb{M} of machine real numbers is *not dense*.

Theorem 6.5 (Nondensity of Machine Real Numbers). *Let \mathbb{M}_n be the set of machine real numbers with n significant digits. The set \mathbb{M}_n is not dense with respect to the strict real ordering $<$, that is*

$$(\exists x_m \in \mathbb{M}_n) (\exists y_m \in \mathbb{M}_n) (x_m < y_m \wedge \neg ((\exists z_m \in \mathbb{M}_n) (x_m < z_m \wedge z_m < y_m))).$$

Proof. The statement of the theorem immediately follows, by definition 2.7, from the fact that the set of machine real numbers is the closure of finite precision numbers under machine rounding. □

In order to clarify, let x_m be an element of \mathbb{M}_n . Then x_m can be written as

$$x_m = x_0 + \frac{x_1}{10} + \frac{x_2}{10^2} + \dots + \frac{x_n}{10^n} = \sum_{k=0}^n \frac{x_k}{10^k},$$

where $x_0, x_1, x_2, \dots, x_n$ are nonnegative integers.

Accordingly, if y_m is an element of \mathbb{M}_n such that

$$y_m = x_0 + \frac{x_1}{10} + \frac{x_2}{10^2} + \dots + \frac{x_n + 1}{10^n} = \frac{1}{10^n} + \sum_{k=0}^n \frac{x_k}{10^k},$$

then y_m is the element of \mathbb{M}_n exactly next to x_m , and therefore, there is no $z_m \in \mathbb{M}_n$ such that $x_m < z_m \wedge z_m < y_m$.

In consequence of theorem 6.5, the set \mathcal{M} of machine intervals is also not dense, from the fact that \mathcal{M} is a proper subset of the powerset of \mathbb{M} .

Corollary 6.1 (Nondensity of Machine Intervals). *Let \mathcal{M}_n be the set of machine interval numbers with n significant digits. The set \mathcal{M}_n is not dense with respect to Moore’s strict partial ordering $<_M$.*

Let, for instance, \mathbb{M}_1 be the set of machine real numbers with one significant digit. Obviously, there is no $z_m \in \mathbb{M}_1$ such that $1.1 < z_m < 1.2$, and there is no $Z_m \in \mathcal{M}_1$ such that $[1.1, 1.1] <_M Z_m <_M [1.2, 1.2]$.

Moreover, unlike the sets \mathbb{R} and $\mathcal{I}_{\mathbb{R}}$, the sets \mathbb{M} and \mathcal{M} , of machine real numbers and machine interval numbers, are countably finite. These are established in the following theorem and its corollary.

Theorem 6.6 (Countability of Machine Real Numbers). *The set \mathbb{M} of machine real numbers is countably finite.*

Proof. The set \mathbb{M} of machine real numbers is, by definition, a finite proper subset of the set \mathbb{Q} of rational numbers. Since \mathbb{Q} is countably infinite, it follows, by definition 2.2, that the set \mathbb{M} is countably finite. □

This theorem, by the fact that \mathcal{M} is a proper subset of the powerset of \mathbb{M} , has as a consequence the following corollary.

Corollary 6.2 (Countability of Machine Intervals). *The set \mathcal{M} of machine interval numbers is countably finite.*

Thus, we can easily determine the number of machine interval numbers between any two elements of \mathbb{M} . This is made precise in the following theorem.

Theorem 6.7 (Count of Machine Interval Numbers). *Let \mathbb{M}_n be the set of machine real numbers with n significant digits, and let x_m and y_m be elements of \mathbb{M}_n such that $x_m \leq y_m$. Then the count of machine interval numbers between x_m and y_m is given by*

$$C_{\mathcal{M}(x_m, y_m)} = \sum_{k=1}^{C_{\mathbb{M}(x_m, y_m)}} k = \frac{C_{\mathbb{M}(x_m, y_m)}^2 + C_{\mathbb{M}(x_m, y_m)}}{2},$$

where

$$C_{\mathbb{M}(x_m, y_m)} = 10^n \times (y_m - x_m) + 1,$$

is the count of machine real numbers between x_m and y_m .

Proof. Obviously, the count of singleton machine intervals between x_m and y_m is $C_{\mathbb{M}(x_m, y_m)}$. The count of non-singleton machine intervals is computed as follows. First, the count of machine intervals with a lower endpoint equal to x_m is $C_{\mathbb{M}(x_m, y_m)} - 1$. The count of machine intervals with a lower endpoint equal to the machine successor of x_m is $C_{\mathbb{M}(x_m, y_m)} - 2$. Following this, the count of machine intervals with a lower endpoint equal to the machine predecessor of y_m is 1. Summing up, we have

$$\begin{aligned} C_{\mathcal{M}(x_m, y_m)} &= C_{\mathbb{M}(x_m, y_m)} + (C_{\mathbb{M}(x_m, y_m)} - 1) + (C_{\mathbb{M}(x_m, y_m)} - 2) + \dots + 1 \\ &= \frac{C_{\mathbb{M}(x_m, y_m)} (C_{\mathbb{M}(x_m, y_m)} + 1)}{2}, \end{aligned}$$

and the theorem follows. □

The following example makes this clear.

Example 6.3 (Counting Machine Numbers). Let \mathbb{M}_2 be the set of machine real numbers with two significant digits. The count of machine real numbers between 1.23 and 1.32 is

$$\begin{aligned} C_{\mathbb{M}(1.23,1.32)} &= 10^2 \times (1.32 - 1.23) + 1 \\ &= 10, \end{aligned}$$

and the count of machine interval numbers between 1.23 and 1.32 is

$$C_{\mathcal{M}(1.23,1.32)} = \frac{10^2 + 10}{2} = 55.$$

Now we turn to some important monotonicity properties of machine interval arithmetic. With the help of definitions 5.1 and 5.2, the following theorem is derivable.

Theorem 6.8 (Non-Strict Real Order is Machine Monotonic). The non-strict real ordering \leq is monotonic with respect to downward and upward roundings. That is, for any two real numbers x and y , we have

- (i) $x \leq y \Rightarrow \nabla x \leq \nabla y$,
- (ii) $x \leq y \Rightarrow \Delta x \leq \Delta y$.

However, in contrast to the case for \leq , the strict real ordering $<$ is not machine monotonic. This is established in the next theorem.

Theorem 6.9 (Strict Real Order is not Machine Monotonic). The strict real ordering $<$ is monotonic with respect neither to downward nor to upward roundings. That is, there exist two real numbers x and y such that

- (i) $x < y \wedge \nabla x \not< \nabla y$,
- (ii) $x < y \wedge \Delta x \not< \Delta y$.

Proof. To prove the theorem, it suffices to give a counterexample. Let \mathbb{M}_1 be the set of machine-representable real numbers with one significant digit. For the two real numbers 0.92 and 0.93, we have $0.92 < 0.93$.

Rounding the two numbers downward, we get $\nabla(0.92) = \nabla(0.93) = 0.9$, and accordingly $\nabla(0.92) \not< \nabla(0.93)$.

Analogously, rounding the two numbers upward, we get $\Delta(0.92) = \Delta(0.93) = 1.0$, and accordingly $\Delta(0.92) \not< \Delta(0.93)$.

The strict real ordering $<$ is therefore not machine monotonic. \square

Finally, by means of definition 5.6, it is not difficult to prove the well-known inclusion monotonicity theorem for machine interval numbers.

Theorem 6.10 (Inclusion Monotonicity in Machine Intervals). Machine interval arithmetic is inclusion monotonic. That is, for any two interval numbers X and Y , we have

- (i) $X \subseteq Y \Rightarrow \square X \subseteq \square Y$,
- (ii) $X \circ Y \subseteq \square(X \circ Y)$,
- (iii) $\diamond X \subseteq \square(\diamond X)$.

Accordingly, we have as a consequence the following corollary.

Corollary 6.3 (Membership Monotonicity for Machine Intervals). Machine interval arithmetic is membership monotonic. That is, for any two interval numbers X and Y with $x \in X$ and $y \in Y$, we have

- (i) $\diamond x \in \square(\diamond X)$,
- (ii) $x \circ y \in \square(X \circ Y)$.

Thus, outward rounding provides an efficient implementation of interval arithmetic, with the property of inclusion monotonicity still satisfied.

To illustrate this, we give two numerical examples.

Example 6.4 (Monotonicity of Machine Intervals). Let \mathbb{M}_3 be the set of machine-representable real numbers with three significant digits.

(i) We have

$$\begin{aligned}\square_3([1, 2] \div [2, 3]) &= [\nabla_3(1/3), \Delta_3(1)] \\ &= [0.333, 1],\end{aligned}$$

and

$$([1, 2] \div [2, 3]) \subset [0.333, 1].$$

(ii) We have

$$\begin{aligned}\square_3([0, 1] + [2.7182, 3.3841]) &= [\nabla_3(2.7182), \Delta_3(4.3841)] \\ &= [2.718, 4.385],\end{aligned}$$

and

$$([0, 1] + [2.7182, 3.3841]) \subset [2.718, 4.385].$$

7 Conclusion

Interval arithmetic is based on the very simple and elegant idea that of expressing uncertain real-valued quantities as real closed intervals. For the great degree of reliability it provides, interval arithmetic is usually a part of all other methods that deal with uncertainty. Thus, in view of this computational power against error and imprecision, machine implementations of interval arithmetic are of great importance. This article has been then devoted to investigating some mathematical notions concerning the algebraic system of machine interval arithmetic. In the first place, after formalizing some algebraic and order-theoretical ingredients of importance to our purpose, we gave a formal characterization of an interval algebra over the real field. Next, we provided a discussion of the limitations of machine real arithmetic along with a clarification of the need for the infinite precision of machine interval arithmetic. Thereupon, we gave an algebraic characterization of the fundamental notions of machine real arithmetic and machine interval arithmetic. Finally, we proved some algebraic and order-theoretic results concerning the structure of machine intervals.

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