

**Theories of**  
**INTERVAL ARITHMETIC**  
Mathematical Foundations and Applications

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**HEND DAWOOD**

*Department of Mathematics  
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ISBN: 978-3-8465-0154-2

Publisher:

LAP LAMBERT Academic Publishing GmbH & Co. KG  
Dudweiler Landstr. 99  
66123 Saarbrücken  
Germany

Register court/number: Handelsregister Amtsgericht Saarbrücken HRA 10752  
Identification Number (Verkehrsnummer): 12917

Partner with unlimited liability/Persönlich haftende Gesellschafterin: VDM Management GmbH  
Register court/number: Handelsregister Amtsgericht Saarbrücken HRB 18918

Managing directors/Geschäftsführer: Thorsten Ohm (CEO), Dr. Wolfgang Philipp Müller, Esther von Krosigk, Christoph Schulligen

Responsible in accordance with Art. 10 par. 3 MDStV (German Interstate Agreement on Media Services): Dr. Wolfgang Philipp Müller

“This new book by Hend Dawood is a fresh introduction to some of the basics of interval computation. It stops short of discussing the more complicated subdivision methods for converging to ranges of values, however it provides a bit of perspective about complex interval arithmetic, constraint intervals, and modal intervals, and it does go into the design of hardware operations for interval arithmetic, which is something still to be done by computer manufacturers.”

-Ramon E. Moore

The Founder of Interval Computations

*Professor Emeritus of Computer and Information Science,  
Department of Mathematics, The Ohio State University,  
Columbus, U.S.A.*

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*Dedicated to my family.*

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# Preface

Scientists are, all the time, in a struggle with uncertainty which is always a threat to a trustworthy scientific knowledge. A very simple and natural idea, to defeat uncertainty, is that of enclosing uncertain measured values in real closed intervals. On the basis of this idea, interval arithmetic is constructed. The idea of calculating with intervals is not completely new in mathematics: the concept has been known since the third century BC, when Archimedes used guaranteed lower and upper bounds to compute his constant  $\pi$ . Interval arithmetic is now a broad field in which rigorous mathematics is associated with scientific computing. This connection makes it possible to solve uncertainty problems that cannot be efficiently solved by floating-point arithmetic. Today, application areas of interval methods include electrical engineering, control theory, remote sensing, experimental and computational physics, chaotic systems, celestial mechanics, signal processing, computer graphics, robotics, and computer-assisted proofs.

The purpose of this monograph is to be a concise but informative introduction to the *theories of interval arithmetic* as well as to some of their computational and scientific applications, for undergraduates and early graduates. It is primarily intended for students of mathematics, computer science, and engineering. The book tries to represent a reasonable portion of the theory of intervals, but still contains only a fraction of the results and applications. So, I believe that it may help to introduce students to the elementary concepts of the interval theory without exhaustively analyzing every single aspect or delving into more advanced topics of the theory.

The book is a greatly expanded edition of my earlier graduate report entitled “*Interval Arithmetic: An Accurate Self-Validating Arithmetic for Digital Computing*”. This edition is completely rewritten for improved clarity and readability. With the various additions and improvements, some stylistic changes, and the corrections of some minor errors I have made to my original account, I

could not resist a change in the original title. The new title, therefore, is more truly representative of the expanded contents of this text.

## Outline of the Book

The book opens with a brief prologue intended to provide a bit of motivation and perspective about the field of interval arithmetic, its history, and how it is a potential weapon against uncertainty in science and technology.

Chapter 2 introduces the key concepts of the classical interval theory and then defines the algebraic and point operations for interval numbers. In section 2.3, we carefully construct the algebraic system of classical interval arithmetic and deduce its fundamental properties.

The first extension of the classical interval theory appears in chapter 3. Here we introduce how the use of classical interval arithmetic can be extended, via *complex interval numbers*, to determine regions of uncertainty in computing with complex numbers. We start with the key concepts of complex interval arithmetic and then move on to carefully construct the algebraic system of complex interval arithmetic and deduce its fundamental properties.

Chapter 4, after describing the *interval dependency problem*, is intended to provide a bit of perspective about two important alternate theories of interval arithmetic: *constraint interval arithmetic*, and *modal interval arithmetic*.

In chapter 5, we delve into some computational applications of the interval theory. Here we explain, in some detail, the interval estimates of the images of continuous real functions on closed intervals, how interval methods are used to bound the error term in Taylor's theorem, and how interval arithmetic is applied to evaluate definite integrals.

In chapter 6, we begin with some key concepts of machine real arithmetic, and then carefully construct the algebraic system of *machine interval arithmetic* and deduce its fundamental properties. In section 6.2, we design a simple circuitry that shows how to realize interval addition on a machine. The final section of this chapter introduces how to use a floating-point multiplier and a comparator to design a simple interval squarer circuitry.

The book closes with a brief epilogue intended to provide a view to the future of the interval theory. Here we provide a concluding look ahead and further information resources.

Appendix A introduces how to use four half adders and eight full adders to

design a 4-by-4 bit multiplier circuitry. We provide detailed Verilog descriptions of the circuitry and simulation process of the 4-by-4 bit multiplier.

Appendix [B](#) introduces how to use two 4-by-4 bit multipliers and one 8-bit comparator to design an interval squarer circuitry. We provide detailed Verilog descriptions of the circuitry and simulation process of the interval squarer.

## Acknowledgments

I owe great thanks to Assoc. Prof. Dr. *Hossam A. H. Fahmy*, Department of Electronics and Communication Engineering, Faculty of Engineering, Cairo University, for encouraging my early interests in interval arithmetic and for his fruitful discussions and valuable suggestions. Dr. Hossam spent a lot of his time introducing me to the concepts of the interval theory.

I am deeply indebted to my teacher Assoc. Prof. Dr. *Hassan A. M. Aly*, Department of Mathematics, Faculty of Science, Cairo University, for his continual support and wise advice and for so many most fruitful discussions. My discussions with Dr. Hassan inspired me to think in a radically new way that permeated the present work.

Finally it is a pleasure to acknowledge the courtesy and substantial help which I have received from *Hannah Olsen*, my editor, and the staff of the LAP Lambert Academic Publishing at all stages in the production of the book.

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Cairo, September, 2011.  
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# Notations and Conventions

Most of our notation is standard but, for the purpose of legibility, we give here a consolidated list of symbols for the entire text.

## Logical Symbols

Symbol	Meaning
$\neg$	Logical Negation ( <i>not</i> ).
$\Rightarrow$	Implication ( <i>if ..., then ...</i> ).
$\Leftrightarrow$	Equivalence ( <i>if, and only if</i> ).
$\wedge$	Conjunction ( <i>and</i> ).
$\vee$	Inclusive disjunction ( <i>or</i> ).
$\forall$	Universal quantifier ( <i>for all</i> ).
$\exists$	Existential quantifier ( <i>there exists</i> ).
$=$	Identity ( <i>equality</i> ).

## Non-logical Symbols

Symbol	Meaning
$\emptyset$	The empty set.
$\mathbb{R}$	The set of real numbers.
$x, y, z$	Real variable symbols ( <i>with or without subscripts, and with or without lower or upper hyphens</i> ).

Symbol	Meaning
$a, b, c$	Real constant symbols ( <i>with or without subscripts, and with or without lower or upper hyphens</i> ).
$\circ \in \{+, \times\}$	A binary algebraic operator.
$\diamond \in \{-, ^{-1}\}$	A unary algebraic operator.
$\wp(S)$	The powerset of a set $S$ .
$[\mathbb{R}]$	The set of real interval numbers.
$[\mathbb{R}]_p$	The set of point ( <i>singleton</i> ) interval numbers $[x, x]$ .
$[\mathbb{R}]_s$	The set of symmetric interval numbers $[-x, x]$ .
$[\mathbb{R}]_{\tilde{0}}$	The set of interval numbers that do not contain 0.
$X, Y, Z$	Interval variable symbols ( <i>with or without subscripts</i> ).
$A, B, C$	Interval constant symbols ( <i>with or without subscripts</i> ).
$\mathbb{C}$	The set of ordinary complex numbers.
$i = \sqrt{-1}$	The ordinary imaginary unit.
$\mathbf{x}, \mathbf{y}, \mathbf{z}$	Complex variable symbols ( <i>with or without subscripts</i> ).
$\mathbf{a}, \mathbf{b}, \mathbf{c}$	Complex constant symbols ( <i>with or without subscripts</i> ).
$[\mathbb{C}]$	The set of complex interval numbers.
$[\mathbb{C}]_{\tilde{0}}$	The set of nonzero complex interval numbers.
$[\mathbb{C}]_p$	The set of point complex interval numbers.
$\mathbf{i} = [i, i]$	The interval imaginary unit.
$\mathbf{X}, \mathbf{Y}, \mathbf{Z}$	Complex interval variable symbols ( <i>with or without subscripts</i> ).
$\mathbf{A}, \mathbf{B}, \mathbf{C}$	Complex interval constant symbols ( <i>with or without subscripts</i> ).
$\widehat{\mathbf{X}}$	The conjugate of a complex interval number $\mathbf{X}$ .
$\mathbb{M} \subset \mathbb{R}$	The set of machine-representable real numbers.
$\mathbb{M}_n$	The set of machine real numbers with $n$ significant digits.
$[\mathbb{M}]$	The set of machine interval numbers.
$\nabla$	Downward rounding operator.
$\triangle$	Upward rounding operator.
$\diamond$	Outward rounding operator.



Symbol	Meaning
${}^c [\mathbb{R}]$	The set of constraint interval numbers.
${}^c [\mathbb{R}]_{\tilde{0}}$	The set of nonzero constraint interval numbers.
${}^c [\mathbb{R}]_p$	The set of point constraint interval numbers.
$\mathcal{M}$	The set of modal intervals.
$\mathcal{M}_{\exists}$	The set of existential (proper) modal intervals.
$\mathcal{M}_{\forall}$	The set of universal (improper) modal intervals.
${}^m X, {}^m Y, {}^m Z$	Modal interval variable symbols ( <i>with or without subscripts</i> ).
${}^m A, {}^m B, {}^m C$	Modal interval constant symbols ( <i>with or without subscripts</i> ).

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## Prologue: *A Weapon Against Uncertainty*

*You cannot be certain about uncertainty.*

—Frank Knight (1885-1972)

Scientists are, all the time, in a struggle with *error* and *uncertainty*. Uncertainty is the quantitative estimation of errors present in measured data; all measurements contain some uncertainty generated through many types of error. Error (“*mistaken result*”, or “*mistaken outcome*”) is common in all scientific practice, and is always a serious threat to the search for a trustworthy scientific knowledge and to certain epistemic foundations of science.

This brief chapter is intended to provide a bit of motivation and perspective about the field of interval arithmetic, its history, and how it is a potential weapon against uncertainty in science and technology (For further background, the reader can consult, e.g., [Allchin2000], [Allchin2007], [Kline1982], and [Pedrycz2008]).

### 1.1 What Interval Arithmetic is and Why it is Considered

In real-life computations, uncertainty naturally arises because we process values which come from *measurements* and from *expert estimations*. From both the *epistemological* and *physical* viewpoints, neither measurements nor expert estimations can be exact for the following reasons:

- The actual value of the measured quantity is a real number; so, in general, we need *infinitely* many bits to describe the exact value, while after every measurement, we gain only a *finite* number of bits of information (e.g., a finite number of binary digits in the binary expansion of the number).

- There is always some difficult-to-delete noise which is mixed with the measurement results.
- Expert estimates cannot be absolutely exact, because an expert generates only a finite amount of information.
- Experts are usually even less accurate than are measuring instruments.

So, there are usually three sources of error while performing numerical computations with real numbers; *rounding*, *truncation* and *input errors*. A mistaken outcome is always a concern in scientific research because errors inevitably accumulate during calculations. Interval arithmetic keeps track of all error types simultaneously, because an interval arithmetic operation produces a *closed interval* within which the true *real-valued* result is guaranteed to lie.

Interval arithmetic (also known as “*interval mathematics*”, “*interval analysis*”, and “*interval computation*”) is an arithmetic defined on sets of real intervals, rather than sets of real numbers. It specifies a precise method for performing arithmetic operations on closed intervals (*interval numbers*). The concept is simple: in the interval number system, each interval number represents some fixed real number between the *lower* and *upper endpoints* of the closed interval. So, an interval arithmetic operation produces two values for each result. The two values correspond to the lower and upper endpoints of the resulting interval such that the true result certainly lies within this interval, and the *accuracy* of the result is indicated by the *width* of the resulting interval.

As a result of error, we all the time have to face situations in which scientific measurements give *uncertain values*. Let  $x$  be a real number whose value is uncertain, and assume a measurement gives adequate information about an acceptable range,  $\underline{x} \leq x \leq \bar{x}$ , in which the true value of  $x$  is estimated to lie. The closed interval (interval number),

$$[\underline{x}, \bar{x}] = \{x \in \mathbb{R} | \underline{x} \leq x \leq \bar{x}\},$$

is called the “*interval of certainty*” (or the “*interval of confidence*”) about the value of  $x$ . That is, we are certain that the true value of  $x$  lies within the interval  $[\underline{x}, \bar{x}]$ .

If it is the case that  $\vec{\bar{x}}_n = \langle x_1, x_2, \dots, x_n \rangle$  is a *multidimensional quantity* (*real-valued vector*) such that for each  $x_i$  there is an interval of certainty  $X_i = [\underline{x}_i, \bar{x}_i]$ , then the quantity  $\vec{\bar{x}}_n$  has an “*n-dimensional parallelotope of certainty*”,  $\mathcal{X}_n$ , which is the Cartesian product of the intervals  $X_1, X_2, \dots, X_n$ .

Figure 1.1 illustrates the case  $n = 2$ ; with  $\vec{x}_2 = \langle x_1, x_2 \rangle$ ,  $x_1 \in [\underline{x}_1, \bar{x}_1]$ ,  $x_2 \in [\underline{x}_2, \bar{x}_2]$ , and  $\mathcal{X}_2$  is a *rectangle of certainty*.

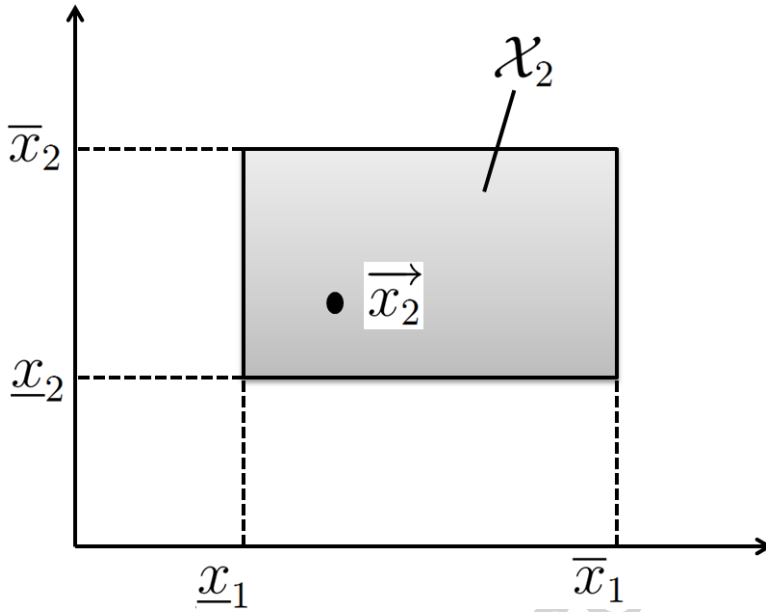


Figure 1.1: A 2-dimensional parallelootope of certainty.

To illustrate this, we give two numerical examples.

**Example 1.1** *The Archimedes's constant,  $\pi$ , is an irrational number, which means that its value cannot be expressed exactly as a fraction. Consequently, it has no certain decimal representation because its decimal representation never ends or repeats. Since  $314 \times 10^{-2} \leq \pi \leq 315 \times 10^{-2}$ , the number  $\pi$  can be represented as the interval number*

$$[\underline{\pi}, \bar{\pi}] = [314 \times 10^{-2}, 315 \times 10^{-2}].$$

*That is, we are certain that the true value of  $\pi$  lies within the interval  $[\underline{\pi}, \bar{\pi}]$  whose width indicates the maximum possible error,*

$$\begin{aligned} \text{Error} &= \text{width}([\underline{\pi}, \bar{\pi}]) \\ &= \bar{\pi} - \underline{\pi} \\ &= 315 \times 10^{-2} - 314 \times 10^{-2} \\ &= 10^{-2}. \end{aligned}$$

**Example 1.2** *Suppose two independent scientific measurements give different uncertain results for the same quantity  $q$ . One measurement gives  $q = 1.4 \pm 0.2$ . The other gives  $q = 1.5 \pm 0.2$ . These uncertain values of  $q$  can be represented*

as the interval numbers  $X = [1.2, 1.6]$  and  $Y = [1.3, 1.7]$ , respectively. Since  $q$  lies in both, it certainly lies in their intersection  $X \cap Y = [1.3, 1.6]$ . So, if  $X \cap Y \neq \emptyset$ , we can get a better (tighter) “interval of certainty”. If not, we can be certain that at least one of the two measurements is wrong.

After we present a rigorous construction of the theoretical foundations of interval arithmetic in chapters [2](#), [3](#), and [4](#), more advanced examples and applications shall be discussed, in detail, in chapter [5](#).

## 1.2 A History Against Uncertainty

The term “*interval arithmetic*” is reasonably recent: it dates from the 1950s, when the works of Paul S. Dwyer, R. E. Moore, R. E. Boche, S. Shayer, and others made the term popular (see, [Dwyer1951](#), [Moore1959](#), [Boche1963](#), and [Shayer1965](#)). But the notion of calculating with intervals is not completely new in mathematics: in the course of history, it has been invented and re-invented several times, under different names, and never been abandoned or forgotten. The concept has been known since the third century BC, when Archimedes used guaranteed lower and upper bounds to compute his constant,  $\pi$  (see [Archimedes2002](#)).

Early in the twentieth century, the idea seemed to be rediscovered. A form of interval arithmetic perhaps first appeared in 1924 by J. C. Burkill in his paper “*Functions of Intervals*” ([Burkill1924](#)), and in 1931 by R. C. Young in his paper “*The Algebra of Many-Valued Quantities*” ([Young1931](#)) that gives rules for calculating with intervals and other sets of real numbers; then later in 1951 by Paul S. Dwyer in his book “*Linear computations*” ([Dwyer1951](#)) that discusses, in a heuristic manner, certain methods for performing basic arithmetic operations on real intervals, and in 1958 by T. Sunaga in his book “*Theory of an Interval Algebra and its Application to Numerical Analysis*” ([Sunaga1958](#)).

However, it was not until 1959 that new formulations of interval arithmetic were presented. Modern developments of the interval theory began in 1959 with R. E. Moore’s technical report “*Automatic Error Analysis in Digital Computation*” ([Moore1959](#)) in which he developed a number system and an arithmetic dealing with closed real intervals. He called the numbers “*range numbers*” and the arithmetic “*range arithmetic*” to be the first synonyms of “*interval numbers*” and “*interval arithmetic*”. Then later in 1962, Moore developed a theory for exact or infinite precision interval arithmetic in his very influential dissertation titled “*Interval Arithmetic and Automatic Error Analysis in Digital Computing*” ([Moore1962](#)) in which he used a modified digital (*rounded*)

interval arithmetic as the basis for automatic analysis of total error in a digital computation. Since then, thousands of research papers and numerous books have appeared on the subject.

Interval arithmetic is now a broad field in which rigorous mathematics is associated with scientific computing. The connection between computing and mathematics provided by intervals makes it possible to solve problems that cannot be efficiently solved using floating-point arithmetic and traditional numerical approximation methods. Today, the interval methods are becoming rapidly popular as a prospective weapon against uncertainties in measurements and errors of numerical computations. Nowadays, interval arithmetic is widely used and has numerous applications in scientific and engineering disciplines that deal intensely with *uncertain data* (or *range data*). Practical application areas include electrical engineering, structure engineering, control theory, remote sensing, quality control, experimental and computational physics, dynamical and chaotic systems, celestial mechanics, signal processing, computer graphics, robotics, and computer-assisted proofs (In chapter 5, we shall discuss, in some detail, a number of the computational applications of interval arithmetic).

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# The Classical Theory of Interval Arithmetic

*Algebra is the metaphysics of arithmetic.*

–John Ray (1627-1705)

A very simple and natural idea is that of enclosing real numbers in real closed intervals. Based on this idea, this chapter is devoted to rigorously defining and constructing the number system of *classical interval arithmetic* and developing the most fundamental tools for classical interval analysis (For a brief overview of some alternate theories of interval arithmetic, see chapter [4](#)).

The chapter opens with the key concepts of the classical interval theory and then introduces the algebraic and point operations for interval numbers. In section [2.3](#), we carefully construct the algebraic system of classical interval arithmetic and deduce its fundamental properties (For other constructions of the classical interval theory, the reader may consult, e.g., [\[Jaulin2001\]](#), [\[Moore1962\]](#), [\[Moore2009\]](#), and [\[Shayer1965\]](#)).

## 2.1 Algebraic Operations for Interval Numbers

The basic algebraic operations for real numbers can be extended to interval numbers. In this section, we shall formulate the basic relations and algebraic operations on interval numbers.

We first define what an interval number is.

**Definition 2.1** *Let  $\underline{x}, \bar{x} \in \mathbb{R}$  such that  $\underline{x} \leq \bar{x}$ . An interval number  $[\underline{x}, \bar{x}]$  is a closed and bounded nonempty real interval, that is*

$$[\underline{x}, \bar{x}] = \{x \in \mathbb{R} \mid \underline{x} \leq x \leq \bar{x}\},$$

where  $\underline{x} = \min([\underline{x}, \bar{x}])$  and  $\bar{x} = \max([\underline{x}, \bar{x}])$  are called, respectively, the lower and upper bounds (endpoints) of  $[\underline{x}, \bar{x}]$ .

**Definition 2.2** The set  $[\mathbb{R}]$  of interval numbers is a subset of the powerset of  $\mathbb{R}$  such that

$$[\mathbb{R}] = \{X \in \wp(\mathbb{R}) \mid (\exists \underline{x} \in \mathbb{R}) (\exists \bar{x} \in \mathbb{R}) (X = [\underline{x}, \bar{x}])\}.$$

Since, corresponding to each pair of real constants  $\underline{x}, \bar{x}$  ( $\underline{x} \leq \bar{x}$ ) there exists a closed interval  $[\underline{x}, \bar{x}]$ , the set of interval numbers is *infinite*.

In order to be able easily to speak of different types of interval numbers, it is convenient to introduce some notational conventions.

**Notation 2.1** The set of all interval numbers that do not contain the real number 0 is denoted by  $[\mathbb{R}]_{\tilde{0}}$ , that is

$$[\mathbb{R}]_{\tilde{0}} = \{X \in [\mathbb{R}] \mid 0 \notin X\}.$$

**Notation 2.2** The set of all symmetric interval numbers is denoted by  $[\mathbb{R}]_s$ , that is

$$[\mathbb{R}]_s = \{X \in [\mathbb{R}] \mid (\exists x \in \mathbb{R})(0 \leq x \wedge X = [-x, x])\}.$$

**Notation 2.3** The set of all singleton (point) interval numbers is denoted by  $[\mathbb{R}]_p$ , that is

$$[\mathbb{R}]_p = \{X \in [\mathbb{R}] \mid (\exists x \in \mathbb{R})(X = [x, x])\}.$$

The set  $[\mathbb{R}]_p$  is an infinite subset of  $[\mathbb{R}]$  and is *isomorphically equivalent* to the set  $\mathbb{R}$  of real numbers (see theorem 2.5). That is, every element  $[x, x] \in [\mathbb{R}]_p$  is an *isomorphic copy* of an element  $x \in \mathbb{R}$ . By convention, and being less pedantic, we agree to *identify* a point interval number  $[x, x]$  with its *real* isomorphic copy  $x$ . In this sense, throughout the text, we may write  $x = [x, x]$ .

Hereafter, the upper-case Roman letters  $X, Y$ , and  $Z$  (with or without subscripts), or equivalently  $[\underline{x}, \bar{x}]$ ,  $[\underline{y}, \bar{y}]$ , and  $[\underline{z}, \bar{z}]$ , shall be employed as variable symbols to denote elements of  $[\mathbb{R}]$ .

The equality relation on  $[\mathbb{R}]$  is characterized in the following definition.

**Definition 2.3** (Equality on  $[\mathbb{R}]$ ). The equality relation for interval numbers is defined as

$$[\underline{x}, \bar{x}] = [\underline{y}, \bar{y}] \Leftrightarrow \underline{x} = \underline{y} \wedge \bar{x} = \bar{y}.$$

Thus equality in  $[\mathbb{R}]$  is defined in terms of equality in  $\mathbb{R}$ . This definition is a special case of the axiom of extensionality<sup>1</sup> of axiomatic set theory, from the fact that every interval number is an ordered set.

Only a partial order (can only compare certain elements) can be defined on  $[\mathbb{R}]$  as follows.

**Definition 2.4** (Ordering on  $[\mathbb{R}]$ ). *A strict partial order on  $[\mathbb{R}]$  with respect to the inequality,  $<$ , is defined as*

$$[\underline{x}, \bar{x}] < [\underline{y}, \bar{y}] \Leftrightarrow \bar{x} < \underline{y}.$$

Thus the ordering on  $[\mathbb{R}]$  is defined in terms of the ordering on  $\mathbb{R}$ . The relation  $<$  is *irreflexive*, *asymmetric*, and *transitive* on  $[\mathbb{R}]$ .

We observe that not every two elements of  $[\mathbb{R}]$  can be compared by the relation  $<$  (if  $X \cap Y \neq \emptyset$ ): this is why the ordering by  $<$  is *strictly partial*<sup>2</sup> on  $[\mathbb{R}]$ . If  $S$  is a subset of  $[\mathbb{R}]$  such that for every two elements  $X$  and  $Y$  of  $S$ , it is the case that either  $X = Y$  or  $X \cap Y = \emptyset$ ; then we have a *total order*<sup>3</sup> on  $S$  with respect to the relation  $\leq$ , that is,  $X \leq Y$  or  $Y \leq X$ .

<sup>1</sup> The axiom of extensionality asserts that two sets are equal if, and only if they have precisely the same elements (see, e.g., [Causey1994], [Devlin1993], and [Kleene1952]), that is

$$(\forall X)(\forall Y)(X = Y \Leftrightarrow (\forall z)(z \in X \Leftrightarrow z \in Y)).$$

<sup>2</sup> A relation  $\prec$  is a *strict partial ordering* on a set  $S$  iff  $\prec$  is

- *irreflexive*:  $(\forall x \in S)(\neg(x \prec x))$ ,
- *asymmetric*:  $(\forall x \in S)(\forall y \in S)(x \prec y \Rightarrow \neg(y \prec x))$ , and
- *transitive*:  $(\forall x \in S)(\forall y \in S)(\forall z \in S)(x \prec y \wedge y \prec z \Rightarrow x \prec z)$ .

If a relation  $\prec$  is a strict partial ordering on a set  $S$ , then  $S$  has at least one pair which is *non-comparable* (see, e.g., [Causey1994], [Devlin1993], and [Gleason1992]), in symbols

$$(\exists x \in S)(\exists y \in S)(\neg(x \prec y) \wedge \neg(y \prec x)).$$

<sup>3</sup> A relation  $\preceq$  is a *total ordering* on a set  $S$  iff  $\preceq$  is

- *reflexive*:  $(\forall x \in S)(x \preceq x)$ ,
- *antisymmetric*:  $(\forall x \in S)(\forall y \in S)(x \preceq y \wedge y \preceq x \Rightarrow x = y)$ ,
- *transitive*:  $(\forall x \in S)(\forall y \in S)(\forall z \in S)(x \preceq y \wedge y \preceq z \Rightarrow x \preceq z)$ , and
- *connected (total)*:  $(\forall x \in S)(\forall y \in S)(x \neq y \Rightarrow x \preceq y \vee y \preceq x)$ .

**Example 2.1** For three given interval numbers  $A = [1, 2]$ ,  $B = [1, 2]$ , and  $C = [4, 7]$ , we have  $A = B < C$ .

We now proceed to define the basic algebraic operations for interval numbers: two binary operations, namely *addition* (“+”) and *multiplication* (“×”), and two unary operations, namely *negation* (“−”) and *reciprocal* (“<sup>−1</sup>”).

According to the fact that interval numbers are *sets*, the binary and unary algebraic operations for interval numbers can be characterized, respectively, in the following two *set-theoretic* definitions.

**Definition 2.5** (Binary Interval Operations). For any two interval numbers  $X$  and  $Y$ , the binary algebraic operations are defined by

$$X \circ Y = \{z \in \mathbb{R} \mid (\exists x \in X) (\exists y \in Y) (z = x \circ y)\},$$

where  $\circ \in \{+, \times\}$ .

**Definition 2.6** (Unary Interval Operations). For any interval number  $X$ , the unary algebraic operations are defined by

$$\diamond X = \{z \in \mathbb{R} \mid (\exists x \in X) (z = \diamond x)\},$$

where  $\diamond \in \{-, ^{-1}\}$  and  $0 \notin X$  if  $\diamond$  is “<sup>−1</sup>”.

By means of the above set-theoretic definitions and from the fact that interval numbers are *ordered sets* of real numbers, the following four theorems are immediate.

**Theorem 2.1** (Interval Addition). For any two interval numbers  $[\underline{x}, \bar{x}]$  and  $[\underline{y}, \bar{y}]$ , interval addition is formulated in terms of the intervals’ endpoints as

$$[\underline{x}, \bar{x}] + [\underline{y}, \bar{y}] = [\underline{x} + \underline{y}, \bar{x} + \bar{y}].$$

---

If a relation  $\preceq$  is a total ordering on a set  $S$ , then  $S$  has *no non-comparable* pairs (see, e.g, [Causey1994], [Devlin1993], and [Gleason1992]), in symbols

$$(\forall x \in S) (\forall y \in S) (x \preceq y \vee y \preceq x).$$

**Theorem 2.2** (Interval Multiplication). *For any two interval numbers  $[\underline{x}, \bar{x}]$  and  $[\underline{y}, \bar{y}]$ , interval multiplication is formulated in terms of the intervals' endpoints as*

$$[\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}] = [\min\{\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\}].$$

**Theorem 2.3** (Interval Negation). *For any interval number  $[\underline{x}, \bar{x}]$ , interval negation is formulated in terms of the interval endpoints as*

$$-[\underline{x}, \bar{x}] = [-\bar{x}, -\underline{x}].$$

**Theorem 2.4** (Interval Reciprocal). *For any interval number  $[\underline{x}, \bar{x}] \in [\mathbb{R}]_{\tilde{0}}$  (that is,  $0 \notin [\underline{x}, \bar{x}]$ ), interval reciprocal is formulated in terms of the interval endpoints as*

$$[\underline{x}, \bar{x}]^{-1} = [\bar{x}^{-1}, \underline{x}^{-1}].$$

In accordance with the above theorems, we can now define *subtraction*, *division*, and *exponentiation* for interval numbers.

**Definition 2.7** (Interval Subtraction). *For any two interval numbers  $X$  and  $Y$ , interval subtraction is defined by*

$$X - Y = X + (-Y).$$

**Definition 2.8** (Interval Division). *For any  $X \in [\mathbb{R}]$  and any  $Y \in [\mathbb{R}]_{\tilde{0}}$ , interval division is defined by*

$$X \div Y = X \times (Y^{-1}).$$

**Definition 2.9** (Interval Exponentiation). *For any interval number  $X$  and any integer  $n$ , the integer exponents of  $X$  are recursively defined by:*

- (i)  $X^0 = [1, 1]$ ,
- (ii)  $0 < n \Rightarrow X^n = X^{n-1} \times X$ ,
- (iii)  $0 \notin X \wedge 0 \leq n \Rightarrow X^{-n} = (X^{-1})^n$ .

Since the interval algebraic operations are defined in terms of the real algebraic operations, and as long as division by zero is disallowed; it follows that the result of an interval algebraic operation is always an interval number.

**Example 2.2** For three given interval numbers  $[1, 2]$ ,  $[3, 4]$ , and  $[-2, 2]$ , we have

- (i)  $[1, 2] + [3, 4] = [4, 6]$ ,
- (ii)  $[1, 2] \times [3, 4] = [3, 8]$ ,
- (iii)  $[-2, 2]^2 = [-2, 2] \times [-2, 2] = [-4, 4]$ .

The result  $[-4, 4]$  of  $[-2, 2]^2$ , in the above example, is not natural in the sense that a square is always nonnegative. Generally, for any *non-point* interval number  $[\underline{x}, \bar{x}]$ , with  $0 \in [\underline{x}, \bar{x}]$ , the square of  $[\underline{x}, \bar{x}]$  is given by

$$\begin{aligned} [\underline{x}, \bar{x}]^2 &= [\underline{x}, \bar{x}] \times [\underline{x}, \bar{x}] \\ &= \left[ \min\{\underline{x}^2, \underline{x}\bar{x}, \bar{x}^2\}, \max\{\underline{x}^2, \underline{x}\bar{x}, \bar{x}^2\} \right] \\ &= \left[ \underline{x}\bar{x}, \max\{\underline{x}^2, \bar{x}^2\} \right], \end{aligned}$$

which is consistent with interval multiplication (theorem 2.2), but it is *not consistent* with the fact that a square is always nonnegative, for the case when  $\underline{x}\bar{x} < 0$ . However, if we changed it to be

$$\begin{aligned} [\underline{x}, \bar{x}]^2 &= \{z \in \mathbb{R} \mid (\exists x \in [\underline{x}, \bar{x}]) (z = x^2)\} \\ &= \left[ 0, \max\{\underline{x}^2, \bar{x}^2\} \right], \end{aligned}$$

then it would be inconsistent with interval multiplication. This is not a problem if interval arithmetic is regarded as a *numerical approximation method*, for real-valued problems, such that the result of an interval operation contains all possible solutions. But if interval arithmetic is regarded as a *definitional extension*<sup>4</sup> of the theory of real numbers (this is the case in almost all interval literature), in the logical sense; then the theory of interval arithmetic is *not*

<sup>4</sup> A theory  $T_E$  is called a *definitional extension* of a theory  $T$  iff  $T_E$  is obtained from  $T$  by adding new relation symbols and function symbols defined in terms of symbols of  $T$  (see, e.g., [Kleene1952], [Rasiowa1963] and [Tarski1965]).

**Defining relation symbols.** Let  $\varphi(x_1, \dots, x_n)$  be a formula of  $T$  such that  $x_1, \dots, x_n$  occur free in  $\varphi$ . A new theory  $T_E$  can be obtained from  $T$  by adding a new  $n$ -ary relation symbol  $R$ , the logical axioms featuring  $R$ , and the new *definitional axiom* of  $R$

$$(\forall x_1) \dots (\forall x_n) (R(x_1, \dots, x_n) \Leftrightarrow \varphi(x_1, \dots, x_n)).$$

An example of such a definition of relation symbols is the definition of a *closed interval* in terms of the order relation  $\leq$  of the theory of real numbers.

**Defining function symbols.** Let  $\varphi(y, x_1, \dots, x_n)$  be a formula of  $T$  such that  $y, x_1, \dots, x_n$

*consistent*. In this book, we shall regard interval arithmetic as a numerical approximation tool of guaranteed efficiency against computation errors, in the sense we discussed in chapter [1](#).

## 2.2 Point Operations for Interval Numbers

A *point* (or *scalar*) interval operation is an operation whose operands are interval numbers, and whose result is a *point* interval (or, equivalently, a real number). This is made precise in the following definition.

**Definition 2.10** (Point Interval Operations). *Let  $[\mathbb{R}]^{\langle n \rangle}$  be the  $n$ -th Cartesian power of  $[\mathbb{R}]$ . An  $n$ -ary point interval operation,  $\omega_n$ , is a function that maps elements of  $[\mathbb{R}]^{\langle n \rangle}$  to the set  $[\mathbb{R}]_p$  of point interval numbers, that is*

$$\omega_n : [\mathbb{R}]^{\langle n \rangle} \mapsto [\mathbb{R}]_p.$$

Several point interval operations can be defined. Next we define some *unary* and *binary* point interval operations.

**Definition 2.11** (Interval Infimum). *The infimum of an interval number  $[\underline{x}, \bar{x}]$  is defined to be*

$$\inf([\underline{x}, \bar{x}]) = \min([\underline{x}, \bar{x}]) = \underline{x}.$$

**Definition 2.12** (Interval Supremum). *The supremum of an interval number  $[\underline{x}, \bar{x}]$  is defined to be*

$$\sup([\underline{x}, \bar{x}]) = \max([\underline{x}, \bar{x}]) = \bar{x}.$$

---

occur free in  $\varphi$ . Assume that the sentence

$$(\forall x_1) \dots (\forall x_n) (\exists! y) (\varphi(y, x_1, \dots, x_n)),$$

is provable in  $T$ ; that is, for all  $x_1, \dots, x_n$ , there exists a unique  $y$  such that  $\varphi(y, x_1, \dots, x_n)$ . A new theory  $T_E$  can be obtained from  $T$  by adding a new  $n$ -ary relation symbol  $F$ , the logical axioms featuring  $F$ , and the new definitional axiom of  $F$

$$(\forall x_1) \dots (\forall x_n) (\varphi(F(x_1, \dots, x_n), x_1, \dots, x_n)).$$

An example of such a definition of function symbols is the definition of the *interval algebraic operations* in terms of the algebraic operations of the theory of real numbers.

If  $T_E$  is a *consistent* definitional extension of  $T$ , then for any formula  $\psi$  of  $T_E$  we can form a formula  $\varphi$  of  $T$ , called a translation of  $\psi$  into  $T$ , such that  $\psi \Leftrightarrow \varphi$  is provable in  $T_E$ . Such a formula is *not unique*, but any two formulas  $\varphi_1$  and  $\varphi_2$  can be proved to be equivalent in  $T$ . That is, for  $T_E$  to be a consistent definitional extension of  $T$ ; if  $\psi_1 \Leftrightarrow \varphi_1$ ,  $\psi_2 \Leftrightarrow \varphi_2$ , and  $\varphi_1 \Leftrightarrow \varphi_2$ , then  $\psi_1 \Leftrightarrow \psi_2$ .

**Definition 2.13** (Interval Width). *The width of an interval number  $[\underline{x}, \bar{x}]$  is defined to be*

$$w([\underline{x}, \bar{x}]) = \bar{x} - \underline{x}.$$

Thus, the width of a point interval number is *zero*, that is

$$(\forall x \in \mathbb{R}) (w([x, x]) = x - x = 0).$$

**Definition 2.14** (Interval Radius). *The radius of an interval number  $[\underline{x}, \bar{x}]$  is defined to be*

$$r([\underline{x}, \bar{x}]) = w([\underline{x}, \bar{x}]) / 2 = (\bar{x} - \underline{x}) / 2.$$

**Definition 2.15** (Interval Midpoint). *The midpoint (or mean) of an interval number  $[\underline{x}, \bar{x}]$  is defined to be*

$$m([\underline{x}, \bar{x}]) = (\underline{x} + \bar{x}) / 2.$$

Hence, the midpoint of a point interval number is its *real isomorphic copy*, that is

$$(\forall x \in \mathbb{R}) (m([x, x]) = (x + x) / 2 = x).$$

We observe that any interval number  $X$  can be expressed, in terms of its width and its midpoint, as the sum of its midpoint and a corresponding symmetric interval, that is

$$X = m(X) + [-w(X) / 2, w(X) / 2],$$

where, by convention,  $m(X) = [m(X), m(X)]$ .

**Definition 2.16** (Interval Absolute Value). *The absolute value of an interval number  $[\underline{x}, \bar{x}]$  is defined, in terms of the absolute values of its real endpoints, to be*

$$|[\underline{x}, \bar{x}]| = \max(\{|\underline{x}|, |\bar{x}|\}).$$

Thus, the absolute value of a point interval number is the usual absolute value of its *real isomorphic copy*, that is

$$(\forall x \in \mathbb{R}) (|[x, x]| = \max(\{|x|, |x|\}) = |x|).$$

All the above point interval operations are unary operations. An important definition of a binary point interval operation is that of the *interval metric* (or the distance between two interval numbers).



**Definition 2.17** (Interval Distance). *The distance (or metric) between two interval numbers  $[\underline{x}, \bar{x}]$  and  $[\underline{y}, \bar{y}]$  is defined to be*

$$d([\underline{x}, \bar{x}], [\underline{y}, \bar{y}]) = \max(\{|\underline{x} - \underline{y}|, |\bar{x} - \bar{y}|\}).$$

The importance of this definition is that starting from the distance function for interval numbers, we can verify that it induces a *Hausdorff metric space*<sup>5</sup> for interval numbers which is a generalization of the usual metric space of real numbers. Thus, the notions of a sequence, convergence, continuity, and a limit can be defined for interval numbers in the standard way. These notions give rise to what we may call a “*measure theory for interval numbers*”. An interval measure theory is beyond the scope of this book.

**Example 2.3** *For three given interval numbers  $[1, 2]$ ,  $[3, 4]$ , and  $[-5, 3]$ , we have*

- (i)  $w([1, 2]) = w([3, 4]) = 1,$
- (ii)  $m([1, 2]) = 3/2, m([3, 4]) = 7/2,$
- (iii)  $|[-5, 3]| = \max(\{|-5|, |3|\}) = 5,$
- (iv)  $d([1, 2], [3, 4]) = \max(\{|1 - 3|, |2 - 4|\}) = 2.$

## 2.3 Algebraic Properties of Interval Arithmetic

By means of the notions prescribed in sections 2.1 and 2.2, we shall now inquire into some fundamental theorems concerning interval arithmetic. By virtue of our definition of an interval number, the properties of real numbers are naturally assumed priori.

A first important theorem we shall now prove is the *isomorphism theorem* for interval arithmetic.

---

<sup>5</sup> A *metric space* is an ordered pair  $(S, d)$ , where  $S$  is a set and  $d$  is a metric on  $S$ , that is

$$d : S^{(2)} \mapsto \mathbb{R},$$

such that for any  $x, y, z \in S$ , the following holds:

- $x = y \Leftrightarrow d(x, y) = 0,$
- $d(x, y) = d(y, x),$
- $d(x, z) \leq d(x, y) + d(y, z).$

A *Hausdorff space* is a metric space in which distinct points have disjoint neighbourhoods (see [Bryant1985]).

**Theorem 2.5** *The ordered structure  $\langle [\mathbb{R}]_p; +_{[\mathbb{R}]}, \times_{[\mathbb{R}]}; <_{[\mathbb{R}]} \rangle$  is isomorphically equivalent to the field  $\langle \mathbb{R}; +_{\mathbb{R}}, \times_{\mathbb{R}}; <_{\mathbb{R}} \rangle$  of real numbers.*

**Proof.** Let  $\iota : \mathbb{R} \hookrightarrow [\mathbb{R}]_p$  be a mapping from  $\mathbb{R}$  to  $[\mathbb{R}]_p$  such that

$$(\forall x \in \mathbb{R}) \left( \exists \iota(x) \in [\mathbb{R}]_p \right) (\iota(x) = [x, x]).$$

The following conditions for  $\iota$  are satisfied:

- $\iota$  is a bijection from  $\mathbb{R}$  onto  $[\mathbb{R}]_p$  since, by definitions 2.2 and 2.3, the range of  $\iota$  is  $[\mathbb{R}]_p$ , and

$$(\forall x \in \mathbb{R}) (\forall y \in \mathbb{R}) (\iota(x) = \iota(y) \Rightarrow x = y).$$

- $\iota$  is function-preserving for “+” since, by theorem 2.1, we have

$$\begin{aligned} \iota(x +_{\mathbb{R}} y) &= [x +_{\mathbb{R}} y, x +_{\mathbb{R}} y] \\ &= [x, x] +_{[\mathbb{R}]} [y, y] \\ &= \iota(x) +_{[\mathbb{R}]} \iota(y). \end{aligned}$$

- $\iota$  is function-preserving for “ $\times$ ” since, by theorem 2.2, we have

$$\begin{aligned} \iota(x \times_{\mathbb{R}} y) &= [x \times_{\mathbb{R}} y, x \times_{\mathbb{R}} y] \\ &= [x, x] \times_{[\mathbb{R}]} [y, y] \\ &= \iota(x) \times_{[\mathbb{R}]} \iota(y). \end{aligned}$$

- $\iota$  is relation-preserving for “ $<$ ” since, by definition 2.4, we have

$$\begin{aligned} x <_{\mathbb{R}} y &\Leftrightarrow [x, x] <_{[\mathbb{R}]} [y, y] \\ &\Leftrightarrow \iota(x) <_{[\mathbb{R}]} \iota(y). \end{aligned}$$

The mapping  $\iota$  thus is an isomorphism from  $\mathbb{R}$  onto  $[\mathbb{R}]_p$  and  $\langle \mathbb{R}; +_{\mathbb{R}}, \times_{\mathbb{R}}; <_{\mathbb{R}} \rangle$  is isomorphically equivalent to  $\langle [\mathbb{R}]_p; +_{[\mathbb{R}]}, \times_{[\mathbb{R}]}; <_{[\mathbb{R}]} \rangle$ . ■

That is, up to isomorphism, the sets  $\mathbb{R}$  and  $[\mathbb{R}]_p$  are equivalent, and each element of  $[\mathbb{R}]_p$  is an *isomorphic copy* of an element of  $\mathbb{R}$ . In other words, *everything that is true for real numbers is certainly true for point interval numbers.*

The properties of the interval operations are similar to, but not the same as, those of the real operations. The algebraic properties of the interval operations are prescribed by the following theorems.

**Theorem 2.6** (Absorbing Element). *The interval number  $[0, 0]$  is an absorbing element for interval multiplication, that is*

$$(\forall X \in [\mathbb{R}]) ([0, 0] \times X = X \times [0, 0] = [0, 0]).$$

**Proof.** For any interval number  $X$ , according to definition 2.5 and assuming the properties of real multiplication, we have

$$\begin{aligned} [0, 0] \times X &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in [0, 0]) (r = y \times x)\} \\ &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in [0, 0]) (r = x \times y)\} \\ &= \{r \in \mathbb{R} \mid (\exists x \in X) (r = x \times 0)\} \\ &= X \times [0, 0] = [0, 0], \end{aligned}$$

and therefore, the point interval number  $[0, 0]$  *absorbs* any interval number  $X$  by interval multiplication. ■

**Theorem 2.7** (Identity for Addition). *The interval number  $[0, 0]$  is both a left and right identity for interval addition, that is*

$$(\forall X \in [\mathbb{R}]) ([0, 0] + X = X + [0, 0] = X).$$

**Proof.** For any interval number  $X$ , according to definition 2.5 and assuming the properties of real addition, we have

$$\begin{aligned} [0, 0] + X &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in [0, 0]) (r = y + x)\} \\ &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in [0, 0]) (r = x + y)\} \\ &= \{r \in \mathbb{R} \mid (\exists x \in X) (r = x + 0)\} \\ &= X + [0, 0] = X, \end{aligned}$$

and therefore,  $[0, 0]$  is both a left and right identity for interval addition. ■

**Theorem 2.8** (Identity for Multiplication). *The interval number  $[1, 1]$  is both a left and right identity for interval multiplication, that is*

$$(\forall X \in [\mathbb{R}]) ([1, 1] \times X = X \times [1, 1] = X).$$

**Proof.** For any interval number  $X$ , according to definition 2.5 and assuming the properties of real multiplication, we have

$$\begin{aligned}
 [1, 1] \times X &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in [1, 1]) (r = y \times x)\} \\
 &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in [1, 1]) (r = x \times y)\} \\
 &= \{r \in \mathbb{R} \mid (\exists x \in X) (r = x \times 1)\} \\
 &= X \times [1, 1] = X,
 \end{aligned}$$

and therefore, it is shown that  $[1, 1]$  is both a left and right identity for interval multiplication. ■

**Theorem 2.9** (Commutativity). *Both interval addition and multiplication are commutative, that is*

- (i)  $(\forall X, Y \in [\mathbb{R}]) (X + Y = Y + X)$ ,
- (ii)  $(\forall X, Y \in [\mathbb{R}]) (X \times Y = Y \times X)$ .

**Proof.** (i) For any two interval numbers  $X$  and  $Y$ , according to definition 2.5 and assuming the properties of real addition, we have

$$\begin{aligned}
 X + Y &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in Y) (r = x + y)\} \\
 &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in Y) (r = y + x)\} \\
 &= Y + X.
 \end{aligned}$$

(ii) In a manner analogous to (i), assuming the properties of real multiplication, we have

$$\begin{aligned}
 X \times Y &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in Y) (r = x \times y)\} \\
 &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in Y) (r = y \times x)\} \\
 &= Y \times X.
 \end{aligned}$$

Therefore, both addition and multiplication are commutative in  $[\mathbb{R}]$ . ■

**Theorem 2.10** (Associativity). *Both interval addition and multiplication are associative, that is*

- (i)  $(\forall X, Y, Z \in [\mathbb{R}]) (X + (Y + Z) = (X + Y) + Z)$ ,
- (ii)  $(\forall X, Y, Z \in [\mathbb{R}]) (X \times (Y \times Z) = (X \times Y) \times Z)$ .

**Proof.** (i) For any three interval numbers  $X$ ,  $Y$ , and  $Z$ , according to definition 2.5 and assuming the properties of real addition, we have

$$\begin{aligned}
 X + (Y + Z) &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists s \in (Y + Z)) (r = x + s)\} \\
 &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in Y) (\exists z \in Z) (r = x + (y + z))\} \\
 &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in Y) (\exists z \in Z) (r = (x + y) + z)\} \\
 &= \{r \in \mathbb{R} \mid (\exists t \in (X + Y)) (\exists z \in Z) (r = t + z)\} \\
 &= (X + Y) + Z.
 \end{aligned}$$

(ii) In a manner analogous to (i), assuming the properties of real multiplication, we have

$$\begin{aligned}
 X \times (Y \times Z) &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists s \in (Y \times Z)) (r = x \times s)\} \\
 &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in Y) (\exists z \in Z) (r = x \times (y \times z))\} \\
 &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in Y) (\exists z \in Z) (r = (x \times y) \times z)\} \\
 &= \{r \in \mathbb{R} \mid (\exists t \in (X \times Y)) (\exists z \in Z) (r = t \times z)\} \\
 &= (X \times Y) \times Z.
 \end{aligned}$$

Therefore, both addition and multiplication are associative in  $[\mathbb{R}]$ . ■

**Theorem 2.11** (Inverses). *Additive and multiplicative inverses do not always exist for interval numbers except for point interval numbers, that is*

- (i)  $(\exists X \in [\mathbb{R}]) (X + (-X) \neq [0, 0])$ ,
- (ii)  $(\exists X \in [\mathbb{R}]) (0 \notin X \wedge X \times (X^{-1}) \neq [1, 1])$ ,
- (iii)  $(\forall X \in [\mathbb{R}]_p) (X + (-X) = [0, 0])$ ,
- (iv)  $(\forall X \in [\mathbb{R}]_p) (0 \notin X \Rightarrow X \times (X^{-1}) = [1, 1])$ .

**Proof.** For (i) and (ii), let  $X = [\underline{x}, \bar{x}]$  be a nonpoint interval, that is  $\underline{x} \neq \bar{x}$ . According to theorems 2.1 and 2.2, we have

$$\begin{aligned}
 [\underline{x}, \bar{x}] + (-[\underline{x}, \bar{x}]) &= [\underline{x}, \bar{x}] + [-\bar{x}, -\underline{x}] \\
 &= [\underline{x} - \bar{x}, \bar{x} - \underline{x}] \\
 &\neq [0, 0],
 \end{aligned}$$

and, if  $0 \notin [\underline{x}, \bar{x}]$ , we have

$$\begin{aligned} [\underline{x}, \bar{x}] \times ([\underline{x}, \bar{x}]^{-1}) &= [\underline{x}, \bar{x}] \times [1/\bar{x}, 1/\underline{x}] \\ &= [\underline{x}/\bar{x}, \bar{x}/\underline{x}] \\ &\neq [1, 1]. \end{aligned}$$

(iii) and (iv) are immediate from the fact that  $[\mathbb{R}]_p$  and  $\mathbb{R}$  are *isomorphically equivalent* (theorem 2.5). ■

The result formulated in the following theorem is an important property of interval arithmetic.

**Theorem 2.12** (Subdistributivity). *The distributive law does not always hold in interval arithmetic, that is*

$$(\exists X, Y, Z \in [\mathbb{R}]) (Z \times (X + Y) \neq Z \times X + Z \times Y),$$

except when

- (i)  $Z$  is a point interval number, or
- (ii)  $X = Y = [0, 0]$ , or
- (iii)  $(\forall x \in X) (\forall y \in Y) (xy \geq 0)$ .

In general, interval arithmetic only has the subdistributive law:

$$(\forall X, Y, Z \in [\mathbb{R}]) (Z \times (X + Y) \subseteq Z \times X + Z \times Y).$$

**Proof.** Let  $[\underline{z}, \bar{z}]$ ,  $[x, x]$ , and  $[-x, -x]$  be in  $[\mathbb{R}]$ . First adding and then multiplying, we have

$$[\underline{z}, \bar{z}] \times ([x, x] + [-x, -x]) = [\underline{z}, \bar{z}] \times [0, 0] = [0, 0].$$

But if we first multiply and then add, we get

$$\begin{aligned} &([\underline{z}, \bar{z}] \times [x, x]) + ([\underline{z}, \bar{z}] \times [-x, -x]) \\ &= [\min(\underline{z}x, \bar{z}x), \max(\underline{z}x, \bar{z}x)] + [\min(-\underline{z}x, -\bar{z}x), \max(-\underline{z}x, -\bar{z}x)] \\ &\neq [0, 0], \end{aligned}$$

unless  $\underline{z} = \bar{z}$ , or  $x = -x = 0$ , or both.

Thus, there are some interval numbers for which the distributive law does not hold.

(i) Let  $Z = [a, a]$  be a point interval number, for some real constant  $a$ . According to definition 2.5, we have

$$\begin{aligned}
 Z \times (X + Y) &= \{r \in \mathbb{R} \mid (\exists z \in [a, a]) (\exists s \in (X + Y)) (r = z \times s)\} \\
 &= \{r \in \mathbb{R} \mid (\exists s \in (X + Y)) (r = a \times s)\} \\
 &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in Y) (r = a \times (x + y))\} \\
 &= \{r \in \mathbb{R} \mid (\exists x \in X) (\exists y \in Y) (r = a \times x + a \times y)\} \\
 &= \{r \in \mathbb{R} \mid (\exists t \in (Z \times X)) (\exists u \in (Z \times Y)) (r = t + u)\} \\
 &= Z \times X + Z \times Y.
 \end{aligned}$$

(ii) By theorems 2.6 and 2.7, we immediately have

$$\begin{aligned}
 Z \times ([0, 0] + [0, 0]) &= Z \times [0, 0] \\
 &= [0, 0] \\
 &= Z \times [0, 0] + Z \times [0, 0].
 \end{aligned}$$

(iii) Let  $X = [\underline{x}, \bar{x}]$  and  $Y = [\underline{y}, \bar{y}]$ . Without loss of generality, we consider only the case when  $\underline{x} \geq 0$  and  $\underline{y} \geq 0$ . We have three cases for  $Z = [\underline{z}, \bar{z}]$ .

*Case 1.* If  $\underline{z} \geq 0$ , then we have

$$\begin{aligned}
 Z \times (X + Y) &= [\underline{z}(\underline{x} + \underline{y}), \bar{z}(\bar{x} + \bar{y})] \\
 &= [\underline{z}\underline{x}, \bar{z}\bar{x}] + [\underline{z}\underline{y}, \bar{z}\bar{y}] \\
 &= Z \times X + Z \times Y.
 \end{aligned}$$

*Case 2.* If  $\bar{z} \leq 0$ , we obtain the same result by considering  $-Z$ .

*Case 3.* If  $\underline{z}\bar{z} < 0$ , then we have

$$\begin{aligned}
 Z \times (X + Y) &= [\underline{z}(\bar{x} + \bar{y}), \bar{z}(\underline{x} + \underline{y})] \\
 &= [\underline{z}\bar{x}, \bar{z}\underline{x}] + [\underline{z}\bar{y}, \bar{z}\underline{y}] \\
 &= Z \times X + Z \times Y,
 \end{aligned}$$

and the last case is thus proved.

Now let  $r \in (Z \times (X + Y))$ . From the distributive property of real numbers, we have

$$r = z(x + y) = zx + zy,$$

for some  $x \in X$ ,  $y \in Y$ , and  $z \in Z$ .

Thus  $r \in (Z \times X + Z \times Y)$ , and therefore

$$(\forall X, Y, Z \in [\mathbb{R}]) (Z \times (X + Y) \subseteq Z \times X + Z \times Y),$$

which completes the proof. ■

It should be borne in mind that interval arithmetic does not even have the distributive law

$$Z \times ([x, x] + [y, y]) = Z \times [x, x] + Z \times [y, y],$$

for point interval numbers  $[x, x]$  and  $[y, y]$ .

An important and more general result we shall next prove is the *inclusion monotonicity* of interval arithmetic.

**Theorem 2.13** (Inclusion Monotonicity). *Let  $X_1, X_2, Y_1,$  and  $Y_2$  be interval numbers such that  $X_1 \subseteq Y_1$  and  $X_2 \subseteq Y_2$ . Then for any interval operation  $\circ \in \{+, \times\}$ , we have*

$$X_1 \circ X_2 \subseteq Y_1 \circ Y_2.$$

**Proof.** By hypothesis, we have  $X_1 \subseteq Y_1$  and  $X_2 \subseteq Y_2$ . Then, according to definition 2.5, we have

$$\begin{aligned} X_1 \circ X_2 &= \{r \in \mathbb{R} \mid (\exists x_1 \in X_1) (\exists x_2 \in X_2) (r = x_1 \circ x_2)\} \\ &\subseteq \{s \in \mathbb{R} \mid (\exists y_1 \in Y_1) (\exists y_2 \in Y_2) (s = y_1 \circ y_2)\} \\ &= Y_1 \circ Y_2, \end{aligned}$$

and the theorem follows. ■

An immediate consequence is the following important special case.

**Corollary 2.1** *Let  $X$  and  $Y$  be interval numbers with  $x \in X$  and  $y \in Y$ . Then for any interval operation  $\circ \in \{+, \times\}$ , we have*

$$x \circ y \in X \circ Y.$$

Finally, we prove the following result about the algebraic system of interval arithmetic.

**Theorem 2.14** *Let  $\langle [\mathbb{R}], \circ \rangle$  be the algebraic system of interval numbers. Then for any interval operation  $\circ \in \{+, \times\}$ , the system  $\langle [\mathbb{R}], \circ \rangle$  is an abelian monoid.*



**Proof.** For  $\circ \in \{+, \times\}$ , the following four criteria are satisfied.

- **Closure.** By definition, the set  $[\mathbb{R}]$  is closed under both addition and multiplication.
- **Associativity.** Both addition and multiplication are associative, by theorem 2.10.
- **Commutativity.** Both addition and multiplication are commutative, by theorem 2.9.
- **Identity Elements.** The interval numbers  $[0, 0]$  and  $[1, 1]$  are identity elements respectively for addition and multiplication, by theorems 2.7 and 2.8.

Therefore, the set  $[\mathbb{R}]$  of interval numbers forms an abelian monoid under both addition and multiplication. ■

Two important properties, peculiar to the classical theory of interval arithmetic, figure in the theorems of this section: additive and multiplicative inverses do not always exist for interval numbers, and there is no distributivity between addition and multiplication except for certain special cases. Then, we have to sacrifice some useful properties of ordinary arithmetic, if we want to use the interval weapon against uncertainty.

Preprint Excerpt

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