A Logical Formalization of the Notion of Interval Dependency
Towards Reliable Intervalizations of Quantifiable Uncertainties

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Abstract

Progress in scientific knowledge discloses an increasingly paramount use of quantifiable properties in the description of states and processes of the real-world physical systems. Through our encounters with the physical world, it reveals itself to us as systems of uncertain quantifiable properties. One approach proved to be most fundamental and reliable in coping with quantifiable uncertainties is interval mathematics. A main drawback of interval mathematics, though, is the persisting problem known as the “interval dependency problem”. This, naturally, confronts us with the question: Formally, what is interval dependency? Is it a ‘meta-concept’ or an ‘object-ingredient’ of interval and fuzzy computations? In other words, what is the fundamental defining properties that characterize the notion of interval dependency as a formal mathematical object? Since the early works on interval mathematics by John Charles Burkill and Rosalind Cecily Young in the dawning of the twentieth century, this question has never been touched upon and remained a question still today unanswered. Although the notion of interval dependency is widely used in the interval and fuzzy literature, it is only illustrated by example, without explicit formalization, and no attempt has been made to put on a systematic basis its meaning, that is, to indicate formally the criteria by which it is to be characterized. Here, we attempt to answer this long-standing question. This article, therefore, is devoted to presenting a complete systematic formalization of the notion of interval dependency, by means of the notions of Skolemization and quantification dependence. A novelty of this formalization is the expression of interval dependency as a logical predicate (or relation) and thereby gaining the advantage of deducing its fundamental properties in a merely logical manner. Moreover, on the strength of the generality of the logical apparatus we adopt, the results of this article are not only about classical intervals, but they are meant to apply also to any possible theory of interval arithmetic. That being so, our concern is to shed new light on some fundamental problems of interval mathematics and to take one small step towards paving the way for developing alternate dependency-aware interval theories and computational methods.

Keywords: Interval mathematics; Interval dependency; Functional dependence; Skolemization; Guaranteed bounds; Interval enclosures; Interval functions; Quantifiable uncertainty; Scientific knowledge; Reliability; Fuzzy mathematics; InCLosure

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In logic there is no such thing as a hidden connection. You can’t get behind the rules because there isn’t any behind.

–Ludwig Wittgenstein (1889–1951)

1 Introduction

Acquiring scientific knowledge discloses an ever-increasing use of quantifiable features of real-world physical systems. Many features of the physical world represent themselves to us as numerical values. Three farads capacitance, six becquerels radioactivity, and nine metres depth are common examples of quantifiable properties. The numerical values of quantifiable properties are established by means of measurement. Through our encounters with the physical world, measurements cannot, nevertheless, provide perfect exactitude and measurable magnitudes are usually uncertain. In
the effort to deal with quantifiable uncertainties, various theoretical approaches have been developed. Among these, one can mention: probabilization, fuzzification, and intervalization. One approach that proved to be subtle, reliable, and most fundamental in all of mathematics of uncertainty is interval mathematics.

The term “interval arithmetic” is reasonably recent: it dates from the 1950s, when the works of Paul S. Dwyer, Ramon Edgar Moore, Raymond E. Boche, Sidney Shayer, and others made the term popular (see, [30], [55], [7], and [64]). But the idea of calculating with intervals is not completely new in mathematics: in the course of history, it has been invented and re-invented several times, under different names, and never been abandoned or forgotten. The concept has been known since the third century BC, when Archimedes used guaranteed lower and upper bounds to compute his constant, π (see [38]).

Early in the twentieth century, the idea seemed to be rediscovered. A form of interval arithmetic perhaps first appeared in 1924 by John Charles Burkill in his paper “Functions of Intervals” ([8]), and in 1931 by Rosalind Cecily Young in her paper “The Algebra of Many-Valued Quantities” ([73]) that gives rules for calculating with intervals and other sets of real numbers; then later in 1951 by Paul S. Dwyer in his book “Linear computations” ([30]) that discusses, in a heuristic manner, certain methods for performing basic arithmetic operations on real intervals, and in 1958 by Teruo Sunaga in his book “Theory of an Interval Algebra and its Application to Numerical Analysis” ([66]).

However, it was not until 1959 that new formulations of interval arithmetic were presented. Modern developments of the interval theory began in 1959 with Moore’s technical report “Automatic Error Analysis in Digital Computation” ([55]) in which he developed a number system and an arithmetic dealing with closed real intervals. He called the numbers “range numbers” and the arithmetic “range arithmetic” to be the first synonyms of “interval numbers” and “interval arithmetic”. Then later in 1962, Moore developed a theory for exact or infinite precision interval arithmetic in his very influential dissertation entitled “Interval Arithmetic and Automatic Error Analysis in Digital Computing” ([56]) in which he used a modified digital (rounded) interval arithmetic as the basis for automatic analysis of total error in a digital computation. In his comprehensive book “Interval Analysis” ([57]), Moore was the first to define interval analysis in its modern sense and recognize its practical power as a viable computational tool for bounding errors and intervalizing uncertainty. Since then, thousands of research papers and numerous books have appeared on the subject.

By integrating the complementary powers of rigorous mathematics and scientific computing, interval arithmetic is able to offer highly reliable accounts of uncertainty. Not surprisingly, therefore, that the interval theory has been fruitfully applied in diverse areas that deal intensely with uncertain quantitative data (see, e.g., [17, 19, 29, 28, 34, 45, 47, and 59]). In view of its computational power against error, machine realizations of interval arithmetic are of great importance. As a matter of course, there are various software implementations of interval arithmetic. As instances, we may mention INTLAB, Sollya, InClosuralure and others (see, e.g., [63, 11, 16, and 53]). Fortunately, computers are getting faster and most existing parallel processors provide a tremendous computing power. So, with little extra hardware, it is very possible to make interval computations as fast as floating point computations (For further reading about machine arithmeticizations and hardware circuitries for interval arithmetic, see, e.g., [15, 19, 44, 43, 48, 60, 40], and [41]).

Despite all of the above mentioned advantages of interval mathematics, it has its disadvantages as well. A main drawback of interval mathematics is the persisting problem known as the “interval dependency problem”. This, naturally, confronts us with the crucial question: Formally, what is interval dependency? Is it a meta-concept or an object-ingredient of interval and fuzzy mathematics? In other words, what is the fundamental defining properties that characterize the notion of interval dependency as a formal mathematical object? Since the early works on interval mathematics by Burkill [8] and Young [73] in the dawning of the twentieth century, this question has never been touched upon and remained a question still today unanswered. Although the notion of interval dependency is widely used in the interval and fuzzy literature, it is only illustrated by example, without explicit formalization, and no attempt has been made to put on a systematic basis its meaning, that is, to indicate formally the criteria by which it is to be characterized. Here, we attempt to answer this long-standing question.

To reiterate, what exactly is the sense of saying that two intervals are dependent (independent)? and how does the dependency of two intervals X and X differ from that of X2 and X? Our aim here is to propose a precise logical characterization of the notion of interval dependency and to begin the development of a rigorous mathematical theory which formally characterizes and explains the differences between all cases of interval dependency. That being so, the problem to be dealt with in this text is that of the possibility and the scope of a symbolic formalization of interval dependency. The indispensable role of symbolic formalizations can neither be ignored nor denigrated. Kleene, in [42], best described this by saying:

“Anyone who doubts the advantages of symbols (in their proper place) is invited to solve the equation x2 + 3x − 2 = 0 by completing the square (as taught in high school), but doing all the work in words. We start him off by stating the equation in words: The square of the unknown, increased by three times the unknown, and diminished by two, is equal to zero.”
With an eye toward making the adopted formal approach clearly comprehensible, before moving on to the main business of the text, we set the stage by fixing our formalized apparatus early in section 2. In section 3, we give a characterization of the classical interval theory as a many-sorted algebra over the real field and deduce some of the fundamental properties of interval numbers. Section 4 is devoted to providing a bit of perspective on the need for interval mathematics to deal with quantifiable uncertainties, along with giving a description of the interval dependency problem and the challenges thereof. The objective of section 5 is to justify a symbolic notation suitable for our purpose and to delve into some important fundamentals concerning the logical formulation of the notion of functional dependence and some related notions. In section 6, we attempt to answer the long-standing question concerning the defining properties that characterize the notion of interval dependency as a formal mathematical object. We present a complete systematication of the notion of interval dependency, by means of the logical notions of Skolemization and quantification dependence. Finally, in section 7, we discuss briefly how to compute useful guaranteed enclosures of real-valued functions under functional dependence. The numerical examples of section 7 are computed using version 2.0 of InCLosure. The InCLosure commands¹ to compute the results of the examples are described and an InCLosure input file and its corresponding output containing, respectively, the code and results of the examples are also available as a supplementary material to this text (see Supplementary Materials).

What is then the fundamental importance of a logical formalization of interval dependency? The authors believe that such a formalization, hopefully, might have a worthwhile impact on both fundamental research and real world applications. Effort in pursuit of this aim can have many fruitful consequences. A novelty of this formalization is the expression of interval dependency as a logical predicate (or relation) and thereby gaining the advantage of deducing its fundamental properties in a merely logical manner. The mathematical theory developed in the present article formally characterizes and explains the differences between all cases of interval dependency, and thus sheds new light on many fundamental problems of interval mathematics. Moreover, taking the passage from the informal treatments to the formal technicalities of mathematical logic, a breakthrough behind our systematization of interval dependency is that it paves the way and provides the systematic apparatus for developing alternate dependency-aware interval theories and computational methods with mathematical constructions that better account for dependencies between the quantifiable uncertainties of the real world. Noteworthy also is that on the strength of the generality of the logical apparatus we adopt, the results of this article are not only about Moore’s classical intervals, but they are meant to apply also to any possible theory of interval arithmetic.

2 A Bit of Formalism: Setting the Stage

In order to be able to give a complete systematic formalization of the notion of interval dependency and related notions, we do need readily available a formal apparatus. So, the adaptation of a particular formal approach, other than that of natural language, is of the utmost importance and has been forced on us by the pursuit of formulating the underlying ideas in a strictly accurate manner that all the results of this work can be generated from clear and distinct elementary concepts. Therefore, before moving on to the main business of this article, we begin in this section by specifying some notational conventions and formalizing some purely logical and algebraic ingredients we shall need throughout this text (For further details about the notions prescribed here, the reader may consult, e.g., [5], [22], [23], [26], [19], [28], and [54]).

Most of our notions are characterized in terms of ordinals and ordinal tuples. So, we first define what an ordinal is. An ordinal is the well-ordered set of all ordinals preceding it. That is, for each ordinal \( n \), there exists an ordinal \( S(n) \) called the successor of \( n \) such that

\[
(\forall n)(\forall k)((k \in S(n) \iff (\forall m)(m \in k \iff m \in n \lor m = n))).
\]

In other words, we have \( S(n) = n \cup \{n\} \). Accordingly, the first infinite (transfinite) ordinal is the set \( \omega = \{0, 1, 2, \ldots\} \). All ordinals preceding \( \omega \) (all elements of \( \omega \)) are finite ordinals. The idea of transfinite counting (counting beyond the finite) is due to Cantor (See [9]).

With the aid of ordinals, the notions of countably finite, countably infinite and uncountably infinite sets can be characterized as follows. A set \( S \) is countably finite if there is a bijective mapping from \( S \) onto some finite ordinal \( n \in \omega \). A set \( S \) is countably infinite (or denumerable) if there is a bijective mapping from \( S \) onto the infinite ordinal \( \omega \). For example the set \( \{a_0, a_1, a_2\} \) is countably finite because it can be bijectively mapped onto the finite ordinal \( 3 = \{0, 1, 2\} \), while the set \( \{a_0, a_1, a_2, \ldots\} \) is denumerable because it can be bijectively mapped onto the infinite ordinal \( \omega = \{0, 1, 2, \ldots\} \). An uncountably infinite set is an infinite set which is not countably infinite. For example the set \( \mathbb{R} \) of real numbers is uncountably infinite.

The notion of an \( n \)-tuple is characterized in the following definition.

¹InCLosure (interval enCLosure) is a language and environment for reliable scientific computing, which is coded entirely in Lisp. Latest version of InCLosure is available for free download via https://doi.org/10.5281/zenodo.2702404.
Definition 2.1 (Ordinal Tuple). For an ordinal \( n = S(k) \), an \( n \)-tuple (ordinal tuple) is any mapping \( \tau \) whose domain is \( n \). A finite \( n \)-tuple is an \( n \)-tuple for some finite ordinal \( n \). That is

\[
\tau_{S(k)} = \langle \tau(0) , \tau(1) , \ldots , \tau(k) \rangle \ = \langle (0, \tau(0)) , (1, \tau(1)) , \ldots , (k, \tau(k)) \rangle.
\]

If \( n = 0 = \emptyset \), then, for any set \( S \), there is exactly one mapping (the empty mapping) \( \tau_{\emptyset} = \emptyset \) from \( \emptyset \) into \( S \).

An important definition we shall need is that of the Cartesian power of a set.

Definition 2.2 (Cartesian Power). Let \( \emptyset \) denote the empty set. For a set \( S \) and an ordinal \( n \), the \( n \)-th Cartesian power of \( S \) is the set \( S^n \) of all mappings from \( n \) into \( S \), that is

\[
S^n = \left\{ \{ \emptyset \} \right\} \ \text{the set of all } n \text{-tuples of elements of } S \quad n = 0,
\]

\[
\emptyset^n = \left\{ \emptyset \right\} \ \text{and} \ \emptyset^{\emptyset} = \left\{ \emptyset \right\} \quad n = 1 \ \forall 1 \in n.
\]

In accordance with the preceding definitions, a set-theoretical relation is a particular type of sets. Let \( S^2 \) be the binary Cartesian power of a set \( S \). A binary relation on \( S \) is a subset of \( S^2 \). That is, a set \( \mathcal{R} \) is a binary relation on a set \( S \) iff \( \forall r \in \mathcal{R} \) \( \left\{ (x, y) \in S, y \right\} \).

We will continue to follow the formal tradition of Suppes [67] and Tarski [69] in defining, within a set-theoretical framework, the notion of a finitary relation and some related concepts. Let \( \mathcal{U}^n \) be the \( n \)-th Cartesian power of a set \( \mathcal{U} \).

A set \( \mathcal{R} \subseteq \mathcal{U}^n \) is an \( n \)-ary relation on \( \mathcal{U} \) iff \( \mathcal{R} \) is a binary relation from \( \mathcal{U}^{n-1} \) to \( \mathcal{U} \). That is, for \( v = (x_1 , \ldots , x_{n-1}) \in \mathcal{U}^{n-1} \) and \( y \in \mathcal{U} \), an \( n \)-ary relation \( \mathcal{R} \) is defined to be \( \mathcal{R} \subseteq \mathcal{U}^n \) iff \( \forall y \in \mathcal{U} \ \exists (v) \mathcal{R} \). In this sense, an \( n \)-ary relation is a binary relation (or simply a relation); then its domain, range, field, and converse are defined to be, respectively dom(\( \mathcal{R} \)) = \( \{ v \in \mathcal{U}^{n-1} | \exists y \in \mathcal{U} \mathcal{R} \} \), ran(\( \mathcal{R} \)) = \( \{ y \in \mathcal{U} | \exists v \in \mathcal{U}^{n-1} \mathcal{R} \} \), fld(\( \mathcal{R} \)) = dom(\( \mathcal{R} \)) ∪ ran(\( \mathcal{R} \)), and \( \mathcal{R} \) = \( \{ (y, v) \in \mathcal{U}^n | \mathcal{R} \} \).

Two important points, for the purpose of having an image and preimage of a set, with respect to an \( n \)-ary relation. These are defined as follows [(16) and (28)].

Definition 2.3 (Image and Preimage of a Relation). Let \( \mathcal{R} \) be an \( n \)-ary relation on a set \( \mathcal{U} \), and for \( (v, y) \in \mathcal{R} \), let \( v = (x_1 , \ldots , x_{n-1}) \), with each \( x_k \) is restricted to vary on a set \( X_k \subseteq \mathcal{U} \), that is, \( v \) is restricted to vary on a set \( \mathcal{V} \subseteq \mathcal{U}^{n-1} \).

Then, the image of \( \mathcal{V} \) (or the image of the sets \( X_k \)) with respect to \( \mathcal{R} \), denoted \( I_{\mathcal{R}} \), is defined to be

\[
Y = I_{\mathcal{R}}(\mathcal{V}) = I_{\mathcal{R}}(X_1 \times \cdots \times X_{n-1}) = \{ y \in \mathcal{U} | \exists v \in \mathcal{V} (v \mathcal{R} y) \}
\]

where the set \( \mathcal{V} \), called the preimage of \( Y \), is defined to be the image of \( Y \) with respect to the converse relation \( \mathcal{R} \), that is

\[
\mathcal{V} = I_{\mathcal{R}}^*(Y) = \{ v \in \mathcal{U}^{n-1} | \exists y \in \mathcal{Y} (y \mathcal{R} v) \}.
\]

In accordance with this definition and the fact that \( y \mathcal{R} v \Leftrightarrow v \mathcal{R} y \), we obviously have \( Y = I_{\mathcal{R}}(\mathcal{V}) \Leftrightarrow \mathcal{V} = I_{\mathcal{R}}^*(Y) \).

Now for a mathematically satisfactory characterization of a finitary function. Within this set-theoretical framework, a completely general definition of the notion of an \( n \)-ary function can be formulated. A set \( f \) is an \( n \)-ary function on a set \( \mathcal{U} \) iff \( f \) is an \( (n + 1) \)-ary relation on \( \mathcal{U} \) and \( \forall v \in \mathcal{U}^{n} (\forall z \in \mathcal{U} (v f y \land v f z \Rightarrow y = z) \). Thus, an \( n \)-ary function is a many-one \( (n + 1) \)-ary relation; that is, a relation, with respect to which, any element in its domain is related exactly to one element in its range. Getting down from relations to the particular case of functions, we have at hand the standard notation: \( y = f(v) \) in place of \( v f y \). From the fact that an \( n \)-ary function is a special kind of relation, then all the preceding definitions and results, concerning the domain, range, field, and converse of a relation, apply to functions as well.

With some criteria satisfied, a function is called invertible. A function \( f \) has an inverse, denoted \( f^{-1} \), iff its converse relation \( f \) is a function, in which case \( f^{-1} = f \). In other words, \( f \) is invertible if, and only if, it is an injection from

\[\text{Amer in [3]} \text{ used the } n \text{-th Cartesian power of } \emptyset \text{ to define empty structures, and axiomatized their first-order theory.}\]
its domain to its range, and obviously the inverse \( f^{-1} \) is unique, from the fact that the converse relation is always definable and unique.

A formalized theory is characterized by two things; an object language in which the theory is formalized (the symbolism of the theory), and a set of axioms. Let \( \mathcal{L} \) be an object formal language. A formalized theory in \( \mathcal{L} \) (or an \( \mathcal{L} \)-theory) is a set of \( \mathcal{L} \)-sentences which is closed under its associated deductive apparatus. Let \( \Lambda_{\mathcal{L}} \) denote a finite set of \( \mathcal{L} \)-sentences, and let \( \varphi \) denote an \( \mathcal{L} \)-sentence. The formalized \( \mathcal{L} \)-theory \( \mathcal{T} \) of the set \( \Lambda_{\mathcal{L}} \) under logical consequence, that is
\[
\mathcal{T} = \{ \varphi \in \mathcal{L} | \varphi \text{ is a consequence of } \Lambda_{\mathcal{L}} \}.
\]

The set \( \Lambda_{\mathcal{L}} \) is called the set of axioms (or postulates) of \( \mathcal{T} \).

A model (or an interpretation) of a theory \( \mathcal{T} \) is some particular (algebraic or relational) structure that satisfies every formula of \( \mathcal{T} \). Finally, we close this section by characterizing some algebraic structures of particular importance to our purpose (see [19] and [22]).

**Definition 2.4 (Ringoid).** A ringoid (or a ring-like structure) is a structure \( \mathcal{R} = (\mathcal{R}; +_\mathcal{R}, \times_\mathcal{R}) \) with \( +_\mathcal{R} \) and \( \times_\mathcal{R} \) are total binary operations on the universe set \( \mathcal{R} \). The operations \( +_\mathcal{R} \) and \( \times_\mathcal{R} \) are called respectively the addition and multiplication operations of the ringoid \( \mathcal{R} \).

**Definition 2.5 (S-Ringoid).** An S-ringoid (or a subdistributive ringoid) is a ringoid that satisfies at least one of the following subdistributive criteria.

(i) \( \forall x, y, z \in \mathcal{R} \) \( (x \times_\mathcal{R} (y +_\mathcal{R} z) \subseteq x \times_\mathcal{R} y +_\mathcal{R} x \times_\mathcal{R} z) \).

(ii) \( \forall x, y, z \in \mathcal{R} \) \( ((y +_\mathcal{R} z) \times_\mathcal{R} x \subseteq y \times_\mathcal{R} x +_\mathcal{R} z \times_\mathcal{R} x) \).

Criteria (i) and (ii) in the preceding definition are called respectively left and right subdistributivity (or S-distributivity).

**Definition 2.6 (Semiring).** Let \( \mathcal{R} = (\mathcal{R}; +_\mathcal{R}, \times_\mathcal{R}) \) be a ringoid. \( \mathcal{R} \) is said to be a semiring iff

(i) \( (\mathcal{R}; +_\mathcal{R}) \) is a commutative monoid with identity element \( 0_\mathcal{R} \).

(ii) \( (\mathcal{R}; \times_\mathcal{R}) \) is a monoid with identity element \( 1_\mathcal{R} \).

(iii) Multiplication, \( \times_\mathcal{R} \), left and right distributes over addition, \( +_\mathcal{R} \).

(iv) \( 0_\mathcal{R} \) is an absorbing element for \( \times_\mathcal{R} \).

A commutative semiring is one whose multiplication is commutative.

**Definition 2.7 (S-Semiring).** An S-semiring (or a subdistributive semiring) is an S-ringoid that satisfies criteria (i), (ii), and (iv) in definition 2.6. A commutative S-semiring is one whose multiplication is commutative.

At this point, let us note that the notion of S-semiring is a generalization of the notion of a near-semiring; a near-semiring is a ringoid that satisfies the criteria of a semiring except that it is either left or right distributive (For detailed discussions of near-semirings and related concepts, the interested reader may consult, e.g., [71], [61], and [12]).

3 The Theory of Interval Algebra over the Real Field

With the formalized apparatus of section 2 at our disposal, the main business of this section is to give a formalized characterization of the theory \( \mathcal{T}_{\mathcal{H}} \) of classical interval arithmetic over the real field. There are many theories of interval arithmetic (see, e.g., [37], [48], [36], [51], [50], [40], [41], [15], [25], [21], and [28]). We are here interested in characterizing classical interval arithmetic as introduced in, e.g., [55], [64], [59], [17], [19], and [27]. Notwithstanding, on the strength of the generality of the logical apparatus we adopt in this work, the formalization of interval dependency presented in the succeeding sections and the results thereof are not only about Moore’s classical intervals, but they are meant to apply also to any possible theory of interval arithmetic.

Our adopted strategy to obtain a concrete system of classical interval numbers is to start with the field of real numbers and to “intervalize” it, by defining new interval relations and operations. In other words, the theory \( \mathcal{T}_{\mathcal{H}} \) of real intervals will be constructed as a definitional extension of the theory of real numbers. Hereafter and throughout this work, the machinery used, and assumed priori, is the standard (classical) predicate calculus and axiomatic set theory. Moreover, in all the proofs, elementary facts about operations and relations on the real numbers are usually used without explicit reference.

A theory \( \mathcal{T}_{\mathcal{H}} \) of a real interval algebra (a classical interval algebra or an interval algebra over the real field) is characterized in the following definition (see [17] and [19]).
Definition 3.1 (Theory of Real Interval Algebra). Take $\sigma = \{+, \times, -; ^{-1}; 0, 1\}$ as a set of non-logical constants and let $\mathbb{R} = (\mathbb{R}; \sigma^R)$ be the totally \(\leq\)-ordered field of real numbers. The theory $\mathbb{Th}_2$ of an interval algebra over the field $\mathbb{R}$ is the theory of a two-sorted structure $\mathcal{I}_R = (\mathcal{I}_R; \mathcal{S}; \sigma^3_R)$ prescribed by the following set of axioms.

1. $\forall X \in \mathcal{I}_R \ (X = \{ x \in \mathbb{R} | \exists y \in \mathbb{R} (x \leq y \leq X) \})$.
2. $\forall X, Y \in \mathcal{I}_R \ (X + Y = \{ z \in \mathbb{R} | \exists x \in X, y \in Y (z = x + y) \})$.
3. $\forall X \in \mathcal{I}_R \ (X * Y = \{ z \in \mathbb{R} | \exists x \in X, y \in Y (z = x * y) \})$.
4. $\forall X \in \mathcal{I}_R \ (X = \{ x \in \mathbb{R} | \exists y \in \mathbb{R} (x \leq y \leq X) \})$.
5. $\forall X \in \mathcal{I}_R \ (X = \{ x \in \mathbb{R} | \exists y \in \mathbb{R} (x \leq y \leq X) \})$.
6. $\forall X \in \mathcal{I}_R \ (X = \{ x \in \mathbb{R} | \exists y \in \mathbb{R} (x \leq y \leq X) \})$.

The sentence (1) of definition 3.1 characterizes what an interval number (or a closed interval) is. The sentences (2) and (3) prescribe, respectively, the binary and unary operations for $\mathbb{R}$-intervals. Hereafter, the upper-case Roman letters $X$, $Y$, and $Z$ (with or without subscripts), or equivalently $[a, b]$, $[y, z]$, and $[c, d]$, shall be employed as variable symbols to denote real interval numbers. A (point singleton) interval number $\{ x \}$ shall be denoted by $[x]$. The letters $A$, $B$, and $C$, or equivalently $[a, b]$, $[y, z]$, and $[c, d]$, shall be used to denote constants of $\mathcal{I}_R$. Also, we shall single out the symbols $I_1$ and $0_1$ to denote, respectively, the singleton $\mathbb{R}$-intervals $\{1\}$ and $\{0\}$. For the purpose at hand, it is convenient to define two proper subsets of $\mathcal{I}_R$: the sets of symmetric interval numbers and point interval numbers. Respectively, these are defined and denoted by

\[ \mathcal{I}_S = \{ X \in \mathcal{I}_R | (\exists x \in \mathbb{R} (0 \leq x \leq X)) \}, \]
\[ \mathcal{I}_1 = \{ X \in \mathcal{I}_R | (\exists x \in \mathbb{R} (X = [x, x])) \}. \]

From the fact that real intervals are totally $\leq$-ordered subsets of $\mathbb{R}$, equality of $\mathbb{R}$-intervals follows immediately from the axiom of extensionality \(^3\) of set theory. That is,

\[ [a, b] = [c, d] \iff a = c \land b = d. \]

Since $\mathbb{R}$-intervals are ordered sets of real numbers, it follows that the next theorem is derivable from definition 3.1 (see \([17]\) and \([20]\)).

Theorem 3.1 (Interval Operations). For any two interval numbers $[x, y]$ and $[y, z]$, the binary and unary interval operations are formulated in terms of the intervals’ endpoints as follows.

1. $\forall x, y, z \in \mathbb{R} \ (x + y = \{ x + z \ | \ z \in [x, y] \})$,  
2. $\forall x, y, z \in \mathbb{R} \ (x \times y = \{ x \times z \ | \ z \in [x, y] \})$,  
3. $\forall x, y, z \in \mathbb{R} \ (x^{-1} = \{ x^{-1} \ | \ z \in [x, y] \})$.

where min and max are respectively the $\leq$-minimal and $\leq$-maximal.

Wherever there is no confusion, we shall drop the subscripts $\mathcal{I}$ and $\mathcal{R}$. It is obvious that all the interval operations, except interval reciprocal, are total operations. The additional operations of interval subtraction and division can be defined respectively as $X - Y = X + (-Y)$ and $X : Y = X \times (Y^{-1})$.

Classical interval arithmetic has a number of peculiar algebraic properties: The point intervals $0_1$ and $1_1$ are identity elements for addition and multiplication, respectively; interval addition and multiplication are both commutative and associative; interval addition is cancellative; interval multiplication is cancellative only for zeroless intervals; an interval number is invertible for addition (respectively, multiplication) if and only if it is a point interval (respectively, a nonzero point interval); and interval multiplication left and right subdistributes over interval addition (see definition 2.5 of section 2). To sum up, according to definition 2.7, the structure $\langle \mathcal{I}_R; +_I, \times_I; 0_1, 1_1 \rangle$ of classical interval numbers is a commutative $S$-semiring (\([17]\) and \([19]\)).

Throughout this text, we shall employ the following theorem and its corollary (see, \([16]\), and \([17]\)).

Theorem 3.2 (Inclusion Monotonicity in Classical Intervals). Let $X_1$, $X_2$, $Y_1$, and $Y_2$ be interval numbers such that $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$. Then for any binary operation $\circ \in \{+, \times\}$ and any definable unary operation $\circ \in \{-, ^{-1}\}$, we have

1. $X_1 \circ X_2 \subseteq Y_1 \circ Y_2$.  

\(^3\)The axiom of extensionality asserts that two sets are equal if, and only if they have precisely the same elements, that is, for any two sets $S$ and $T$, $S = T \iff \forall z \in S \exists z \in T$.  

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\( \diamond \subseteq X \subseteq \diamond Y \).

In consequence of this theorem, from the fact that \([x, x] \subseteq X \Leftrightarrow x \in X\), we have the following important special case.

**Corollary 3.1 (Membership Monotonicity for Classical Intervals).** Let \( X \) and \( Y \) be real interval numbers with \( x \in X \) and \( y \in Y \). Then for any binary operation \( \circ \in \{+ , \times \} \) and any definable unary operation \( \circ \in \{ - , ^{-1} \} \), we have

(i) \( x \circ_R y \in X \circ_O Y \),

(ii) \( \circ_R x \in \circ_O X \).

If we endow the classical interval algebra \( \langle I_R; + , \times , 0 , 1 \rangle \) with the compatible partial ordering \( \subseteq \), then we have a partially-ordered commutative S-semiring. In addition to ordering intervals by the set inclusion relation \( \subseteq \), there are many orders presented in the interval literature. Among these is Moore’s partial ordering which is defined by \( [x, y] \preceq_M [v, w] \iff x \leq_R y \wedge v \leq_R w \). In contrast to the case for \( \subseteq \), Moore’s partial ordering \( \preceq_M \) is not compatible with the algebraic operations on \( I_R \) (see [17] and [24]).

Some numerical examples are shown below.

**Example 3.1 (Classical Interval Operations).** For three given interval numbers \([1, 2], [3, 4], \) and \([-2, 2] \), we have

(i) \([1, 2] + [3, 4] = [4, 6],\)

(ii) \([1, 2] \times [3, 4] = [3, 8],\)

(iii) \([1, 2]^{-1} = [1/2, 1],\)

(iv) \([-2, 2]^2 = [-2, 2] \times [-2, 2] = [-4, 4].\)

**Example 3.2 (Moore’s and Inclusion Orders).** For four given interval numbers \( A = [1, 2], B = [1, 2], C = [1, 3], \) and \( D = [4, 7], \) we have \( A = B <_M D \) and \( A \subseteq B \subseteq C \).

4 Intervalization of Physical Uncertainties and the Dependency Problem

When modelling and predicting a real-world or physical phenomena, we often face the problem that, through our way to acquire this knowledge, scientists and engineers always perform one or both of the following two inquiries:

- Measuring identifiable features through experiments, to acquire quantitative knowledge about the real-world phenomena, it reveals its present and future states to us as systems of uncertain quantifiable properties.
- Formulating hypotheses and making inferences about the world through deductive processes of formal reasoning.

The epistemology of scientific knowledge, that is the study of how our knowledge of real-world systems (which can be physical, chemical, biological, economical, social, and so forth) is acquired through observing, experimenting, measuring identifiable features, and formulating hypotheses with the aid of formal reasoning ([17] and [24]). In their way to acquire this knowledge, scientists and engineers always perform one or both of the following two inquiries:

- **Present-State Inquiry.** To quantify an identifiable property \( \rho \) that provides some information about the present state of the real-world system.

In this context, the search for a trustworthy scientific knowledge and to reliable epistemic foundations of science.
• **Future-State Inquiry.** To quantify an identifiable property $\tau$ that provides some information about a future state of the real-world system.

In most practical problems, the properties $\rho$ and $\tau$ are not directly identifiable, that is, they are not quantifiable by direct measurements or expert estimations. However, we usually know a present-state function (or algorithm) $\rho$ that relates the present property $\rho$ to some quantifiable auxiliary properties $x_1, \ldots, x_n$, and a future-state function (or algorithm) $\tau$ that relates the future property $\tau$ to some quantifiable auxiliary properties $y_1, \ldots, y_n$. So, to quantify the properties $\rho$ and $\tau$, we measure or estimate the related quantifiable auxiliary properties, and then apply the known present-state and future-state functions

\[
\rho = p(x_1, \ldots, x_n), \\
\tau = f(y_1, \ldots, y_n).
\]

At this point the crucial question is: Do measurements or expert estimations provide reliable information about the quantifiable properties? The answer is: "Unfortunately, no". In practical situations, uncertainty naturally arises when processing values which come from measurements or from expert estimations. From both the epistemological and physical viewpoints, neither measurements nor expert estimations can be exact for the following reasons ([17] and [46]):

- The actual value of the measured quantity is a real number; so, in general, we need infinitely many bits to describe the exact value, while after every measurement, we gain only a finite number of bits of information (e.g., a finite number of binary digits in the binary expansion of the number).
- Truncation Errors. Truncation errors arise when replacing a continuous or infinite operation by a computable discrete operation.
- Rounding Errors. Rounding errors arise when doing arithmetic on a machine. This error is the difference between the result obtained using exact arithmetic and the result computed using finite precision arithmetic.
This problem of machine subtraction has dangerous consequences in numerical computations. To illustrate, consider the problem of calculating the derivative of a real-valued function \( f \) at a given point. The method of finite differences is a numerical method which can be performed by a computer to approximate the derivative. For a differentiable function \( f \), the first derivative \( f^{(1)} \) can be approximated by

\[
\frac{f(x + dx) - f(x)}{dx},
\]

for a small nonzero value of \( dx \). As \( dx \) approaches zero, the derivative is better approximated. But as \( dx \) gets smaller, the rounding error increases because of the finite precision of machine real arithmetic, and we accordingly get the problematic situation of \( f(x + dx) - f(x) = 0 \). That is, for small enough values of \( dx \), the derivative will be always computed as zero, regardless of the rule of the function \( f \). Using interval enclosures of the function \( f \) instead, we can find a way out of this problem, by virtue the infinite precision of machine interval arithmetic. For further details on interval enclosures of derivatives, see, e.g., \([17]\) and \([29]\).

Another problem of finite precision arises when truncating an infinite operation by a computable finite operation. For example, The exponential function \( e^{x} \) may be written as a Taylor series

\[
e^{x} = 1 + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{n}}{n!}.
\]

In order to compute this infinite series on a machine, we have to truncate it to the partial sum

\[
S_{k} = \sum_{n=0}^{k} \frac{x^{n}}{n!},
\]

for some finite \( k \), and the truncation error then is \( |e^{x} - S_{k}| \). Using interval bounds for this error term, machine interval arithmetic can provide a guaranteed enclosure of the exact value of the exponential function \( e^{x} \).

In addition to the problematic situations described above, the loss of precision of machine real arithmetic can even lead to fatal and costly disasters, for example, the failure of the Patriot anti-missile batteries during the Gulf War which is due to round-off error in floating-point calculations. Another example is the explosion of the Ariane 5 rocket on June 4, 1996 (for further details, see \([43]\)).

The preceding examples shed light on the fact that taking the passage from real arithmetic to interval arithmetic opens the way to the rich technicalities and the infinite precision of interval computations. With the power of intervals at our disposal, let us revisit the uncertainty problem that we discussed in the beginning of this section, that of acquiring knowledge about the real world. When applying traditional numerical methods to estimate the error in the measurable auxiliary properties \( x_{1}, \ldots, x_{m} \) and \( y_{1}, \ldots, y_{n} \), we get approximate (non-guaranteed) bounds to the measurement errors that are, in many cases, not sufficient. Moreover, we sometimes face situations in which the probability distribution for the measurement errors cannot be determined, and consequently probabilization is not valid. Furthermore, in some practical situations, fuzzifying the problem suffers from many limitations, and therefore the efficacy of fuzzification, in such situations, is questionable.

Now, let us intervalize the problems of inquiring about the present and future states of a real-world system. We can measure or estimate the quantifiable auxiliary properties \( x_{1}, \ldots, x_{m} \) and \( y_{1}, \ldots, y_{n} \). With intervals at hand, we have intervals of certainty \( X_{i} \) and \( Y_{k} \) for the auxiliary properties \( x_{i} \) and \( y_{k} \) respectively. Knowing the present-state function \( p \) and the future-state function \( f \) that relate the directly-unquantifiable present property \( p \) and future property \( f \) to their auxiliary properties \( x_{i} \) and \( y_{k} \) respectively, we need to compute the images of \( X_{i} \) and \( Y_{k} \) with respect to the functions \( p \) and \( f \). These images are defined and denoted by

\[
I_{p}(X_{1}, \ldots, X_{m}) = \{ \rho \in \mathbb{R} \mid (x_{1}, \ldots, x_{m}) \in X_{i} \},
\]

\[
I_{f}(Y_{1}, \ldots, Y_{n}) = \{ \tau \in \mathbb{R} \mid (x_{1}, \ldots, x_{m}) \in X_{i} \}.
\]

The functions \( p \) and \( f \) are usually continuous, and therefore the images are in turn real closed intervals. That is, there are intervals of certainty \( [p, \bar{p}] \) and \( [\tau, \bar{\tau}] \) within which the desirable values of the properties \( \rho \) and \( \tau \) are guaranteed to lie respectively.

It is thus natural to think of extending the ordinary arithmetic on real-valued quantities to interval-valued quantities in such a way that we can do arithmetic on the intervals \( X_{i} \) and \( Y_{k} \) to get, respectively, the intervals \( [p, \bar{p}] \) and \( [\tau, \bar{\tau}] \) as results. So, the uncertainty problem of inquiring about real-world systems is now the problem of interval enclosures of images of real-valued functions. Now the question is that to what extent interval enclosures can be useful. Toward answering this, we conclude the present section with a peculiarity of interval arithmetic that seems quite strange at first.
Note that the set-theoretic characterization of the interval operations (definition 3.1) implies that interval arithmetic considers all instances of variables as independent. Accordingly, for two interval variables $X$ and $Y$ assigned the same interval constant $A$, both the interval operations $X \circ A X$ and $X \circ A Y$ are equal and they are the same as the image of the multivariate real function $f_{\text{ind}}(x, y) = x \circ A y$, with $x \in A$ and $y \in A$. In fact, this is one of the strengths of interval mathematics: since images of real functions are inclusion monotonic (see, e.g., [16], [28], and [62]), it follows that the image of the function $f_{\text{ind}}$ is an enclosure of the image of a unary real function $f_{\text{dep}}(x) = x \circ A x$, with $x \in A$, and therefore $X \circ A X = X \circ A Y$ is a guaranteed enclosure of the image of $f_{\text{dep}}$. However, in many situations, this enclosure might be too wide to be useful. This phenomenon is known as the interval dependency problem. The notion of interval dependency and the problems thereof will be logically characterized in the succeeding sections.

5 From Quantifiers and Skolemizations to Functional Dependence

In order to be able to formalize the notion of interval dependency using elementary logical concepts, it is imperative to justify a symbolic notation suitable for the precise expression of ideas and for carrying out proofs in a merely logical manner. An early locus where the indispensability of a symbolic apparatus is emphasized is Frege’s “Begriffsschrift” [33]. In the spirit of the quote from Wittgenstein given in the beginning of this article, Frege opened his “Begriffsschrift” by saying:

“The most reliable way of carrying out a proof, obviously, is to follow pure logic, a way that, disregarding the particular characteristics of objects, depends solely on those laws upon which all knowledge rests.”

Our goal here is thus to fix a symbolic apparatus suitable and adequate for the purpose at hand. Plainly, we mean to reduce the notion of functional dependence to the pure logical concepts of Skolemization and quantification dependence.

Before getting down to particulars, it is apt to commence this section with historic and epistemic generalities at which level the concept of dependency is framed. The notion of dependency is one of the most fundamental ingredients that underlie mathematical reasoning and scientific reasoning in general. The notion of dependency comes from the notion of a function. Not surprisingly, therefore, there is scarcely a mathematical theory which does not involve the notion of a function (See, e.g., [16] and [28]). In ancient mathematics the idea of functional dependence was not expressed explicitly and was not an independent object of research, although a wide range of specific functional relations were known and were studied systematically. The concept of a function appears in a rudimentary form in the works of scholars in the Middle Ages, but only in the work of mathematicians in the 17th century, and primarily in those of Pierre de Fermat, Rene Descartes, Isaac Newton, and Gottfried Leibniz, did it begin to take shape as an independent concept. Later, in the 18th century, Euler had a more general approach to the concept of a function as “dependence of one variable quantity on another” [31]. By the year 1834, Lobachevskii was writing: “The general concept of a function requires that a function of $x$ is a number which is given for each $x$ and gradually changes with $x$. The value of a function can be given either by an analytic expression or by a condition which gives a means of testing all numbers and choosing one of them; or finally a dependence can exist and remain unknown” [49].

On the epistemic side, when scientists observe the world to formulate the defining properties of some physical phenomenon, these defining properties figure as attributes (variables) depending on some other attributes. Translating this dependence into a formal mathematical language, gives rise to the notion of functional dependence: “a variable $y$ is absolutely determined by some given variables $x_1, ..., x_n$”, or “a variable $y$ is a function of some given variables $x_1, ..., x_n$”, symbolically $y = f(x_1, ..., x_n)$. In some cases, such a translation can deterministically result in a certain rule for the function $f$, for instance $y = x_1 + ... + x_n$. In other cases, we have an approximate rule for $f$, or we know that a dependence exist but the rule cannot be determined, in which case we write the general usual notation $y = f(x_1, ..., x_n)$, without specifying explicitly a rule for the function $f$. So, in mathematics, a dependence is formally a function (For further exhaustive details about the notion of dependence, from the logical and epistemological viewpoints, see, e.g., [4], [39], [70], [15], [22], and [28]).

In the theory of real closed intervals, the notion of interval dependency naturally comes from the idea of functional dependence of real variables. Despite the fact that dependency is an essential and useful notion of real variables, interval dependency is the main unsolved problem of the classical theory of interval arithmetic and its modern generalizations (see [17], and [28]). So, a first step in our way to solve (or to cope with) the dependency problem is to answer the question: Formally, what is interval dependency? or, in other words, what is the fundamental defining properties that characterize the notion of interval dependency as a formal mathematical object? Providing an answer for this question is the main business of the succeeding section. For now, we will delve deep into some semantical and syntactical fundamentals concerning the logical formulation of the notion of functional dependence and some related notions.
Two pure logical notions we shall need are those of a quantification matrix and a prenex sentence. A quantification matrix \( Q \) is a sequence \((Q_1, x_1) \ldots (Q_n, x_n)\), where \( x_1, \ldots, x_n \) are variable symbols and each \( Q_i \) is \( \forall \) or \( \exists \). A prenex sentence is a sentence of the form \( Q \varphi \), where \( Q \) is a quantification matrix and \( \varphi \) is a quantifier-free formula.

As is well known, the most fundamental part of all mathematical sciences is formal logic. So, getting down to the most elementary fundamentals, it can be clarified that in all mathematical theories, any type of dependence can be reduced to the following simple logical definition (see, e.g., [16] and [28]).

**Definition 5.1 (Quantification Dependence).** Let \( Q \) be a quantification matrix and let \( \varphi(x_1, \ldots, x_m; y_1, \ldots, y_n) \) be a quantifier-free formula. For any universal quantification \( (\forall x_i) \) and any existential quantification \( (\exists y_j) \) in \( Q \), the variable \( y_j \) is dependent on the variable \( x_i \) in the prenex sentence \( Q \varphi \) iff \( (\exists y_j) \) is in the scope of \( (\forall x_i) \) in \( Q \). Otherwise \( x_i \) and \( y_j \) are independent.

That is, the order of quantifiers in a quantification matrix determines the mutual dependence between the variables in a sentence.

Let us illustrate this by the following two examples.

**Example 5.1.** Consider the prenex sentence

\[
(\exists x)(\forall y)(\exists z)(y = x \circ y \land x = z \circ y),
\]

which asserts that there exists an identity element \( x \), for the operation \( \circ \), with respect to which every element possesses an inverse \( z \).

According to the order in which quantifiers are written, the variable \( z \) depends only on \( y \), while there is no dependency between \( x \) and \( y \) or between \( x \) and \( z \).

**Example 5.2.** In the prenex sentence

\[
(\forall x)(\exists y)(\forall z)(\exists u) \varphi(x, y, z, u),
\]

the variable \( y \) depends on \( x \), and the variable \( u \) depends on both \( x \) and \( z \).

By means of a Skolem equivalent form or a Skolemization\(^4\), a quantification dependence is translated into a functional dependence. The notion of a Skolem equivalent form is characterized in the following definition (see, e.g., [28], [32], and [70]).

**Definition 5.2 (Skolem Equivalent Form).** Let \( \sigma \) be a sentence that takes the prenex form

\[
(\forall y_1=1 x_1)(\exists y_2=1 y_j) \varphi(x_1, \ldots, x_m; y_1, \ldots, y_n),
\]

where \( \varphi \) is a quantifier-free formula.

The Skolem equivalent form of \( \sigma \) is defined to be

\[
(\exists y_1=1 f_j)(\forall y_1=1 x_1) \varphi(x_1, \ldots, x_m; f_1, \ldots, f_n),
\]

where \( f_j(x_1, \ldots, x_m) = y_j \) are the dependency functions of \( y_j \) upon \( x_1, \ldots, x_m \), for \( i \in \{1, \ldots, m\} \) and \( j \in \{1, \ldots, n\} \).

It comes therefore as no surprise that in all mathematics, any instance of a dependence is, in fact, a functional dependence.

In order to clarify the matters, let us consider the following example.

**Example 5.3 (Skolemization of a Sentence).** Let a sentence \( \sigma \) take the prenex form

\[
(\forall x)(\exists y)(\forall z)(\exists u) \varphi(x, y, z, u).
\]

The Skolem equivalent form of \( \sigma \) is

\[
(\exists f)(\exists g)(\forall x)(\forall z) \varphi(x, f(x), z, g(x, z)).
\]

\(^4\)Skolemization is named after the Norwegian logician Thoralf Skolem (1887–1963), who first presented the notion in [66].
6 What Interval Dependency Formally is: Putting on a Systematic Basis All Together

Now we are proposing to study interval dependency, and indeed by the formal apparatus we fixed in the preceding sections. With the logical and set-theoretical fundamentals of sections 2 and 5 at our disposal, this section, is devoted to presenting a formalized treatment of the notion of interval dependency, that is, putting on a systematic basis its meaning, and thus gaining the advantage of indicating formally the criteria by which it is to be characterized and, accordingly, deducing its fundamental properties in a merely logical manner. On the strength of the generality of the logical apparatus we adopt, the results of this section are not only about Moore’s classical intervals, but they are meant to apply also to any possible theory of interval arithmetic.

Before we proceed, it is convenient here to introduce some notational conventions. By a finitary real-valued function in real arguments (in short, a real function or \( \mathbb{R} \)-function), we understand a function \( f_\mathbb{R} : D_\mathbb{R} \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \), and by an interval function (or \( I \)-function) we understand a function \( f_I : D_I \subseteq I^n \rightarrow I \). The \( \mathbb{R} \)-subscripted letters \( f_\mathbb{R}, g_\mathbb{R}, h_\mathbb{R} \) shall be employed to denote real-valued functions, while the \( I \)-subscripted letters \( f_I, g_I, h_I \) shall be employed to denote interval-valued functions. If the type of function is clear from its arguments, and if confusion is likely to ensue, we shall usually drop the subscripts “\( \mathbb{R} \)” and “\( I \)”. Thus, we may, for instance, write \( f(x_1, ..., x_n) \) and \( f \bigl(x_1, ..., x_n\bigr) \) for, respectively, a real-valued function and an interval-valued function, which are both defined by the same rule.

An important notion we shall need is that of the image set of real closed intervals, under an \( n \)-ary real-valued function. This notion is a special case of that of the corresponding \((n+1)\)-ary relation on \( \mathbb{R} \). More precisely, we have the following definition.

**Definition 6.1 (Image of Real Closed Intervals).** Let \( f \) be an \( n \)-ary function on \( \mathbb{R} \), and for \((v, y) \in f\), let \( v = (x_1, ..., x_n) \), with each \( x_k \) is restricted to vary on a real closed interval \( X_k \subset \mathbb{R} \), that is, \( v \) is restricted to vary on a set \( V \subset \mathbb{R}^n \). Then, the image of the closed intervals \( X_k \) with respect to \( f \), denoted \( I_f \), is defined to be

\[
Y = I_f (V) = I_f \bigl((X_1, ..., X_n)\bigr) = \{ y \in \mathbb{R} | \exists \forall v \in V \exists (v f y) \} = \{ y \in \mathbb{R} | (\exists_{k=1}^n x_k \in X_k) (y = f(x_1, ..., x_n)) \} \subseteq \mathbb{R},
\]

where the set \( V \), called the preimage \(^5\) of \( Y \), is defined to be the image of \( Y \) with respect to the converse relation \( \hat{f} \), that is

\[
V = I_f (Y) = \{ v \in \mathbb{R}^n | (\exists \forall y \in Y \exists (y \hat{f} v)) \}.
\]

The fact formulated in the following theorem is well-known (see, e.g., [16] and [28]).

**Theorem 6.1 (Extreme Value Theorem).** Let \( X_k \) be real closed intervals and let \( f(x_1, ..., x_n) \) be an \( n \)-ary real-valued function with \( x_k \in X_k \). If \( f \) is continuous in \( X_k \), in symbols \( \text{Cont}(f, X_k) \), then \( f \) must attain its minimum and maximum value, that is

\[
(\forall f) \ (\text{Cont}(f, X_k) \Rightarrow (\exists_{a_1, ..., a_n} \in X_k) (\exists_{b_1, ..., b_n} \in X_k) (\forall_{k=1}^n x_k \in X_k) (f(a_1, ..., a_n) \leq f(x_1, ..., x_n) \leq f(b_1, ..., b_n))),
\]

where \( \min f = f(a_1, ..., a_n) \) and \( \max f = f(b_1, ..., b_n) \) are respectively the minimum and maximum of \( f \).

An immediate consequence of definition 6.1 and theorem 6.1, is the following important property [28].

**Theorem 6.2 (Main Theorem of Image Evaluation).** Let an \( n \)-ary real-valued function \( f \) be continuous in the real closed intervals \( X_k \). The (accurate) image \( I_f \bigl((X_1, ..., X_n)\bigr) \), of \( X_k \), is in turn a real closed interval such that

\[
I_f (X_1, ..., X_n) = \left[ \min_{x_k \in X_k} f(x_1, ..., x_n), \max_{x_k \in X_k} f(x_1, ..., x_n) \right]
\]

A cornerstone result from the above theorem, that should be stressed at once, is that the best way to evaluate the accurate image of a continuous real-valued function is to apply minimization and maximization directly to determine the exact lower and upper endpoints of the image. For rational \(^6\) real-valued functions, this optimization problem is,

\(^5\)From the fact that the converse relation \( \hat{f} \) is always definable, the preimage of a function \( f \) is always definable, regardless of the definability of the inverse function \( f^{-1} \).

\(^6\)A rational real-valued function is a function obtained by means of a finite number of the basic real algebraic operations \( \circ \in \{+,-,\times\} \) and \( \circ \in \{-,+,\times\} \).
in general, computationally solvable, by applying Tarski’s algorithm, which is also known as Tarski’s real quantifier elimination (see, e.g., [10] and [84]). For algebraic \(^7\) real-valued functions, the problem is computable, by applying the cylindrical algebraic decomposition algorithm (CAD algorithm, or Collins’ algorithm) \(^8\), which is a more effective version of Tarski’s algorithm (see, e.g., [13] and [52]).

Before turning to the notion of interval dependency, we first prove the following indispensable result.

**Theorem 6.3 (Image Inclusions in Prenex Sentences).** Let \(\sigma_1\) and \(\sigma_2\) be the two prenex sentences such that

\[
\begin{align*}
\sigma_1 & \iff (\forall i = 1, \ldots, m) (\forall x_i \in X_i) (\exists y_i \in Y_i) (\exists z \in \mathbb{R}) (z = f(x_1, \ldots, x_m, y_1, \ldots, y_n)),
\sigma_2 & \iff (\forall i = 1, \ldots, m) (\forall x_i \in X_i) (\forall y_i \in Y_i) (\exists z \in \mathbb{R}) (z = f(x_1, \ldots, x_m, y_1, \ldots, y_n)),
\end{align*}
\]

where \(X_i\) and \(Y_i\) are real closed intervals, and \(f\) is a continuous real-valued function with \(x_i \in X_i\) and \(y_j \in Y_j\).

If \(I_f^\sigma_1\) and \(I_f^\sigma_2\) are the images of \(f\), respectively, in \(\sigma_1\) and \(\sigma_2\), then \(I_f^\sigma_1 \subseteq I_f^\sigma_2\).

**Proof.** According to definition 5.1, in the sentence \(\sigma_1\), all \(y_j\) are dependent upon all \(x_i\), and in the sentence \(\sigma_2\), all \(x_i\) and \(y_j\) are pairwise independent.

By definition 5.2, there are some functions \(g_j(x_1, \ldots, x_m)\) such that \(\sigma_1\) has the Skolem equivalent form

\[
(\exists z)(\forall y_j \in Y_j)(\forall x_i \in X_i)(z = f(x_1, \ldots, x_m, g_1, \ldots, g_n)).
\]

Finally, employing theorem 6.2, we therefore have \(I_f^\sigma_1 \subseteq I_f^\sigma_2\).

From the fact that existential quantification over a nonempty set \(S\) defines a set \(T\) such that \(T \subseteq S\), the previous theorem entails, as a special case, the following important result of “real analysis”.

**Corollary 6.1 (Inclusion Monotonicity of Real Images).** Let \(V_X = (X_1, \ldots, X_n)\) and \(V_Y = (Y_1, \ldots, Y_n)\) be two preimages of a continuous real-valued function \(f\). Then, the image \(I_f\) is inclusion monotonic. That is

\[
(\forall i = 1, \ldots, m)(\forall x_i \in X_i)(\forall y_i \in Y_i)(\forall z \in \mathbb{R})(z = f(x_1, \ldots, x_m, y_1, \ldots, y_n)).
\]

The following example makes the statement of theorem 6.3 clear.

**Example 6.1 (Image Inclusion in Two Prenex Sentences).** Let \(\sigma_1\) and \(\sigma_2\) be the two prenex sentences such that

\[
\begin{align*}
\sigma_1 & \iff (\forall x \in [1, 2])(\exists y \in [1, 2])(\exists z \in \mathbb{R}) (z = f(x, y) = y - x),
\sigma_2 & \iff (\forall x \in [1, 2])(\forall y \in [1, 2])(\exists z \in \mathbb{R}) (z = f(x, y) = y - x).
\end{align*}
\]

In the sentence \(\sigma_1\), the variable \(y\) depends on \(x\), and therefore there is some function \(g(x)\) such that \(\sigma_1\) has the Skolem equivalent form

\[
(\exists z)(\forall x \in [1, 2])(\exists z \in \mathbb{R}) (z = f(x, g(x)) = g(x) - x).
\]

Let \(g\) be the identity function. Consequently, the image of \(f\) in \(\sigma_1\) is \(I_f^{\sigma_1} = \{0\}\).

Obviously, the image of \(f\) in \(\sigma_2\) is \(I_f^{\sigma_2} = [-1, 1]\), and therefore \(I_f^{\sigma_1} \subseteq I_f^{\sigma_2}\).

Next we define the notion of an exact (or generalized) interval operation.

**Definition 6.2 (Exact Interval Operation).** Let \(\circ \in \{+, \times\}\) be a binary real operation, and let \(I_f^\text{Dep} = I_f^\text{Ind}\), where \(I_f^\text{Dep}\) and \(I_f^\text{Ind}\) are the images of a function \(f\) for two real closed intervals \(X\) and \(Y\) in, respectively, two prenex sentences \(\sigma_\text{Dep}\) and \(\sigma_\text{Ind}\) such that

\[
\begin{align*}
\sigma_\text{Dep} & \iff (\forall x \in X)(\exists y \in Y)(\exists z \in \mathbb{R}) (z = f(x, y) = x \circ y),
\sigma_\text{Ind} & \iff (\forall x \in X)(\forall y \in Y)(\exists z \in \mathbb{R}) (z = f(x, y) = x \circ y).
\end{align*}
\]

Then, an exact interval operation \(\circ_f \in \{+, \times\}\) is defined by

\[
X \circ_f Y = I_f(X, Y).
\]

We have then the following obvious result for the classical interval operations.

\(^7\)An algebraic function is a function that satisfies a polynomial equation whose coefficients are polynomials with rational coefficients.

\(^8\)The CAD algorithm is efficient enough for being one of the most important optimization algorithms of computational real algebraic geometry (see, e.g., [8]).
Theorem 6.4 (Inexactness of Classical Interval Operations). The value of a classical interval operation \( X \circ_T Y \) is exact only when the real variables \( x \in X \) and \( y \in Y \) are independent, that is

\[
X \circ_T Y = I^n_{\text{id}}(X, Y).
\]

\( \Box \)

Proof. The theorem is immediate from definition 3.1 of the classical interval algebra.

With the help of the preceding notions and the deductions from them, we are now ready to pass to our formal characterization of the notion of interval dependency.

Definition 6.3 (Interval Dependency Relation). Let \( S_1, ..., S_n \) be some arbitrary real closed intervals. For two interval variables \( X \) and \( Y \), we say that \( Y \) is dependent on \( X \), in symbols \( Y \sqsubseteq X \), iff there is some given real-valued function \( f \) such that \( Y \) is the image of \( (X; S_1, ..., S_n) \) with respect to \( f \). That is

\[
Y \sqsubseteq X \iff Y = I_f(X; S_1, ..., S_n),
\]

where \( f \) is called the dependency function of \( Y \) on \( X \). Otherwise \( Y \) is not dependent on \( X \), in symbols \( Y \not\sqsubseteq X \), that is

\[
Y \not\sqsubseteq X \iff \neg(Y = I_f(X; S_1, ..., S_n)).
\]

From now on, and throughout the text, the following notational convention shall be adopted. We write \( Y \sqsubseteq X \) (with the subscript \( f \)) to mean that \( Y \) is dependent on \( X \) by some given dependency function \( f \), and we write \( \exists(X, Y) \) to mean that \( X \) and \( Y \) are mutually independent. In general, the notation \( \exists(X_1, ..., X_n) \) shall be employed to mean “all \( X_1, ..., X_n \) are pairwise mutually independent”. Hereafter, for simplicity of the language, we shall always make use of the following abbreviation.

\[
\exists_{k=1}^n(X_k) \equiv \exists(X_1, ..., X_n).
\]

So, to say that an interval variable \( Y \) is dependent on an interval variable \( X \), we must be given some real-valued function \( f \) such that \( Y \) is the image of \( X \) under \( f \). This characterization of interval dependency is completely compatible with the concept of functional dependence of real variables: for two real variables \( x \) and \( y \), the variable \( y \) is functionally dependent on \( x \) if there is some given function \( f \) such that \( y = f(x) \), and to keep the dependency information, between \( x \) and \( y \), in an algebraic expression \( x \circ_R y \), it suffices to write \( x \circ_R f(x) \). If \( x \) and \( y \) are mutually dependent by an idempotence \( y = f(x) \) and \( x = g(y) \), then, to keep the dependency information, it suffices to write either \( x \circ_R f(x) \) or \( g(y) \circ_R y \). In case there is neither such a given function \( f \) nor such a given function \( g \), then it is obvious that the real variables \( x \) and \( y \) are not functionally dependent. Definition 6.3 extends this concept to the set of real closed intervals.

The preceding definition, along with two deductions that we shall presently make (theorem 6.5 and corollary 6.2), touches the notion of interval dependency in a way which copes with all possible cases. Noteworthy also is that the dependency relation characterized in definition 6.3 is a meta-concept, not an object-ingredient of the numerous interval (and fuzzy) theories heretofore presented in the literature. In other words, on the strength of the generality of the logical apparatus we adopt, the results of our formalization of interval dependency are not only about Moore’s classical interval theory, but they are meant to apply also to any possible theory of interval arithmetic.

To illustrate, let us give the following example.

Example 6.2 (Dependency Relation for Two Variables). Let \( X \) and \( Y \) be two interval variables that both are assigned the same individual constant \([0, 1] \). Then, we may have one of the following cases.

(i) \( Y \) is not dependent on \( X \) (there is no given dependency function).

(ii) \( Y \) is dependent on \( X \), by the identity function \( y = f(x) = x \).

(iii) \( Y \) is dependent on \( X \), by the square function \( y = f(x) = x^2 \).

This example shows that if two interval variables \( X \) and \( Y \) both are assigned the same individual constant (both have the same value), it does not necessarily follow that \( X \) and \( Y \) are identical, unless they are dependent by the identity function. Thereupon, for an interval theory to be dependency-aware, it must incorporate in its symbolism the dependency relation as an object-ingredient in such a way that two intervals are equal iff they are “one and the same”.

\( ^{9} \) The notions of identity and equality are commonly confused and treated as synonyms. However, they are two distinct logical concepts. Despite the fact that equality implies identity in the theory of real numbers, this is not always the case. Two line halves are equal but not identical (one and the same). Every line equals infinitely many other lines, but no line is (identical to) any other line (see [14] and [99]). Identity, which is the most fundamental ingredient of any mathematical theory, is characterized by Leibniz’s principle of the identity of indiscernibles which states that two entities \( x \) and \( y \) are identical iff any property of \( x \) is also a property of \( y \) and vice versa.
This fact itself calls for a new characterization of the equality relation for interval numbers: Two interval variables $X$ and $Y$ are equal (identical) iff they are dependent by identity, that is

$$X = Y \iff X \mathcal{D} Y.$$  

In consequence of this definition, in a possible dependency-aware interval theory, for any two interval variables $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$, it should be the case that

$$[\underline{x}, \overline{x}] = [\underline{y}, \overline{y}] \iff x = y \land \overline{x} = \overline{y} \land (\forall x \in [\underline{x}, \overline{x}]) \left( \exists y \in [\underline{y}, \overline{y}] \right) (y = \text{Id}(x)).$$

In any case, the formalization proposed here provides a tool for investigating the dependency problem rigorously. As a consequence of our characterization of interval dependency, we have the next immediate theorem that establishes that the interval dependency relation is a quasi-ordering relation.

**Theorem 6.5 (Quasi-Orderness of the Dependency Relation).** The interval dependency relation is a quasi-ordering relation on the set of real closed intervals. That is, for any three interval variables $X$, $Y$, and $Z$, the following statements are true:

- (i) $\mathcal{D}$ is reflexive, in symbols $(X \mathcal{D} X)$.
- (ii) $\mathcal{D}$ is transitive, in symbols $(X \mathcal{D} Y \land Y \mathcal{D} Z \Rightarrow X \mathcal{D} Z)$.

In accordance with this theorem and definition 6.2, we also have the following corollary.

**Corollary 6.2 (Dependency Relation Properties).** For any interval operation $\circ_J$, and for any three interval variables $X$, $Y$, and $Z$, the following two assertions are true:

- (i) $(X \circ_J Y) \mathcal{D} X$.
- (ii) $(X \circ_J Y) \mathcal{D} Y$.

The interval dependency problem can now be formulated in the following theorem.

**Theorem 6.6 (Dependency Problem).** Let $X_k$ be real closed intervals and let $f(x_1, \ldots, x_n)$ be a continuous real-valued function with $x_k \in X_k$. Evaluating the accurate image of $f$ for the interval numbers $X_k$, using classical interval arithmetic, is not always possible if there exist $X_i$ and $X_j$ such that $X_i \mathcal{D} X_j$, for $i \neq j$. That is,

- (i) $(\exists f) \left( \text{Id}_f(X_1, \ldots, X_n) \neq f(X_1, \ldots, X_n) \right)$.

In general,

- (ii) $(\forall f) \left( \text{Id}_f(X_1, \ldots, X_n) \subseteq f(X_1, \ldots, X_n) \right)$.

**Proof.** For (i), it suffices to give a counterexample.

For two interval variables $X_1$ and $X_2$ both that are assigned the same individual constant $[-a, a]$, let $f$ be a function defined by the rule $f(x_1, x_2) = x_1 x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. If $X_1 \mathcal{D} X_1$, with $g$ is the identity function $x_2 = g(x_1) = x_1$, then $f$ has the equivalent rule $f(x) = x^2$, with $x \in [-a, a]$.

According to theorem 6.2, the (accurate) image of $[-a, a]$ under the real-valued function $f$ is

$$\text{Id}_f([-a, a]) = \left[ \min_{x \in [-a, a]} x^2, \max_{x \in [-a, a]} x^2 \right] = [0, a^2].$$

If we evaluate the image of $[-a, a]$ using classical interval arithmetic, by theorem 3.1, we obtain the interval-valued function,

$$f([-a, a]) = [-a, a] \times [-a, a] = [-a^2, a^2],$$

which is not the actual image of $[-a, a]$ under $f$, that is, there is some function $f$, for which

$$\text{Id}_f(X_1, \ldots, X_n) \neq f(X_1, \ldots, X_n),$$

and therefore evaluating the accurate image of real-valued functions is not always possible, using classical interval arithmetic.
Toward proving (ii), let
\[ I_f (X_1, \ldots, X_n) = I_f^0 (X_1, \ldots, X_n) \lor I_f (X_1, \ldots, X_n) = I_f^0 (X_1, \ldots, X_n), \]
where \( I_f^0 \) and \( I_f^0 \) are the images of \( f \), respectively, in two prenex sentences \( \sigma_1 \) and \( \sigma_2 \) such that in \( \sigma_1 \), there exist \( X_i \) and \( X_j \) such that \( X_i \subseteq X_j \) for \( i \neq j \), and in \( \sigma_2 \), all \( X_k \) are pairwise independent, that is \( X_k \). Employing theorem 6.3, we accordingly have
\[ I_f^0 (X_1, \ldots, X_n) \subseteq I_f^0 (X_1, \ldots, X_n). \]

According to definition 3.1, of the classical interval algebra, all interval variables are assumed to be independent. We consequently have
\[ f (X_1, \ldots, X_n) = I_f^0 (X_1, \ldots, X_n). \]
Thus
\[ (\forall f) \left( I_f (X_1, \ldots, X_n) \subseteq f (X_1, \ldots, X_n) \right), \]
and therefore (ii) is verified. □

Obviously, the result \([-a^2, a^2]\), obtained using classical interval arithmetic, has an overestimation of
\[ w \left( [-a^2, a^2] \right) - w \left( [0, a^2] \right) = a^2, \]
where \( w \) is the width of the interval. This overestimated result is due to the fact that the classical interval theory assumes independence of all interval variables, even when dependencies exist.

A numerical example is shown below.

**Example 6.3 (Overestimation due to Dependency).** Consider the real-valued function
\[ f (x) = x (x - 1), \]
with \( x \in [0, 1] \).

The actual image of \([0, 1] \) under \( f \) is \([-1/4, 0] \). Evaluating the image using classical interval arithmetic, we get
\[ f ([0, 1]) = [0, 1] \times ([0, 1] - 1) = [-1, 0], \]
which has an overestimation of
\[ w([-1, 0]) - w([-1/4, 0]) = 3/4. \]

Finally, in consequence of theorem 6.6, we are led to the following immediate result.

**Theorem 6.7 (Occurrences of Variables and Dependency).** Let \( f (x) \) and \( g (x) \) be two real functions of different rules such that \( f (x) = g (x) \), and let \( f (X) \) and \( g (X) \) be the interval extensions of \( f (x) \) and \( g (x) \) respectively. Define \( O_f (X) \) and \( O_g (X) \) to be the numbers of occurrences of the variable \( X \) in \( f \) and \( g \) respectively. Then
\[ O_f (X) \leq O_g (X) \Rightarrow f (X) \subseteq g (X). \]

Due to interval dependency, classical interval arithmetic has a number of peculiar properties. For example, we have **subdistribution inverses** (or S-inverses) with respect to interval addition and multiplication. Precisely
\[ (\forall X) \left( 0 \subseteq X - X \right), \]
\[ (\forall X) \left( 0 \subseteq X \Rightarrow 1 \subseteq X/X \right). \]

In general, for two interval numbers \( X \) and \( Y \), if \( X \cap Y \neq \emptyset \), then
\[ (0 \subseteq X - Y) \land (0 \subseteq Y - X), \]
\[ (0 \subseteq Y) \Rightarrow (1 \subseteq X/Y), \]
\[ (0 \subseteq X) \Rightarrow (1 \subseteq Y/X). \]

The problem of computing the image \( I_f (X_1, \ldots, X_n) \), using interval arithmetic, is the main problem of interval computations. This problem is, in general, NP-hard \(^{10} \) (see, e.g., [35], [46], and [62]). That is, for the classical interval theory, there is no efficient algorithm to make the identity
\[ (\forall f) \left( I_f (X_1, \ldots, X_n) = f (X_1, \ldots, X_n) \right), \]

\(^{10}\) In principle, this result is not necessarily applicable to other theories of interval arithmetic (present or future) because each theory has its peculiar set of algorithms, where each algorithm is a sequence of elementary relations and functions of the foundational level of the theory.
always hold unless NP = P, which is widely believed to be false. However, a considerable scientific effort is put into finding a way out from the interval dependency problem. There are many special methods and algorithms, based on the classical interval theory, that successfully compute useful narrow bounds to the desirable accurate image. In the succeeding section, we shall give a bit of perspective on how to compute useful guaranteed interval enclosures under functional dependence.

Beyond the techniques based on the classical interval theory, various proposals for possible alternate theories of interval arithmetic were introduced to reduce the dependency effect or to enrich the algebraic structure of interval numbers. Among these alternate theories of intervals, we can mention as examples: Hansens generalized intervals, Kulisch’s complete intervals, directed intervals, modal intervals, parametric intervals, universal intervals, and others (see, e.g., [37], [48], [51], [36], [50], [16], [27], and [28]).

7 Guaranteed Enclosures Under Interval Dependency

In this section, we shall discuss briefly how to compute useful guaranteed enclosures of real-valued functions under functional dependence. The numerical examples of this section are computed using version 2.0 of InCLosure. The InCLosure commands to compute the results of the examples are described and an InCLosure input file and its corresponding output containing, respectively, the code and results of the examples are also available as a supplementary material to this text (see Supplementary Materials).

As mentioned in the preceding section, there are many special methods and algorithms, based on the classical interval theory, that provide a way out of the dependency problem to get narrower enclosures. With a knowledge of regions of monotonicity, most elementary interval functions can be defined to be the exact images of the corresponding real functions. As instances, for an interval number $X = [x, \bar{x}]$ and a nonnegative integer $n$, we can define

$$e^X = [e^x, e^{\bar{x}}], \quad \ln(X) = [\ln(x), \ln(\bar{x})] \text{ if } x > 0;$$

$$\sqrt{X} = \left[\sqrt{x}, \sqrt{\bar{x}}\right] \text{ if } x \geq 0, \quad \sin(X) = \min_{x \in X} (\sin(x)), \max_{x \in X} (\sin(x));$$

$$X^n = \begin{cases} [x^n, \bar{x}^n] & \text{iff } x > 0 \text{ or } n \text{ is odd}, \\ [\bar{x}^n, x^n] & \text{iff } \bar{x} < 0 \text{ and } n \text{ is even}, \\ [0, |\bar{x}|] & \text{iff } 0 \in X \text{ and } n \text{ is even}; \end{cases}$$

where $|X| = \max\{|x|, |ar{x}|\}$ is the absolute value of the interval number $X$.

Applying the naive (pure) interval operations on these exact evaluations, we can get better (narrower) enclosures of the images of the algebraic combinations of the corresponding real functions. Moreover, many techniques are used to improve the results obtained from the naive method by reducing the width of the resulting interval. Among these techniques centered forms, generalized centered forms, circular complex centered forms, Hansens method, remainder forms, the subdivision method, and many others (see, e.g., [17], [26], [57], [56], [62], [48], [1], and [2]). For instance, we can get arbitrarily narrower enclosures of images of real functions using the subdivision method, which is due to Moore (see, e.g., [67] and [58]). This method can be described as follows. Let $X = [\underline{x}, \bar{x}]$ be an interval number and let $w(X) = \bar{x} - x$ be the width of $X$. First, we subdivide the interval $X$ into $n$ subintervals $X_i$ such that

$$X_i = \left[x + (i - 1)w(X)/n, x + iw(X)/n\right],$$

where $w(X_i) = w(X)/n$. Hence $X = \bigcup_{i=1}^n X_i$. Then, we evaluate the interval-valued function $f$ for each subinterval $X_i$, $f(X_i)$. Accordingly, we have [28]

$$I_f(X) \subseteq \bigcup_{i=1}^n f(X_i) \subseteq f(X).$$

That is, the subdivision method produces better enclosures than the naive method. Moreover, the larger the number of subintervals $n$, the narrower the enclosure of the image $I_f(X)$. Next, we shall describe how to make use of the interval capability of InCLosure to get arbitrarily better enclosures of the images of real functions.

Example 7.1 (Interval Subdivision in InCLosure). Consider the interval function

$$f(X) = X^9 - X^6 - X^3 - 369.$$

InCLosure provides arbitrarily narrower interval results which are limited only by the computational power of the host machine. We can compute the value of the above interval function at the interval $[-3, 6]$, $f([-3, 6])$, using the following InCLosure command.

EvalInt "X^9-X^6-X^3-369" "X=[-3,6]" 1 30
This will result in
\[ [-66924.0, 1.0077354E7].\]

The last parameter, “30”, in the previous command, is the precision of the result (default is 20). The parameter “1” indicates the number of subintervals if the subdivision method is applied (the number “1” means no subdivision of the interval). To get narrower intervals, the number of subintervals can be increased arbitrarily. Table 1 shows InCLosure results obtained for no subdivision, subdivision with 10, 50, 100, 500, and 1000 subintervals of \([-3, 6]\).

<table>
<thead>
<tr>
<th>Number of subintervals</th>
<th>InCLosure result (f([-3, 6]))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (no subdivision)</td>
<td>([-66924.0, 1.0077354E7])</td>
</tr>
<tr>
<td>10</td>
<td>([-20771.739, 1.0059598061199E7])</td>
</tr>
<tr>
<td>50</td>
<td>([-20758.574232, 1.0038266720770032576E7])</td>
</tr>
<tr>
<td>100</td>
<td>([-20756.357829, 1.0034509264991644959E7])</td>
</tr>
<tr>
<td>500</td>
<td>([-20754.483089832, 1.0031290472751452950435776E7])</td>
</tr>
<tr>
<td>1000</td>
<td>([-20754.242271729, 1.0030874303048468185232559E7])</td>
</tr>
</tbody>
</table>

Table 1. InCLosure Result for Different Numbers of Subdivisions

The exact image of the corresponding real function \(f(x) = x^9 - x^6 - x^3 - 369\) on \([-3, 6]\) is ‘approximately’ \([-20754, 10030455]\) according to Wolfram Mathematica [72]. All the results of table 1 are guaranteed enclosures of the exact image, and example 7.1 clearly shows that classical interval arithmetic with the arbitrarily narrower interval results of InCLosure so markedly surpasses the ordinary numerical methods.

8 Conclusion

One approach that proved to be subtle, reliable, and most fundamental in all of mathematics of uncertainty is interval mathematics. By integrating the complementary powers of rigorous mathematics and scientific computing, interval arithmetic is able to offer highly reliable accounts of uncertainty. Despite all of advantages of interval mathematics detailed in the introduction and elsewhere, it has its disadvantages as well. A main drawback of interval mathematics is the persisting problem known as the “interval dependency problem”. This, naturally, confronts us with the crucial question: Formally, what is interval dependency? Is it a meta-concept or an object-ingredient of interval and fuzzy mathematics? In other words, what is the fundamental defining properties that characterize the notion of interval dependency as a formal mathematical object? what exactly is the sense of saying that two intervals are dependent? and how does the dependency of two intervals \(X\) and \(X\) differ from that of \(X^2\) and \(X^2\)? Since the early works on interval mathematics by Burkill and Young in the dawn of the twentieth century, this question has never heretofore been touched upon and remained a question still today unanswered. Although the notion of interval dependency is widely used in the interval and fuzzy literature, it is only illustrated by example, without explicit formalization, and no attempt has been made to put on a systematic basis its meaning, that is, to indicate formally the criteria by which it is to be characterized. This article has been devoted to answering this long-standing question, and, that being so, the problem dealt with in this text is that of the possibility and the scope of a symbolic formalization of interval dependency. We proposed a precise metatheoretic characterization of the notion of interval dependency, deduced its fundamental properties in a merely logical manner, and thereupon we developed a rigorous mathematical theory thereof which formally characterizes and explains the differences between all cases of interval dependency.

We would also remark that, by virtue of our formalization, many nice consequences of real and interval analysis come for free, for example: it has been shown that for an interval theory to be dependency-aware, it must incorporate in its symbolism the dependency relation as an object-ingredient in such a way that two intervals are equal iff they are one and the same and a new definition of the equality relation for interval numbers is proposed accordingly, the main theorem of image evaluation and the inclusion monotonicity of real images follow immediately from a generalized theorem about image inclusions in prenex sentences, and a generalized theorem concerning the dependency problem is easily established.

To sum up, what is so important about a logical formalization of interval dependency? In fact, effort in pursuit of this aim has many fruitful consequences. A novelty of this formalization is the expression of interval dependency as
a logical predicate (or relation) and thus gaining the advantage of deducing its fundamental properties in a merely logical manner. Taking the passage from the informal treatments to the formal technicalities of mathematical logic, this result sheds new light on many fundamental problems of interval mathematics. Moreover, a breakthrough behind our formalization of interval dependency is that it paves the way and provides the systematic apparatus for developing alternate dependency-aware interval theories and computational methods with mathematical constructions that better account for dependencies between the quantifiable uncertainties of the real world. Noteworthy also is that on the strength of the generality of the logical apparatus we adopt, the results of this article are not only about Moore’s classical intervals, but they are meant to apply also to any possible theory of interval arithmetic.

Supplementary Materials

To reproduce the results of the calculations in this article, latest version of InCLosure is available for free download via https://doi.org/10.5281/zenodo.2702404. An InCLosure input file and its corresponding output containing, respectively, the code and results of the examples are also available as a supplementary material to this article, via http://doi.org/10.5281/zenodo.3466032.

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A Logical Formalization of the Notion of Interval Dependency


