1. INTRODUCTION
The process of spontaneous parametric down conversion (SPDC), in which one photon splits into two conjugate photons, is at the core of photonic entanglement. The down-converted photons, conventionally termed signal and idler, are known to memorize phase information about the pump field \[ |\psi> = 1/\sqrt{2}(|H_1H_2> + \phi|V_1V_2>). \] (1)

Since the crystals used in this arrangement are relatively thin, the arising transverse walk-off is negligibly small. However, because the down-conversion domain is generally birefringent and dispersive, the phase \( \phi \) effectively varies with the angle and frequency of down-conversion emission which hurts the purity of the output state.

To obtain a high-fidelity entangled state without restricting the detection to narrow spatial–spectral windows, the phase variation needs to be manipulated by appropriate phase compensation. Altepeter et al. [10] could partially compensate the spatial phase variation by placing two nonlinear crystals into the signal and idler trajectories. This led to a sevenfold improvement in the spatial phase flatness which heavily enhanced the brightness of the output without sacrificing the quality. The same method has been adopted in combination with temporal compensation, while low coherence-time pump and wide collection apertures are in use [12]. A more recent technique suggests the use of a spatial phase modulator to impose spatial polarization-dependent phase differences, pixel by pixel [15].

To set up a compensation that counts for the spatial and spectral phase variations, we must determine the directional-spectral phase function of the two-photon state. One calculation approach [10,13,18] involves a number of approximations, including monochromaticity of the pump beam and SPDC radiation and symmetry of phase differences at the two down-conversion directions. Another approach considers only the first-order expansion of the phase function in a number of contributing parameters [15]. While the first approach includes unjustified approximations (led to significant mismatch between calculations and measurements), the second lacks the accuracy when higher-order phase function or wide acceptance range of frequency or direction is taken into account.

In this paper, we present a tunable compensation method for the spatial and spectral phase variations of the two-photon state generated by the noncollinear two-crystal source. Our study is confined to type-I down-conversion crystals with negative birefringence. We first determine the exact phase of the produced state as a function of frequency and angle of emission, taking into account the finite spectral width of the pump beam. Conditions for flattened phase function with respect to frequency and angle of emission are then employed to engineer the characteristics of the compensation elements. The
compensation profile can be broadly tuned by the tilt angles of the compensation elements. A numerical code is developed to count for the directional-spectral phase function before and after the compensation elements. The results of our model are then compared with recent experimental measurements. While, in the context of this paper, we consider the geometry of two type-I crystals, similar analysis and results apply in the case of hyperentangled photon sources with a single type-I crystal and back-reflecting mirror [19–21].

2. QUANTUM STATE OF SUPERIMPOSED SPDC LIGHT

For a nonlinear type-I domain of infinite transverse extent pumped by a normally incident classical beam of circularly symmetric Gaussian profile, the two-photon state produced by the SPDC process takes the form [24]

$$|\psi\rangle = \int dK_s dK_i \chi^{(2)}_{\text{eff}}(w_s; \theta_s, w_i; \theta_i) A_p(w_s + w_i)$$

$$\times \int_\infty^\infty \int_\infty^\infty dx dy \exp(-i\Delta k_p \cdot \mathbf{r}_T) \exp \left[-\left(\frac{\mathbf{r}_T}{W}\right)^2\right]$$

$$\times \int_0^d dz \exp(-i\Delta k_s z)|K_s\rangle|K_i\rangle,$$

(2)

where $K_s$ and $K_i$ are the signal and idler wave vectors, $\chi^{(2)}_{\text{eff}}$ is the second-order effective susceptibility, the nonlinear interaction has a volume defined by $W$ the waist of the incident pump beam and $d$ the crystal thickness, $\Delta k_T$ and $\Delta k_s$ are the transverse and longitudinal wave vector mismatches, $w_s$ and $w_i$ are the frequencies of the signal and idler photons, and $A_p(w_s + w_i)$ is the spectral distribution function of the pump light.

When a photon pair is generated by superimposing SPDC emissions of two coherently pumped type-I crystals (see Fig. 1), the state within spectral and spatial acceptance ranges $[\Delta w_s, \Delta w_i; \Delta \theta_s, \Delta \theta_i, \Delta \phi_s, \Delta \phi_i]$ can be expressed as

$$|\psi\rangle = C_\alpha \int d\Delta w_s \int d\Delta w_i \chi^{(2)}_{\text{eff}}(w_s, \theta_s, w_i, \theta_i) A_p(w_s + w_i)$$

$$\times \int d\Delta \theta_s \int d\Delta \theta_i \int d\Delta \phi_s \int d\Delta \phi_i \exp \left[-\frac{(W\Delta k_T(w_s, w_i, \theta_s, \theta_i, \phi_s, \phi_i))^2}{2}\right]$$

$$\times \int_0^d dz \exp(-i\Delta k_z z)|\psi\rangle|\psi\rangle$$

$$\times \exp[i\phi_s(w_s, w_i, \theta_s, \phi_s) + i\phi_i(w_s, w_i, \theta_i, \phi_i)]|V\rangle|V\rangle,$$

(3)

where $C_\alpha$ is a normalization factor, $(\theta_s, \phi_s)$ and $(\theta_i, \phi_i)$ are polar and azimuthal angles of signal and idler photons outside the crystals, $\phi_i$ is the initial phase difference between the orthogonal pump components, $\phi_{DC}$ is the relative phase accumulated in the SPDC process, and $\phi_s$ is the global phase. Although the relative phase function $\phi_{DC}$ appears to depend substantially on the azimuthal angles $\phi_s$ and $\phi_i$ because of the inherent variation of refractive indices, careful analysis (details will be given in a future paper) shows that such dependence is ignorable in the case of normal pump incidence and small emission angles.

Hereinafter, the term extraordinary (ordinary) will be used to denote the field polarization with refractive index varying (constant) with the polar angle of the wave vector relative to the dielectric axis $Z$ of uniaxial or biaxial domain. While this is the usual terminology in the case of uniaxial domain, it is applicable to the biaxial case when the wave vector lies in one of the principal dielectric planes $XZ$ and $YZ$ (ordinary refractive indices are thus $n_x$ and $n_z$). Our analysis is confined to these two anisotropic configurations in the case of negative
birefringence (the ordinary ray is slower than the extraordinary ray).

In Eq. (2), the distribution of the relative phase \( \phi_{\text{DC}} \) in the far field of the SPDC emission can be expressed as

\[
\phi_{\text{DC}}(w_s, w_l, \theta_s, \theta_l, z) = K_{op}(w_s + w_l) + K_p(w_s + w_l) \Delta z 
+ \phi_{\text{DC}}(w_s, \theta_s, z) + \phi_{\text{DC}}(w_l, \theta_l, z),
\]

(4)

where \( K_{op}(w_s + w_l) \) and \( K_p(w_s + w_l) \Delta z \) are the phase accumulated by the horizontal pump component in its passage through the first crystal and in the distance \( \Delta z \) between the two crystals. \( \phi_{\text{DC}}(w_s, \theta_s, z) \) and \( \phi_{\text{DC}}(w_l, \theta_l, z) \) are the phase differences between two signal (idler) photons born in the two crystals at interaction depth \( z \). As depicted in Fig. 1, through the crystals, the extraordinary photons pass physically according to the Poynting vectors \( S_m \) which walk with the angle \( \sigma \) off the relevant wave vectors \( K_m \) \( (n = s, i, p, c1, c2) \). This yields an explicit expression for the relative phase function of the form

\[
\phi_{\text{DC}}(w_s, w_l, \theta_s, \theta_l, z) = K_{op}(w_s + w_l) + K_p(w_s + w_l) \Delta z 
- \frac{K_{i}(w_s) \Delta z}{\cos \theta_s - \cos \theta_p} \sin \theta_s \cos \sigma_i 
+ \frac{K_{i}(w_l) \Delta z}{\cos \theta_l - \cos \theta_p} \sin \theta_l \cos \sigma_i 
+ \frac{K_{i}(w_l) \Delta z}{\cos \theta_l + \cos \theta_p} \sin \theta_l \cos \sigma_i 
+ \frac{K_{i}(w_l) \Delta z}{\cos \theta_l - \cos \theta_p} \Delta z \tan \theta_s - z \tan \delta_p 
+ \frac{K_{i}(w_l) \Delta z}{\cos \theta_l + \cos \theta_p} \Delta z \tan \theta_l - z \tan \delta_p.
\]

(5)

It is worth observing that the phase differences at the signal and idler trajectories are not the same. This demonstrates that the symmetric-phase approximation adopted in [10,13,18] was not valid.

In Eq. (5), the transverse walk-off of the pump beam contributes to the relative-phase function by introducing dependence on the interaction depth \( z \). This shows that, when SPDC emission is far from modes with transverse momentum conservation [for which, \( K_s(w_s) \sin \theta_i = K_i(w_i) \sin \theta_i \)], thicker crystals accumulate higher spatial–spectral decoherence (besides the known effect of crystal thickness on the spatial and temporal which-crystal information).

The coherent superposition of noncollinear SPDC emissions from different anisotropic domains entails pumping with a relatively wide beam to eliminate the spatial which-crystal information [7] (such requirement can be safely avoided in the collinear case [25]). The Gaussian term in Eq. (3) is thus highly selective for contribution from spatial modes with a very small transverse wave vector mismatch \( \Delta k_{T} \). In the following, we will assume that the pump beam is wide enough [20] to offer conservation of transverse momentum as a good approximation. Accordingly, the two-photon state in Eq. (3) can be rewritten as

\[
|\psi\rangle = C_b \int_{\omega_s} dw_s \int_{\omega_l} dw_l X_{\text{df}}(w_s, w_l; w_s + w_l) A_p(w_s + w_l) 
\times \int_{0}^{\theta_s} d\theta_s \int_{0}^{\theta_l} d\theta_l \int_{0}^{d} dz \exp[-i \Delta k_z(w_s, w_l, \theta_s, \theta_l)z] 
+ i \phi_{\text{DC}}(w_s, w_l, \theta_s, \theta_l) \times |[H]_{w_s, \theta_s, \phi_s} |[H]_{w_l, \theta_l, \phi_l} 
+ \exp[i \phi_s + i \phi_{\text{DC}}(w_s, w_l, \theta_s)] V_{w_s, \theta_s, \phi_s} |V_{w_l, \theta_l, \phi_l}. \]
\]

(6)

3. OPTIMAL SPATIAL–SPECTRAL PHASE COMPENSATION

After compensation, the directional-spectral relative-phase profile imposed by the compensation crystals C1 and C2 [see Fig. 1(c)], with thicknesses \( L_1 \) and \( L_2 \) and tilt angles \( \rho_{c1} \) and \( \rho_{c2} \) and of the same type as the down-conversion crystals can be written explicitly as

\[
\phi_{C}(w_s, w_l, \theta_s) = \phi_{C1}(w_s, \theta_s) + \phi_{C2}(w_l, \theta_l) 
= K_s(w_s) L_1 \left\{ \frac{\sin \theta_s}{\sin \theta_{oc1} \cos \theta_{oc1}} - \sin \theta_s - \rho_{c1} \tan \theta_{oc1} \right\} 
+ \tan \theta_{oc1} \right\} \right\} 
+ K_s(w_s) L_2 \left\{ \frac{\sin \theta_s}{\sin \theta_{oc2} \cos \theta_{oc2}} - \sin \theta_s - \rho_{c2} \tan \theta_{oc2} \right\} 
+ \tan \theta_{oc2} \right\} \right\}.
\]

(7)

where \( \theta_{oc} = \theta_{oc}(\sigma_{oc}) \) is the angle of the ordinary (extraordinary) wave vector \( K_{oc}(K_{oc}) \), and \( \sigma_{oc} \) are the walk-off angles in the two compensation crystals \( (q = 1, 2) \). In general, the phase profile \( \phi_{C}(w_s, w_l, \theta_s) \) can be shaped with the use of the thicknesses \( L_1, L_2 \) and the angles \( (\rho_{c1}, \rho_{c2}, \theta_{p1}, \theta_{p2}) \) of the compensation crystals to counteract the directional-spectral variations of the phase function \( \phi_{\text{DC}}(w_s, w_l, \theta_s) \); hence the coherence of the state is restored. Our numerical analysis demonstrates that the phase profile \( \phi_{C}(w_s, w_l, \theta_s) \) in Eq. (7) is broadly tunable in space and frequency by means of the tilt angles of the compensation crystals \( (\rho_{c1}, \rho_{c2}) \) while the crystal-cut parameters \( (L_1, L_2, \theta_{p1}, \theta_{p2}) \) are kept fixed.

In the vicinity of its flatness, the compensated phase function \( C^\text{\phi}_{\text{DC},\phi} \phi_{\text{DC},\phi} \) becomes a slowly varying convex/concave function in its arguments. The criteria for optimal compensation can thus be expressed as

\[
C^\text{\phi}_{\text{DC}}(w_s, w_l, \theta_s, \theta_l, \Delta \theta_s) = C^\text{\phi}_{\text{DC}}(w_s, w_l, \theta_s, \theta_l, \Delta \theta_s/2).
\]

(8a)

\[
C^\text{\phi}_{\text{DC}}(w_s, w_l, \theta_s, \theta_l, \Delta \theta_s) = C^\text{\phi}_{\text{DC}}(w_s, w_l, \theta_s, \theta_l, \Delta \theta_s/2).
\]

(8b)

where the superscript “0” denotes the central directions and frequencies of the collected SPDC radiation. Here the spectral width of the pump is assumed much narrower than the acceptance ranges \( \Delta w_s \) and \( \Delta w_l \). For optimal compensation, the six variables \( (L_1, L_2, \rho_{c1}, \rho_{c2}, \theta_{p1}, \theta_{p2}) \) should take the values satisfying Eqs. (8a) and (8b). Numerically, one can find an infinite number of equivalent solutions for this problem. Four variables can thus be chosen as parameters to make the number of unknowns equal the number of equations. Under the assumption of monochromatic pump beam, and the settings \( \theta_{p1} = \theta_{p2} = \theta_{pm} \) and \( \rho_{c1} = \rho_{c2} = 0^\circ \), the above criteria can be reduced to
Table 1. Simulation Results for Some Previous Two-Crystal Setups

<table>
<thead>
<tr>
<th>Publication</th>
<th>Angular Phase Slopes</th>
<th>Optimal Thicknesses of Compensation Crystals (μm)</th>
<th>Phase Range over (1°, 30 nm) (deg.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[7, 10]</td>
<td>157.9</td>
<td>302.2, 297.1</td>
<td>Before: 400.2°, 24.2°</td>
</tr>
<tr>
<td>[3]</td>
<td>−62.8</td>
<td>58.2, 56.8</td>
<td>After: 111.1°, 11.5°</td>
</tr>
<tr>
<td>[2]</td>
<td>−32.0</td>
<td>36.5, 36.1</td>
<td></td>
</tr>
<tr>
<td>[14] (0.5 mm crystals)</td>
<td>−114.4</td>
<td>284.1, 280.9</td>
<td></td>
</tr>
<tr>
<td>[15] (BBO crystals)</td>
<td>−127.5</td>
<td>363.0, 359.0</td>
<td></td>
</tr>
<tr>
<td>[16]</td>
<td>−44.0</td>
<td>773.9, 766.1</td>
<td></td>
</tr>
</tbody>
</table>

\[ \phi_{DC} \left( w_s^0, \theta_s^0 - \frac{\Delta \theta_s}{2} \right) - \phi_{DC} \left( w_s^0, \theta_s^0 + \frac{\Delta \theta_s}{2} \right) = (L_1 + L_2) \left[ \Phi_{C1} \left( w_s^0, \theta_s^0 + \frac{\Delta \theta_s}{2} \right) - \Phi_{C1} \left( w_s^0, \theta_s^0 - \frac{\Delta \theta_s}{2} \right) \right], \]  

(9a)

\[ \phi_{DC} \left( w_s^0 \frac{\Delta w_s}{2}, \theta_s^0 \right) - \phi_{DC} \left( w_s^0 \frac{\Delta w_s}{2}, \theta_s^0 \right) = L_1 \left[ \Phi_{C1} \left( w_s^0 \frac{\Delta w_s}{2}, \theta_s^0 \right) - \Phi_{C1} \left( w_s^0 \frac{\Delta w_s}{2}, \theta_s^0 \right) \right] + L_2 \left[ \Phi_{C2} \left( w_s^0 \frac{\Delta w_s}{2}, \theta_s^0 \right) - \Phi_{C2} \left( w_s^0 \frac{\Delta w_s}{2}, \theta_s^0 \right) \right]. \]  

(9b)

where \( \Phi_{C1} = \phi_{C1}/L_1 \) and \( \Phi_{C2} = \phi_{C2}/L_2 \).

4. RESULTS AND DISCUSSION

We have developed a numerical code (available on our website [27]) to calculate the phase functions, the optimal characteristics of compensation elements, and other parameters associated with the problem. We detail in Table 1 the results of phase calculations for a number of two-crystal setups, assuming irradiation by a monochromatic pump beam. These results are accompanied by the optimal thicknesses of compensation crystals in each case as obtained by Eqs. (9a) and (9b). In each case, we assume that the biphoto flux is collected over many spatial modes.

Let us discuss one of the cases in Table 1 in detail. In [7, 10], Fig. 2 depicts linear increase in the spatial and spectral slopes of the relative-phase function as the two down-conversion crystals are shifted farther from each other. Any desired two-photon state can then be captured only along specific spatial-spectral modes of the noncollinear two-crystal radiation (in agreement with [28, 29]).

The angular phase slopes in Table 1 correspond to spatial slopes of about −8.85°/mm (signal) and 25.86°/mm (idler) at a transverse plane, 120 cm from the down-conversion crystals. The spatial slope of the relative-phase function is thus 17.01°/mm, as depicted in Fig. 3(a). This perfectly agrees with the slope value determined experimentally by taking complete state tomography at different points in the transverse plane (17°/mm in [10]).

In this case, using the compensation setup in Table 1, our calculation model predicts a relative-phase range of about 24.0° over angular-spectral ranges (1°, 30 nm) [see Fig. 3(b)]. Any deviation from the optimal \((L_1 + L_2)\) value determined by Eq. (9a) directly boosts the spatial phase variation. For instance, let us consider the compensation realized by Altepeter et al. using two 245 μm crystals. The residual phase slope, calculated by our code for this suboptimal setup is ≈3.1°/mm, on average [see Fig. 3(c)]. This excellently agrees with the slope ≈3.0°/mm measured experimentally in [10].

On the other hand, the higher the deviation from the ratio \((L_1/L_2)\) obtained by Eqs. (9a) and (9b), the more the phase variation in the frequency dimension. For example, while a 599 μm compensation crystal at one arm eliminates the spatial phase variation [satisfying Eq. (8a)], it increases the spectral phase gradient, as depicted in Fig. 3(d). The above discussion can be generalized to all other cases in Table 1.

Moreover, the increment of the inter-crystal distance \(\Delta z\) that corresponds to a relative-phase shift \(\varphi\) can be given by Eq. (5) as

\[ \Delta z = K_p \left( w_s + w_i \right) - K_s \left( w_s \cos \theta_s \right) \left( w_i \cos \theta_i \right). \]  

(10)

The same relation applies to hyperentangled setups with single type-I crystal and back-reflecting mirror [19–23] (except for a factor of 2 because of the double pass). For a degenerate down-conversion with wavelength 727.6 nm and emission angle 3.3°, we determine \(\Delta z = 55.5 \mu m\) which agrees with the experimental measurements in [19] (\(\Delta z = 55 \mu m\)).

It should be noted that, by virtue of the tilt angles \(\rho_1\) and \(\rho_2\), the phase profile imposed by the compensation crystals is broadly tunable in both frequency and angle of emission. For example, as demonstrated by numerical simulations, the suboptimal compensation using two 245 μm crystals can be made optimal by tilting the compensation crystals with \(\rho_1 = 25.6°\) and \(\rho_2 = 35.1°\) for the 25.6° relative phase range observed experimentally in [10] (see Fig. 3).
and $\rho_{2} = 25.0^\circ$. Equations (8a) and (8b) are thus satisfied, and the compensated phase function becomes similar to the one in Fig. 3(b) with a range of about $24.0^\circ$ over $(1^\circ, 30 \text{ nm})$ window. This shift toward tunable phase compensation is substantially useful to realize optimal compensation for the nondegenerate SPDC arrangements using compensation elements satisfying Eqs. (9a) and (9b) in the degenerate case.

In the discussion above, we describe the optimal spatial–spectral phase compensation under the assumption of monochromatic pump beam. Figure 4(a) shows the spectral phase function for the two-crystal arrangement in [13] when a finite pump line width is taken into account and perfect phase matching is assumed in longitudinal and transverse directions. One can observe that, over wide ranges of signal and idler wavelengths, the variations of the phase mainly come with the pump wavelength. It seems as though the perfect phase matching with underlying conservation of energy and momentum also offers conservation of relative phase all over the emission directions.

The complete phase picture can be visualized as follows. The central spatial–spectral SPDC modes determined by perfect phase matching conditions have fixed relative phase over space and spectrum when monochromatic pump beam is considered. Because of the loose phase-matching condition in the longitudinal direction, there exist spatial–spectral SPDC modes around each central mode which receive relative-phases deviated from the phase of the central modes [as in Fig. 3(a)]. When a broad-bandwidth pump is used, the relative phase of the central spatial–spectral SPDC modes exhibits significant dependence on the pump wavelength. This consequently affects the phase of the noncentral spatial–spectral modes.

To compensate for the spectral phase gradient in the case of a broad-bandwidth pump, one might extend the criteria in Eqs. (8a) and (8b) by involving another equality to address the weakly related signal and idler wavelengths. Values for three compensation variables, rather than two, are thus determined by solving the three equations together. Another way is to supplement the compensation setup satisfying Eqs. (8a) and (8b) by an additional crystal placed in front of the SPDC crystals. The spectral phase profile of the pre-compensation crystal counteracts the spectral phase gradient of the SPDC crystals and post-compensation crystals, as depicted in Figs. 4(b)–4(d). One unknown variable of the pre-compensation crystal (either thickness, cut angle, or tilt angle) can thus be determined by solving the equality

$$c_{\text{DC}} \left[ w_{s}^{0} + \frac{\Delta w_{s}}{2}, w_{i}^{0} + \frac{\Delta w_{i}}{2}, \theta_{p}(w_{s}, w_{i}) \right]$$

$$= c_{\text{DC}} \left[ w_{s}^{0}, w_{i}^{0}, \frac{\Delta w_{s}}{2}, \frac{\Delta w_{i}}{2}, \theta_{p}(w_{s}, w_{i}) \right].$$

where $\theta_{p}(w_{s}, w_{i})$ is the emission angle of a signal photon in the case of perfect phase matching and $c_{\text{DC}}$ is the final compensated phase function. remarkably, any combination of compensation variables satisfying (11) is found also to compensate entirely for the temporal walk-off accumulated over the SPDC crystals and post-compensation crystals.

5. CONCLUSION

The spatial–spectral phase variation of the two-photon state generated by the noncollinear two-crystal SPDC emission limits its output efficiency. In this paper, an experimentally
convenient method for tunable phase compensation has been presented in the case of negative birefringent domains. We have determined the exact directional-spectral relative phase function of the output state. We have then employed optimal flatness criteria for the compensated phase over frequency and direction to engineer the compensation elements. We have also studied the dependence on the distance separating the down-conversion crystals. Our theoretical results, supported by numerical simulations, perfectly agree with preceding experimental measurements.

REFERENCES AND NOTES
26. A plausible condition in this case is \( W \gg 2/\Delta k_T \) for \( \Delta k_T \ll k_T, k_T' \).
27. http://www.zewailcity.edu.eg/research-institutes/cpnsrl/