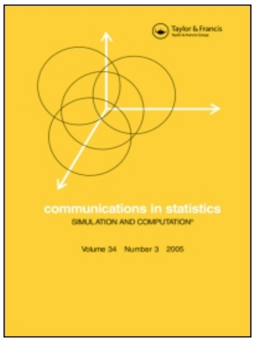

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Communications in Statistics - Simulation and Computation

ISSN: 0361-0918 (Print) 1532-4141 (Online) Journal homepage: http://www.tandfonline.com/loi/lssp20

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To cite this article: Abdulhakim Al-Babtain, Ahmed A. Fattah, A-Hadi N. Ahmed & Faton Merovci (2017) The Kumaraswamy-transmuted exponentiated modified Weibull distribution, Communications in Statistics - Simulation and Computation, 46:5, 3812-3832, DOI: 10.1080/03610918.2015.1011338

To link to this article: https://doi.org/10.1080/03610918.2015.1011338

COMMUNICATIONS IN STATISTICS—SIMULATION AND COMPUTATION®
2017, VOL. 46, NO. 5, 3812–3832
http://dx.doi.org/10.1080/03610918.2015.1011338

The Kumaraswamy-transmuted exponentiated modified Weibull distribution

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ABSTRACT
This article introduces a new generalization of the transmuted exponentiated modified Weibull distribution introduced by Eltehiwy and Ashour in 2013, using Kumaraswamy distribution introduced by Cordeiro and de Castro in 2011. We refer to the new distribution as Kumaraswamy-transmuted exponentiated modified Weibull (Kw-TEMW) distribution. The new model contains 54 lifetime distributions as special cases such as the KumaraswamyWeibull, exponentiated modified Weibull, exponentiated Weibull, exponentiated exponential, transmuted Weibull, Rayleigh, linear failure rate, and exponential distributions, among others. The properties of the new model are discussed and the maximum likelihood estimation is used to evaluate the parameters. Explicit expressions are derived for the moments and examine the order statistics. This model is capable of modeling various shapes of aging and failure criteria.

ARTICLE HISTORY
Received 5 July 2014
Accepted 19 January 2015

KEYWORDS
Exponentiated exponential; Exponentiated weibull; Kumaraswamy distribution; Maximum likelihood estimation; Order statistics; Survival function; Transmutation

MATHEMATICS SUBJECT CLASSIFICATION
62N01; 62N02; 62E10

1. Introduction

For complex electronic and mechanical systems, the failure rate often exhibits non-monotonic (bathtub or upside-down bathtub unimodal) failure rates (Xie and Lai, 2006). Distributions with such failure rates have attracted a considerable attention of researchers in reliability engineering. In software reliability, bathtub shaped failure rate is encountered in firmware, and in embedded software in hardware devices. Firmware plays an important role in functioning of hard drives of large computers, spacecraft and high performance aircraft control systems, advanced weapon systems, or safety critical control systems used for monitoring the industrial process in chemical and nuclear plants (Zhang et al., 2005). The upside down bathtub shaped failure rate is used in data of motor bus failures (Mudholkar et al., 1995), for optimal burn-in decisions (Barreto-Souza et al., 2010; Block and Savits, 1997; Chang, 2000), for aging properties in reliability (Gupta and Gupta, 1983; Gupta and Kundu, 2001; Jiang et al., 2003), and the course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

There are many distributions for modeling such data among the known parametric models; the most popular are the gamma, lognormal, and the Weibull distributions. The Weibull distribution is more popular than the gamma and lognormal distributions because the survival

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functions of the latter cannot be expressed in closed forms, a property that the Weibull distribution enjoys. This distribution does not provide a good fit to datasets with bathtub shaped or upside down bathtub shaped (unimodal) failure rates, often encountered in reliability, engineering, and biological studies. Hence a number of new distributions modeling the data in a better way have been constructed in literature as ramifications of Weibull distribution. Several distributions have been proposed in the literature to extend the Weibull distribution (see Pham and Lai, 2007), which presents a review of some of the generalizations or modifications of Weibull distribution.

Eltehiwy and Ashour (2013) introduced the transmuted exponentiated modified Weibull (TEMW) distribution with cumulative distribution function (cdf) denoted by $G(x, \alpha, \beta, \gamma, \theta, \lambda) \equiv G(x)$ and probability density function (pdf) (for $x > 0$) given by

$$G(x) = [1 - e^{-(\theta x + \gamma x^{\beta})}]^{\alpha}[1 + \lambda - \lambda(1 - e^{-(\theta x + \gamma x^{\beta})})^{\alpha}], \tag{1}$$

and

$$g(x) = \alpha(\theta + \gamma \beta x^{\beta-1})e^{-(\theta x + \gamma x^{\beta})}[1 - e^{-(\theta x + \gamma x^{\beta})}]^{\alpha-1}.[1 + \lambda - 2\lambda(1 - e^{-(\theta x + \gamma x^{\beta})})^{\alpha}], \tag{2}$$

respectively, where$\beta > 0$ and $\alpha > 0$ are shape parameters, $\theta > 0$ and $\gamma > 0$ are scale parameters, and$|\lambda| \leq 1$ is a transmuted parameter. The TEMW model shows flexible properties as it contains a lot of well-known distributions as special cases such as exponentiated Weibull, transmuted Weibull, Weibull, and linear failure rate distributions.

Kumaraswamy (1980) introduced a two-parameter distribution on (0, 1), which will be referred to by "Kw" in the sequel. Its cdf is given by

$$F(x) = 1 - (1 - x^{a})^{b}, \quad x \in (0, 1), \tag{3}$$

where $\alpha > 0$ and $b > 0$ are shape parameters. The model in (3) compares extremely favorably in terms of simplicity with the beta cdf, that is, the incomplete beta function ratio. The pdf corresponding to (3) is given by

$$f(x) = ab(1 - x^{a})^{b-1}, \quad x \in (0, 1). \tag{4}$$

The Kw density function has the same basic shape properties of the beta distribution: $a > 1$ and $b > 1$ (unimodal); $a < 1$ and $b < 1$ (uniantimodal); $a > 1$ and $b \leq 1$ (increasing); $a \leq 1$ and $b > 1$ (decreasing); $a = b = 1$ (constant). The Kw distribution does not seem to be very familiar to statisticians and has not been investigated systematically in much detail before. However, Jones (2009) explored the background and genesis of the Kw distribution and, more importantly, highlighted some advantages and disadvantages of the beta and Kw distributions.

For an arbitrary baseline cdf, $G(x)$, Cordeiro and de Castro (2011) defined the Kw-G distribution by the pdf $f(x)$ and cdf $F(x)$ as

$$f(x) = ab.g(x).G^{a-1}(x)[1 - G^{a}(x)]^{b-1}, \tag{5}$$

and

$$F(x) = 1 - [1 - G^{a}(x)]^{b}, \tag{6}$$

respectively, where $g(x) = dG(x)/dx$ and a and b are two extra positive shape parameters. It follows immediately from (6) that the Kw-G distribution with parent cdf $G(x) = x$ produces the minimax distribution (3). If X is a random variable with pdf (5), we write $X \sim Kw - G(a, b)$, where a and b are additional shape parameters that aim to govern skewness and tail weight of the generated distribution. An attractive feature of this distribution is that the two parameters a and b can afford greater control over the weights both in tails and in its center.

The aim of this article is to introduce a new seven-parameters lifetime distribution to be named Kumaraswamy-transmuted exponentiated modified Weibull distribution (denoted by Kw-TEMW) hoping to decrease the great hole between the statistical models in fitting the different types of data. To this end, we start from the transmuted exponentiated modified Weibull distribution to introduce the new model, by inserting (1) into (6). Then the cumulative distribution function of Kw-TEMW model (for $x > 0$) denoted by $F(x, \alpha, \beta, \gamma, \theta, \lambda, a, b) \equiv F(x)$ becomes

$$F(x) = 1 - \{1 - [1 - e^{-(\theta x+\gamma x^{\beta})}]^{\alpha a}[1 + \lambda - \lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}]^{a}\}^{b}, \tag{7}$$

whereas its pdf can be expressed, from (1), (2), and (5), as

$$\begin{aligned} f(x) = {}& ab\alpha(\theta + \gamma\beta x^{\beta-1})e^{-(\theta x+\gamma x^{\beta})} \times \{1 + \lambda - 2\lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}\} \\ & \times [1 - e^{-(\theta x+\gamma x^{\beta})}]^{\alpha a-1} \times \{1 + \lambda - \lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}\}^{a-1} \\ & \times \{1 - (1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha a}[1 + \lambda - \lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}]^{a}\}^{b-1}, \end{aligned} \tag{8}$$

respectively.

A physical interpretation of (7) is possible when a and b are positive integers. Suppose a system is made up of b independent components in series and that each component is made up of a independent sub-components in parallel. So, the system fails if any of the b components fail and each component fails if all of its a subcomponents fail. If the subcomponent lifetimes have a common TEMW cumulative function, then the lifetime of the entire system will follow the Kw-TEMW distribution (7).

From another view, suppose a system consists of b independent subsystems functioning independently at a given time and that each subsystem consists of a independent parallel components. Suppose too that each component consists of two units. If the two units are connected in series then the overall system will have Kw-TEMW distribution with $\lambda = 1$ whereas if the components are parallel then the overall system will have Kw-TEMW distribution with $\lambda = -1$.

Furthermore, we can interpret the system from the redundancy view. Redundancy is a common method to increase reliability in an engineering design. Barlow and Proschan (1981) indicate that, if we want to increase the reliability of a given system, then redundancy at a component level is more effective than redundancy at a system level. That is, if all components of a system are available in duplicate, it is better to put these component pairs in parallel than it is to build two identical systems and place the systems in parallel.

Suppose a system is made up of b independent subsystems functioning independently at a given time and that each subsystem is made up of a independent parallel components. If we want to improve the reliability of the given system we had to duplicate each component in parallel form, then the time to failure of the given system will have the cumulative distribution function (7) at $\lambda = -1$.

The failure rate function associated with (7) is given by

$$\begin{aligned} h(x) = {}& ab\alpha.\ (\theta + \gamma\beta\, x^{\beta-1})\, e^{-(\theta x+\gamma x^{\beta})} \times \{1 + \lambda - 2\lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}\} \\ & \times \frac{[1 - e^{-(\theta x+\gamma x^{\beta})}]^{\alpha a-1}[1 + \lambda - \lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}]^{a-1}}{\{1 - [1 - e^{-(\theta x+\gamma x^{\beta})}]^{\alpha a}[1 + \lambda - \lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}]^{a}\}}. \end{aligned} \tag{9}$$

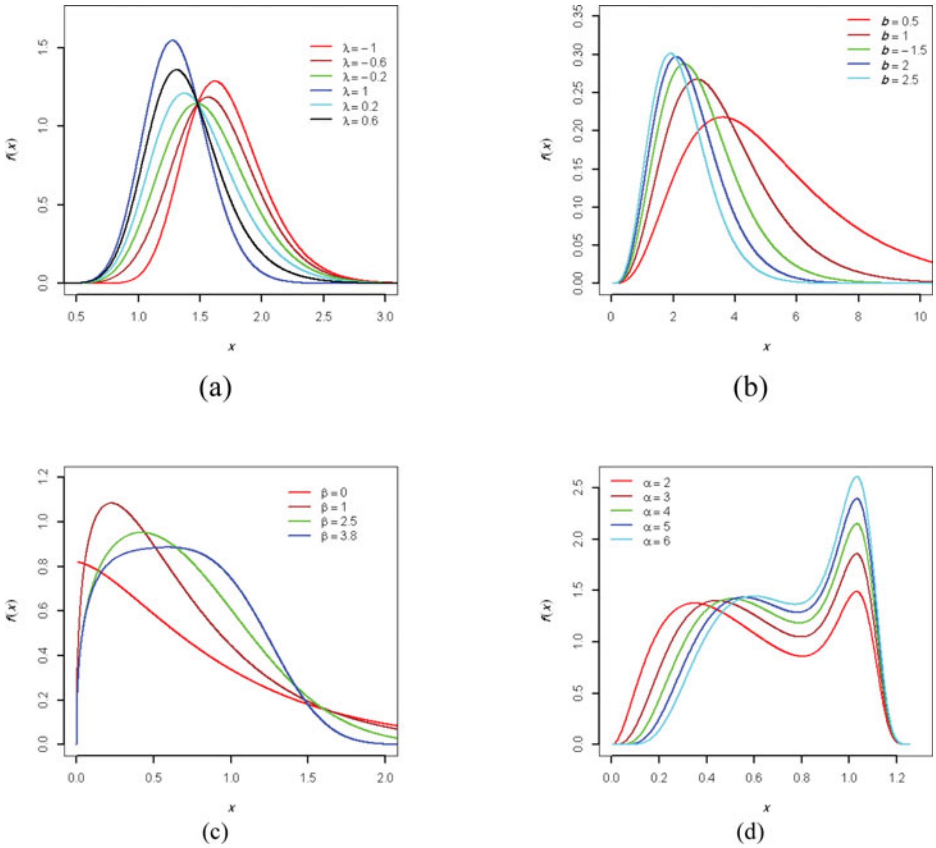

Figure 1. Plots of the Kw-TEMW density function for some parameter values. (a) For different values of λ with $\alpha = 8, \beta = 2, \gamma = 1, \theta = 0.2, a = 1$, and $b = 1$. (b) For different values of b with $\alpha = 3, \beta = 1.3, \gamma = 0.3, \theta = 0.1, \lambda = -1$, and $a = 0.6$. (c) For different values of β with $\alpha = 1, \gamma = 0.32, \theta = 1, \lambda = 1, a = 1.4$, and $b = 1.4$. (d) For different values of α with $\beta = 15, \gamma = 1, \theta = 5, \lambda = -1, a = 0.8$, and $b = 0.4$.

The Kw-TEMW model due to its flexibility in accommodating all forms of the hazard rate function as seen from Fig. 2 (by changing its parameter values) seems to be an important distribution that can be used. Another importance of the proposed Kw-TEMW model is that it is a very flexible model that approaches to different distributions when its parameters are changed. The flexibility of the Kw-TEMW is explained in Table 1 where it has 54 submodels when their parameters are carefully chosen.

Figures 1(a)–1(d) provide some plots of the Kw-TEMW density curves for different values of the parameters $\alpha, \beta, \gamma, \theta, \lambda, a$, and b.

Figure 2 does the same for the associated hazard rate function, showing that it is quite flexible for modeling survival data.

The rest of the article is organized as follows. In Section 2, we derive an expansion to the pdf and the cdf functions. In Section 3, we obtain the r-th non-central moment. Section 4 gives the quantile function for the new model. In Section 5, we introduce the method of likelihood estimation as point estimation and the confidence interval as an interval estimation of the unknown parameters, and then give the equation used to estimate the parameters, using the maximum product spacing estimates and the least square estimates techniques. Finally, we fit the distribution to two real datasets to examine it and to check

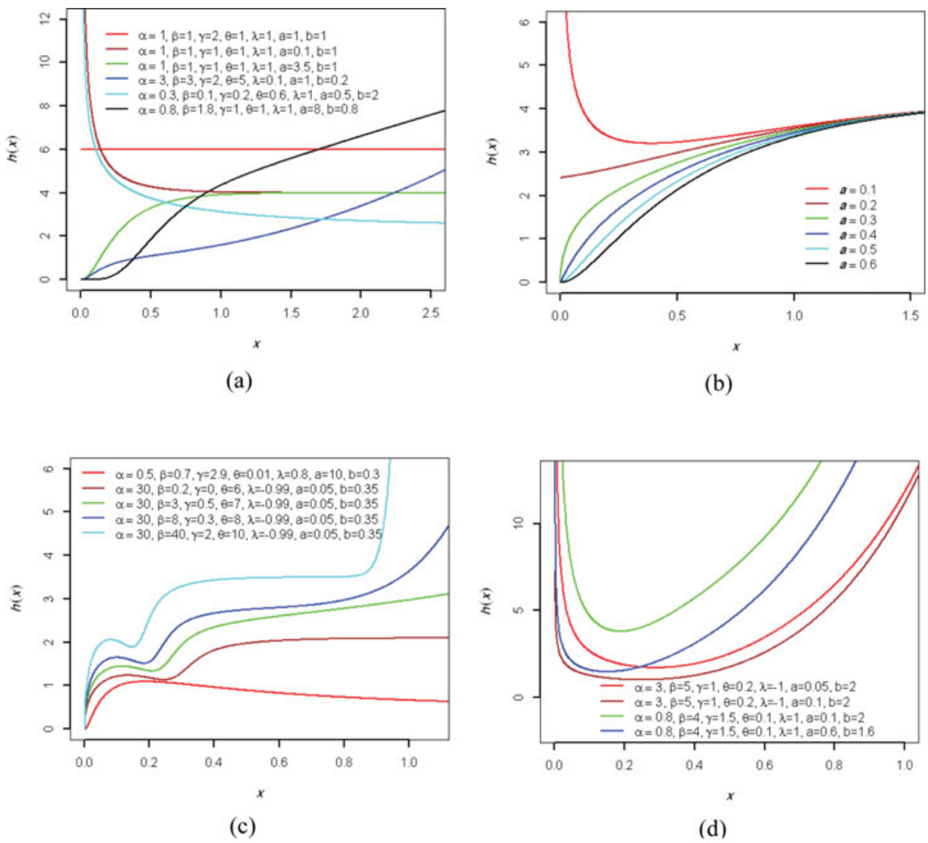

Figure 2. (a), (b) For different values of a with $\alpha = 5, \beta = 1, \gamma = 2, \theta = 0.1, \lambda = 1$ and $b = 1$; (c) and (d) are plots of the hazard rate function for some parameter values.

its suitability with the nested and non-nested models. The estimated densities and the estimated cdfs of the Kw-TEMW distribution and other fitted models for the data are displayed in Figure 3.

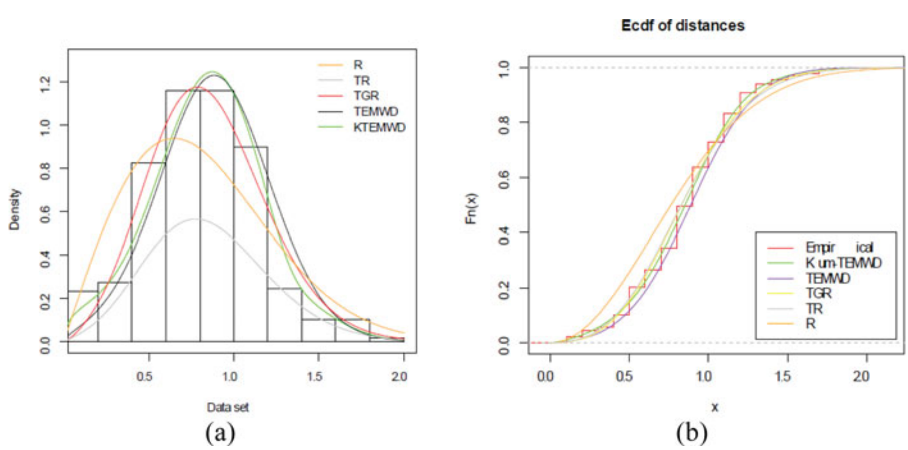

Figure 3. (a) Estimated densities of the Kw-TEMW, TEMW, TER, TR, and R distributions for the data. (b) Estimated cdf function from the fitted Kw-TEMW, TEMW, TER, TR, and R distributions and the empirical cdf for the data.

Table 1. The special cases of the Kw-TEMW distribution.

	Parameters							
Distribution	a	b	λ	α	θ	γ	β	Author
Kw-TELFR							2	New
Kw-TEW					0			New
Kw-TER					0		2	New
Kw-TEE						0		New
Kw-TMW				1				New
Kw-TLFR				1			2	New
Kw-TW				1	0			New
Kw-TR				1	0		2	New
Kw-TE				1		0		New
Kw-EMW			0					New
Kw-ELFR			0				2	Elbatal (2011)
Kw-EW			0		0			New
Kw-ER			0		0		2	Gomes et al. (2014)
Kw-EE			0			0		
Kw-MW			0	1				New
Kw-LFR			0	1			2	
Kw-W			0	1	0			Cordeiro and de Castro (2011)
Kw-R			0	1	0		2	
Kw-E			0	1		0		
ETEMW		1						New
ETELFR		1					2	New
ETEW		1			0			New
ETER		1			0		2	New
ETEE		1				0		New
ETMW		1		1				New
ETLFR		1		1			2	New
ETW		1		1	0			New
ETR		1		1	0		2	New
ETE		1		1		0		New
New-EMW		1	0					New
New-ELFR		1	0				2	New
New-EW		1	0		0			New
New-ER		1	0		0		2	New
New-EE		1	0			0		New
EMW		1	0	1				Elbatal (2011)
EW		1	0	1	0			Mudholkar and Srivastava (1993)
EE		1	0	1		0		Gupta et al. (1998)
TEMW	1	1						Eltehiwy and Ashour (Elbatal, 2013)
TELFR	1	1					2	
TEW	1	1			0			
TER	1	1			0		2	Merovci (2013a)
TEE	1	1				0		Merovci (Marshall and Olkin, 1997)
TMW	1	1		1				Khan and King (2013)
TLFR	1	1		1			2	
TW	1	1		1	0			Aryal and Tsokos (2011)
TR	1	1		1	0		2	Khan and King (Kelton and Law, 2000)
TE	1	1		1		0		Shaw and Buckley (2007)
ELFR	1	1	0				2	Sarhan and Kundu (Pham and Lai, 2007)
ER	1	1	0		0		2	Kundu and Raqab (2005)
MW	1	1	0	1				Sarhan and Zaindin (2009)
LFR	1	1	0	1			2	
W	1	1	0	1	0			Weibull (1939)
R	1	1	0	1	0		2	
E	1	1	0	1		0		

2. Expansion for the pdf and the cdf functions

In this section, we give another expression for the pdf and the cdf functions using the Maclaurin expansion for simplifying the pdf and the cdf forms.

2.1. Expansion for the pdf function

From (8), using the expansion

$$(1-z)^{b-1}=\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(b)}{\Gamma(b-i).i!}z^i,\qquad |z|<1,\tag{10}$$

and applying it to the term $\{1-[1-e^{-(\theta x+\gamma x^\beta)}]^{\alpha a}[1+\lambda-\lambda(1-e^{-(\theta x+\gamma x^\beta)})^\alpha]^a\}^{b-1}$,the pdf of the Kw-TEMW model can be written as

$$\begin{aligned}f(x)&=ab\alpha(\theta+\gamma\beta\,x^{\beta-1})\,e^{-(\theta x+\gamma x^\beta)}\\&\quad\times\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(b)}{\Gamma(b-i).i!}\times\{1+\lambda-2\lambda(1-e^{-(\theta x+\gamma x^\beta)})^\alpha\}\\&\quad\times[1-e^{-(\theta x+\gamma x^\beta)}]^{\alpha a(i+1)-1}\times[1+\lambda-\lambda(1-e^{-(\theta x+\gamma x^\beta)})^\alpha]^{a(i+1)-1}.\end{aligned}\tag{11}$$

Applying (10) to (11) for the term $[1+\lambda-\lambda(1-e^{-(\theta x+\gamma x^\beta)})^\alpha]^{a(i+1)-1}$, (11) can be written as

$$\begin{aligned}f(x)&=ab\alpha(\theta+\gamma\beta\,x^{\beta-1})e^{-(\theta x+\gamma x^\beta)}\\&\quad\times\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}.\Gamma(b).\Gamma(ai+a).(\lambda)^j.(1+\lambda)^{a(i+1)-j-1}}{\Gamma(b-i).\Gamma(ai+a-j).i!.j!}\\&\quad\times\left\{(1+\lambda)(1-e^{-(\theta x+\gamma x^\beta)})^{\alpha(ai+a+j)-1}-2\lambda(1-e^{-(\theta x+\gamma x^\beta)})^{\alpha(ai+a+j+1)-1}\right\}.\end{aligned}\tag{12}$$

Applying (10) again to (12), one finally reaches at the pdf, which can be written as

$$\begin{aligned}f(x)&=ab\alpha\left(\theta+\gamma\beta x^{\beta-1}\right)\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}.\Gamma(b).\Gamma(ai+a).(\lambda)^j.(1+\lambda)^{a(i+1)-j-1}}{\Gamma(b-i).\Gamma\left(ai+a-j\right).i!.j!.k!}\\&\quad\times\left\{\frac{(1+\lambda).\Gamma\left(\alpha ai+\alpha a+\alpha j\right)}{\Gamma\left(\alpha ai+\alpha a+\alpha j-k\right)}-\frac{2\lambda.\Gamma\left(\alpha ai+\alpha a+\alpha j+\alpha\right)}{\Gamma\left(\alpha ai+\alpha a+\alpha j+\alpha-k\right)}\right\}e^{-(k+1)\left(\theta x+\gamma x^\beta\right)}.\end{aligned}\tag{13}$$

The pdf of Kw-TEMW distribution can then be represented as

$$f(x)=\sum_{i,j,k=0}^{\infty}A_{i:k}\times\left(\theta+\gamma\beta x^{\beta-1}\right).\,e^{-(k+1)\left(\theta x+\gamma x^\beta\right)},\tag{14}$$

where $A_{i:k}$ is a constant term given by

$$\begin{aligned}A_{i:k}&=ab\alpha\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}.\Gamma(b).\Gamma(ai+a).(\lambda)^j.(1+\lambda)^{a(i+1)-j-1}}{\Gamma(b-i).\Gamma\left(ai+a-j\right).i!.j!.k!}\\&\quad\times\left\{\frac{(1+\lambda).\Gamma\left(\alpha ai+\alpha a+\alpha j\right)}{\Gamma\left(\alpha ai+\alpha a+\alpha j-k\right)}-\frac{2\lambda.\Gamma\left(\alpha ai+\alpha a+\alpha j+\alpha\right)}{\Gamma\left(\alpha ai+\alpha a+\alpha j+\alpha-k\right)}\right\}.\end{aligned}$$

It can be noticed that the pdf of the Kw-TEMW distribution can be looked at a mixture of the EMW pdfs as

$$f(x) = \sum_{i,j,k=0}^{\infty} A_{i:k} \frac{1}{k+1} \times z(x), \qquad (15)$$

where $z(x) = (k+1)(\theta + \gamma\beta x^{\beta-1})e^{-(k+1)(\theta x+\gamma x^{\beta})}$ is the pdf of the exponentiated modified Weibull distribution.

2.2. Expansion for the cdf function

Applying the expansion in (10) to (7), the cdf function of the Kw-TEMW distribution can be written as

$$F(x) = 1 - \sum_{i,j,k=0}^{\infty} B_{i:k} \times e^{-k\left(\theta x+\gamma x^{\beta}\right)}, \qquad (16)$$

where $B_{i:k}$ is a constant term given by

$$B_{i:k} = \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}.\Gamma(b+1).\Gamma(ai+1).\Gamma\left(\alpha ai+\alpha j+1\right).(\lambda)^{j}.(1+\lambda)^{ai-j}}{i!.j!.k!.\Gamma(b-i+1).\Gamma\left(ai-j+1\right).\Gamma\left(\alpha ai+\alpha j-k+1\right)}.$$

3. Moments

Using (14), the r-th non-central moment, μ^{r}, of the Kw-TEMW model is given by

$$\mu^{r} = \sum_{i,j,k=0}^{\infty} A_{i:k} \times \sum_{h=0}^{\infty} \frac{(-1)^{h}(k+1)^{h}(\gamma)^{h}}{h!}.\left\{\theta\frac{\Gamma(r+h\beta+1)}{[\theta(k+1)]^{r+h\beta+1}} + \gamma\beta\frac{\Gamma(r+h\beta+\beta)}{[\theta(k+1)]^{r+h\beta+\beta}}\right\}. \qquad (17)$$

4. Quantile function

The quantile function is obtained by inverting the cumulative distribution (7), where the p-th quantile x_p of the Kw-TEMW model is the real solution of the following equation:

$$\gamma x_p^{\beta} + \theta x_p + \ln\left\{1 - \left[\left(\frac{-1}{\lambda}\left(1-(1-p)^{\frac{1}{b}}\right)^{\frac{1}{a}} + \frac{1}{4}\left(\frac{1+\lambda}{\lambda}\right)^{2}\right)^{\frac{1}{2}} + \frac{1}{2}\left(\frac{1+\lambda}{\lambda}\right)\right]^{\frac{1}{\alpha}}\right\} = 0. \qquad (18)$$

An expansion for the median M follows by taking $p = 0.5$.

5. Parameter estimation

In this section, the maximum likelihood estimation is used to estimate the unknown parameters. An equation is presented to estimate the parameters using the maximum product spacing estimates as well as the least square estimate.

5.1. Maximum likelihood method

Let $X_1, X_2, \ldots, X_n$ be a sample of size n from a Kw-TEMW distribution. Then the likelihood function (ℓ) is given by

$$\ell = (ab\alpha)^n . \prod_{i=1}^{n}(\theta + \gamma\beta\, x_i^{\beta-1}).\, e^{-\sum_{i=1}^{n}(\theta x_i + \gamma x_i{}^{\beta})} \times \prod_{i=1}^{n}[1 - e^{-(\theta x_i + \gamma x_i{}^{\beta})}]^{\alpha a - 1}$$

$$\times \prod_{i=1}^{n}\{1 - (1 - e^{-(\theta x_i + \gamma x_i{}^{\beta})})^{\alpha a}[1 + \lambda - \lambda(1 - e^{-(\theta x_i + \gamma x_i{}^{\beta})})^{\alpha}]^{a}\}^{b-1}$$

$$\times \prod_{i=1}^{n}[1 + \lambda - 2\lambda(1 - e^{-(\theta x_i + \gamma x_i{}^{\beta})})^{\alpha}] \times \prod_{i=1}^{n}[1 + \lambda - \lambda(1 - e^{-(\theta x_i + \gamma x_i{}^{\beta})})^{\alpha}]^{a-1}. \quad (19)$$

Hence, the log-likelihood function,$\mathcal{L}$, becomes

$$\mathcal{L} = n\ln a + n\ln b + n\ln\alpha + \sum_{i=1}^{n}\ln(\theta + \gamma\beta\, x_i{}^{\beta-1}) - \sum_{i=1}^{n}(\theta x_i + \gamma x_i{}^{\beta}) + (b-1)$$

$$\times \sum_{i=1}^{n}\ln\{1 - (1 - e^{-(\theta x_i + \gamma x_i{}^{\beta})})^{\alpha a}[1 + \lambda - \lambda(1 - e^{-(\theta x_i + \gamma x_i{}^{\beta})})^{\alpha}]^{a}\}$$

$$+(\alpha a - 1)\sum_{i=1}^{n}\ln[1 - e^{-(\theta x_i + \gamma x_i{}^{\beta})}] + \sum_{i=1}^{n}\ln[1 + \lambda - 2\lambda(1 - e^{-(\theta x_i + \gamma x_i{}^{\beta})})^{\alpha}]$$

$$+(a-1)\sum_{i=1}^{n}\ln[1 + \lambda - \lambda(1 - e^{-(\theta x_i + \gamma x_i{}^{\beta})})^{\alpha}]. \quad (20)$$

Therefore, the MLEs of α, β, γ, θ, λ, a, and b must satisfy the following equations:

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \frac{n}{\alpha} + a\sum_{i=1}^{n}\ln\left[1 - e^{-(\theta x_i + \gamma x_i{}^{\beta})}\right]$$

$$-\lambda\left(a-1\right)\sum_{i=1}^{n}\frac{\left(1 - e^{-(\theta x_i + \gamma x_i{}^{\beta})}\right)^{\alpha}.\ln\left(1 - e^{-(\theta x_i + \gamma x_i{}^{\beta})}\right)}{1 + \lambda - \lambda(1 - e^{-(\theta x_i + \gamma x_i{}^{\beta})})^{\alpha}}$$

$$-2\lambda \times \sum_{i=1}^{n} \frac{(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}.\ln(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})}{1+\lambda-2\lambda(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}} - a(b-1)$$

$$\times \sum_{i=1}^{n} \frac{\left(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})}\right)^{\alpha a}\left[1+\lambda-\lambda\left(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})}\right)^{\alpha}\right]^{a-1}}{1-\left(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})}\right)^{\alpha a}\left[1+\lambda-\lambda\left(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})}\right)^{\alpha}\right]^{a}}$$

$$\times \ln\left(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})}\right)\left[1+\lambda-2\lambda\left(1-e^{-(\theta x_i+\gamma x_i{}^{\beta})}\right)^{\alpha}\right], \tag{21}$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = \gamma \sum_{i=1}^{n} \frac{x_i{}^{\beta-1}[1+\beta.\ln x_i]}{\theta+\gamma\beta\, x_i{}^{\beta-1}} - \gamma \sum_{i=1}^{n} x_i{}^{\beta}.\ln x_i \;+\gamma(\alpha a-1)\sum_{i=1}^{n} \frac{e^{-(\theta x_i+\gamma x_i{}^{\beta})}.x_i{}^{\beta}.\ln x_i}{1- e^{-(\theta x_i+\gamma x_i{}^{\beta})}}$$

$$-\alpha\lambda\gamma(a+1)\sum_{i=1}^{n} \frac{(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha-1}\, e^{-(\theta x_i+\gamma x_i{}^{\beta})}.x_i{}^{\beta}.\ln x_i}{1+\lambda-2\lambda(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}} - a\alpha\gamma(b-1)$$

$$\times \sum_{i=1}^{n} \frac{(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha a-1}[1+\lambda-\lambda(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]^{a-1}}{1-(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha a}[1+\lambda-\lambda(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]^{a}}$$

$$\times e^{-(\theta x_i+\gamma x_i{}^{\beta})}.x_i{}^{\beta}.\ln x_i \times \{[1+\lambda-\lambda(1-e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}].$$

$$.-\lambda(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}\}, \tag{22}$$

$$\frac{\partial \mathcal{L}}{\partial \gamma} = \beta \sum_{i=1}^{n} \frac{x_i{}^{\beta-1}}{(\theta+\gamma\beta\, x_i{}^{\beta-1})} - \sum_{i=1}^{n} x_i{}^{\beta} + (\alpha a-1)\sum_{i=1}^{n} \frac{e^{-(\theta x_i+\gamma x_i{}^{\beta})}x_i{}^{\beta}}{[1- e^{-(\theta x_i+\gamma x_i{}^{\beta})}]}$$

$$-\lambda\alpha(a+1)\sum_{i=1}^{n} \frac{(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha-1}\, e^{-(\theta x_i+\gamma x_i{}^{\beta})}x_i{}^{\beta}}{[1+\lambda-2\lambda(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]} - \alpha a(b-1)$$

$$\times \sum_{i=1}^{n} \frac{(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha a-1}[1+\lambda-\lambda(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]^{a-1}}{\{1-(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha a}[1+\lambda-\lambda(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]^{a}\}}$$

$$\times e^{-(\theta x_i+\gamma x_i{}^{\beta})}\;.x_i{}^{\beta} \times \{[1+\lambda-\lambda(1-e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]-\lambda(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}\}, \tag{23}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \sum_{i=1}^{n} \frac{1}{\theta+\gamma\beta\, x_i{}^{\beta-1}} - \sum_{i=1}^{n} x_i \;+(\alpha a-1)\sum_{i=1}^{n} \frac{e^{-(\theta x_i+\gamma x_i{}^{\beta})}\;.x_i}{1- e^{-(\theta x_i+\gamma x_i{}^{\beta})}}$$

$$-\alpha\lambda(a+1)\sum_{i=1}^{n} \frac{(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha-1}\, e^{-(\theta x_i+\gamma x_i{}^{\beta})}\;.x_i}{1+\lambda-2\lambda(1- e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}} - \alpha a(b-1)$$

$$\times \sum_{i=1}^{n} \frac{(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha a-1}[1+\lambda-\lambda(1-e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]^{a-1}}{1-(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha a}[1+\lambda-\lambda(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]^{a}}$$

$$\times\, e^{-(\theta x_i+\gamma x_i{}^{\beta})}\,.x_i \times \{[1+\lambda-\lambda(1-e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]-\lambda(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}\}, \tag{24}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{i=1}^{n} \frac{1-2(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}}{1+\lambda-2\lambda(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}} + (a-1)\sum_{i=1}^{n} \frac{1-(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}}{1+\lambda-\lambda(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}}$$

$$-a(b-1)\times \sum_{i=1}^{n} \frac{(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha a}[1+\lambda-\lambda(1-e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]^{a-1}}{\{1-(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha a}[1+\lambda-\lambda(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]^{a}\}}$$

$$\times\{1-(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}\}, \tag{25}$$

$$\frac{\partial \mathcal{L}}{\partial a} = \frac{n}{a} + \alpha\sum_{i=1}^{n}\ln[1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})}] + \sum_{i=1}^{n}\ln[1+\lambda-\lambda(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]$$

$$-(b-1)\times \sum_{i=1}^{n} \frac{\{(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}[1+\lambda-\lambda(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]\}^{a}}{\{1-(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha a}[1+\lambda-\lambda(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]^{a}\}}$$

$$\times\ln\{(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}[1+\lambda-\lambda(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]\}, \tag{26}$$

and

$$\frac{\partial \mathcal{L}}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\ln\{1-(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha a}[1+\lambda-\lambda(1-\ e^{-(\theta x_i+\gamma x_i{}^{\beta})})^{\alpha}]^{a}\}. \tag{27}$$

The maximum likelihood estimator $\widehat{\underline{\vartheta}} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\theta}, \hat{\lambda}, \hat{a}, \hat{b})$ of $\underline{\vartheta} = (\alpha, \beta, \gamma, \theta, \lambda, a, b)$ is obtained by solving the nonlinear system of Eqs. (21) through (27). It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function. In order to compute the standard error and asymptotic confidence interval, we use the usual large sample approximation in which the maximum likelihood estimators can be treated as being approximately trivariate normal.

5.2. Maximum product spacing estimates

The maximum product spacing (MPS) method has been proposed by Cheng and Amin (1983). This method is based on an idea that the differences (Spacing) of the consecutive points should be identically distributed. The geometric mean of the differences is given as

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i}, \tag{28}$$

where the difference D_i is defined as

$$D_i = \int_{x_{(i-1)}}^{x_{(i)}} f(x, \alpha, \beta, \gamma, \theta, \lambda, a, b)\ dx; \quad i = 1, 2, \ldots, n+1, \tag{29}$$

where $F(x_{(0)}, \alpha, \beta, \gamma, \theta, \lambda, a, b) = 0$ and $F(x_{(n+1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b) = 0$. The MPS estimators $\hat{\alpha}_{PS},\ \hat{\beta}_{PS},\ \hat{\gamma}_{PS},\ \hat{\theta}_{PS},\ \hat{\lambda}_{PS},\ \hat{a}_{PS}$ and $\hat{b}_{PS}$ of $\alpha, \beta, \gamma, \theta, \lambda, a$ and b are obtained by maximizing the geometric mean (GM) of the differences. Substituting pdf of Kw-TEMW distribution in (29) and taking logarithm of the above expression, we will have

$$\log GM = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left[F\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - F\left(x_{(i-1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right)\right]. \tag{30}$$

The MPS estimators $\hat{\alpha}_{PS},\ \hat{\beta}_{PS},\ \hat{\gamma}_{PS},\ \hat{\theta}_{PS},\ \hat{\lambda}_{PS},\ \hat{a}_{PS}$ and $\hat{b}_{PS}$ of $\alpha, \beta, \gamma, \theta, \lambda, a$ and b can be obtained as the simultaneous solution of the following nonlinear equations:

$$\frac{\partial \log GM}{\partial \alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\alpha}\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - F'_{\alpha}\left(x_{(i-1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right)}{F\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - F\left(x_{(i-1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right)}\right] = 0,$$

$$\frac{\partial \log GM}{\partial \beta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\beta}\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - F'_{\beta}\left(x_{(i-1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right)}{F\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - F\left(x_{(i-1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right)}\right] = 0,$$

$$\frac{\partial \log GM}{\partial \gamma} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\gamma}\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - F'_{\gamma}\left(x_{(i-1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right)}{F\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - F\left(x_{(i-1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right)}\right] = 0,$$

$$\frac{\partial \log GM}{\partial \theta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\theta}\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - F'_{\theta}\left(x_{(i-1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right)}{F\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - F\left(x_{(i-1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right)}\right] = 0,$$

$$\frac{\partial \log GM}{\partial \lambda} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\lambda}\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - F'_{\lambda}\left(x_{(i-1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right)}{F\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - F\left(x_{(i-1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right)}\right] = 0,$$

$$\frac{\partial \log GM}{\partial a} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{a}\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - F'_{a}\left(x_{(i-1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right)}{F\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - F\left(x_{(i-1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right)}\right] = 0,$$

and

$$\frac{\partial \log GM}{\partial b} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{b}\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - F'_{b}\left(x_{(i-1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right)}{F\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - F\left(x_{(i-1)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right)}\right] = 0.$$

5.3. *Least square estimates*

Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be the ordered sample of size n drawn by the Kw-TEMW distribution. Then, the expectation of the empirical cumulative distribution function is defined as

$$E[F(X_{(i)})] = \frac{i}{n+1}; \quad i = 1, 2, \ldots, n. \tag{31}$$

The least square estimates $\hat{\alpha}_{LS}, \hat{\beta}_{LS}, \hat{\gamma}_{LS}, \hat{\theta}_{LS}, \hat{\lambda}_{LS}, \hat{a}_{LS}$ and $\hat{b}_{LS}$ of $\alpha, \beta, \gamma, \theta, \lambda, a$ and b are obtained by minimizing

$$Z(\alpha, \beta, \gamma, \theta, \lambda, a, b) = \sum_{i=1}^{n} \left[F\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - \frac{i}{n+1} \right]^2.$$

Therefore, $\hat{\alpha}_{LS}, \hat{\beta}_{LS}, \hat{\gamma}_{LS}, \hat{\theta}_{LS}, \hat{\lambda}_{LS}, \hat{a}_{LS}$ and $\hat{b}_{LS}$ of $\alpha, \beta, \gamma, \theta, \lambda, a$ and b can be obtained as the solution of the following system of equations:

$$\frac{\partial Z(\alpha, \beta, \gamma, \theta, \lambda, a, b)}{\partial \alpha} = \sum_{i=1}^{n} F'_{\alpha}\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) \times \left(F\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - \frac{i}{n+1} \right) = 0,$$

$$\frac{\partial Z(\alpha, \beta, \gamma, \theta, \lambda, a, b)}{\partial \beta} = \sum_{i=1}^{n} F'_{\beta}\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) \times \left(F\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - \frac{i}{n+1} \right) = 0,$$

$$\frac{\partial Z(\alpha, \beta, \gamma, \theta, \lambda, a, b)}{\partial \theta} = \sum_{i=1}^{n} F'_{\theta}\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) \times \left(F\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - \frac{i}{n+1} \right) = 0,$$

$$\frac{\partial Z(\alpha, \beta, \gamma, \theta, \lambda, a, b)}{\partial \lambda} = \sum_{i=1}^{n} F'_{\lambda}\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) \times \left(F\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - \frac{i}{n+1} \right) = 0,$$

$$\frac{\partial Z(\alpha, \beta, \gamma, \theta, \lambda, a, b)}{\partial a} = \sum_{i=1}^{n} F'_{a}\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) \times \left(F\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - \frac{i}{n+1} \right) = 0,$$

and

$$\frac{\partial Z(\alpha, \beta, \gamma, \theta, \lambda, a, b)}{\partial b} = \sum_{i=1}^{n} F'_{b}\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) \times \left(F\left(x_{(i)}, \alpha, \beta, \gamma, \theta, \lambda, a, b\right) - \frac{i}{n+1} \right) = 0,$$

These nonlinear equations can be routinely solved using Newton's method or fixed point iteration techniques. The subroutines to solve nonlinear optimization problem are available in R (Team, 2012) software namely optim(), nlm(), bbmle(), etc. We used nlm() package for optimizing (19).

6. Order statistics

Let $X_1, X_2, \ldots, X_n$ denote n-independent random variables from a distribution function $F_X(x)$ with pdf $f_X(x)$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the ordered sample arrangement. The pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!\,(n-j)!} f_X(x)[F_X(x)]^{j-1}.[1 - F_X(x)]^{n-j}, \qquad j = 1, 2, \ldots, n.$$

Then from (7) and (8) the pdf of $X_{(j)}$ is given by

$$\begin{aligned} f_{X_{(j)}}(x) = {} & \frac{n!}{(j-1)!\,(n-j)!} \times ab\alpha(\theta + \gamma\beta\, x^{\beta-1})\, e^{-(\theta x + \gamma x^{\beta})} \\ & \times [1 + \lambda - 2\lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}] \times [1 - e^{-(\theta x+\gamma x^{\beta})}]^{\alpha a-1} \\ & \times [1 + \lambda - \lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}]^{a-1} \\ & \times [1 - \{1 - (1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha a}[1 + \lambda - \lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}]^{a}\}^{b}]^{j-1} \\ & \times \{1 - (1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha a}[1 + \lambda - \lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}]^{a}\}^{bn-bj+b-1}. \end{aligned} \tag{32}$$

Therefore, the pdfs of the smallest and the largest order statistic are respectively given by

$$\begin{aligned} f_{X_{(1)}}(x) = {} & n.a.b.\,\alpha.(\theta + \gamma\beta\, x^{\beta-1})\, e^{-(\theta x+\gamma x^{\beta})} \times [1 + \lambda - 2\lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}] \\ & \times [1 - e^{-(\theta x+\gamma x^{\beta})}]^{\alpha a-1} \times [1 + \lambda - \lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}]^{a-1} \\ & \times \{1 - (1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha a}[1 + \lambda - \lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}]^{a}\}^{bn-1}, \end{aligned} \tag{33}$$

and

$$\begin{aligned} f_{X_{(n)}}(x) = {} & nab\alpha(\theta + \gamma\beta\, x^{\beta-1})\, e^{-(\theta x+\gamma x^{\beta})} \times [1 + \lambda - 2\lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}] \\ & \times [1 - e^{-(\theta x+\gamma x^{\beta})}]^{\alpha a-1} \times [1 + \lambda - \lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}]^{a-1} \\ & \times [1 - \{1 - (1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha a}[1 + \lambda - \lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}]^{a}\}^{b}]^{n-1} \\ & \times \{1 - (1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha a}[1 + \lambda - \lambda(1 - e^{-(\theta x+\gamma x^{\beta})})^{\alpha}]^{a}\}^{b-1}. \end{aligned} \tag{34}$$

7. Simulation algorithms

In this section we give an algorithm, using R software, to simulate data from the Kw-TEMW model.

7.1. Inverse cdf method

Since the probability integral transformation cannot be applied explicitly, we, therefore, need to follow the following steps for generating a sample of size n from Kw-TEMW $(\alpha, \beta, \gamma, \theta, \lambda, a, b)$:

Table 2. Estimates and mean square errors (in second row of each cell) of the proposed estimators with varying sample size.

	MLE						
n	α	β	γ	θ	λ	a	b
10	1.4442	2.7901	0.7470	1.0175	− 0.4004	2.0532	3.8501
	1.2717	1.4205	0.6798	0.1300	0.1160	0.0884	1.5461
20	1.2293	2.8466	0.8240	1.0022	− 0.40185	2.0302	3.8466
	0.3925	0.4783	0.2969	0.0507	0.0489	0.0522	0.4658
30	1.1573	2.8786	0.8583	− 0.9949	0.4006	2.0181	3.8466
	0.2298	0.2972	0.1902	0.0307	0.0299	0.0364	0.2583
40	1.1215	2.9352	0.8805	0.9952	− 0.5034	4.0146	3.8654
	0.1590	0.2510	0.1385	0.0224	0.0223	0.312	0.2419
50	1.0965	2.9544	0.8932	0.9954	− 0.6013	4.0203	3.8742
	0.1252	0.1682	0.1141	0.0184	0.0181	0.0269	0.1802
60	1.0804	2.9654	0.9031	0.9956	− 0.6017	2.0106	3.9612
	0.0998	0.1379	0.0931	0.0148	0.0149	0.0187	0.1379
70	1.0711	2.9773	0.9140	0.9966	− 0.6020	2.0101	3.9700
	0.0852	0.1175	0.0805	0.0125	0.0129	0.0118	0.1175
80	1.0553	2.9900	0.9138	0.9978	− 0.6027	2.0100	3.9932
	0.0691	0.1009	0.0681	0.0106	0.0110	0.0142	0.1089
90	1.0511	2.9922	0.9219	0.9992	− 0.6238	2.0104	3.9943
	0.0619	0.0889	0.0610	0.0095	0.0099	0.0093	0.0784
100	1.0481	2.9938	0.9290	0.9982	− 0.63025	2.0087	3.9938
	0.0565	0.0785	0.0557	0.0087	0.0091	0.0089	0.0675

Step 1. Set $n,\ \alpha,\ \beta,\ \gamma,\ \theta,\ \lambda,\ a,\ b$ and initial value x^0.
Step 2. Generate $U \sim$ Uniform $(0, 1)$.
Step 3. Update x^0 by using the Newton's formula.

$$x^* = x^0 - R(x^0, \Theta)$$

where, $R(x^0, \Theta) = \frac{F_X(x^0,\Theta)-U}{f_X(x^0,\Theta)}$, $F_X(.)$ and $f_X(.)$ are cdf and pdf of Kw-TEMW distribution, respectively.
Step 4. If $|x^0 - x^*| \le \epsilon$, (very small, $\varepsilon > 0$ tolerance limit), then store $x = x^*$ as a sample from Kw-TEMW distribution.
Step 5. If $|x^0 - x^*| > \epsilon$, then, set $x^0 = x^*$ and go to Step 3.
Step 6. Repeat Steps 3–5, n times for $x_1, x_2, \ldots, x_n$ respectively.

This subsection explores the behaviors of the proposed estimators in terms of their mean square error on the basis of simulated samples from pdf of Kw-TEMW with varying sample sizes. We take $\alpha = 1, \beta = 3, \gamma = 1, \theta = 1, \lambda = -0.65, a = 2, b = 4$ arbitrarily and $n = 10(10)100$. The algorithms are coded in R (Team, 2012), and the algorithm given in 7.1 has been used for simulation purposes. We calculate MLE estimators of $\alpha, \beta, \gamma, \lambda, a$, and b based on each generated sample. This simulation is repeated 1,000 of times, and average estimates with corresponding mean square errors are computed and reported in Table 2.

From Table 2, it can be clearly observed that as sample size increases the mean square error decreases, which proves the consistency of the estimators.

8. Applications

In this section, we use two real datasets to compare the fits of the Kw-TEMW distribution with four submodels and 10 non-nested models. In each case, the parameters are estimated by maximum likelihood as described in Section 7, using the SAS code (PROC NLMIXED).

The first data works with nicotine measurements made from several brands of cigarettes in 1998. The data have been collected by the Federal Trade Commission, which is an independent agency of the US government, whose main mission is the promotion of consumer protection.

The report entitled tar, nicotine, and carbon monoxide of the smoke of 1,206 varieties of domestic cigarettes for the year of 1998 is available at http://www.ftc.gov/reports/tobacco and consists of the datasets and some information about the source of the data, smokers behavior, and beliefs about nicotine, tar, and carbon monoxide contents in cigarettes. The free form dataset can be found at http://pw1.netcom.com/rdavis2/smoke.html. The site http://home.att.net/rdavis2/cigra.html contains $n = 384$ observations. We analyzed data on nicotine, measured in milligrams per cigarette, from several cigarette brands. Some summary statistics for the nicotine data are as follows:

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.1000	0.6000	0.9000	0.8526	1.1000	2.000

In order to compare the two distribution models, we consider criteria like Kolmogorov Smirnov (KS), $-2\mathcal{L}$, Akaike information criterion (AIC), corrected Akaike information criterion (AIC_C), and Bayesian information criterion (BIC) for the dataset. The better distribution corresponds to smaller KS, $-2\mathcal{L}$, AIC, and AIC_C values:

$$\text{AIC} = -2\mathcal{L} + 2k,$$
$$\text{AIC}_C = -2\mathcal{L} + \left(\frac{2kn}{n-k-1}\right),$$

and

$$\text{BIC} = -2\mathcal{L} + k\log(n),$$

where $\mathcal{L}$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters, and n is the sample size.

Also, for calculating the values of KS we use the sample estimates of $\alpha,\ \beta,\ \gamma, \theta,\ \lambda,\ a,$ and b. Table 3 shows the parameter estimation based on the maximum likelihood and least square estimation, and gives the values of the criteria AIC, AIC_C, BIC, and KS test. The values in Table 3 indicate that the Kw-TEMW distribution leads to a better fit over all the other models.

A density plot compares the fitted densities of the models with the empirical histogram of the observed data (1). The fitted density for the Kw-TEMW model is closer to the empirical histogram than the fits of the TEMW submodel.

Table 4 shows the parameter estimation based on the maximum likelihood and least square estimation, and gives the values of the criteria AIC, AIC_C, BIC, and KS test for the Greenwich data. In the following, we shall compare the proposed model with several other lifetime distributions based on the Greenwich data.

The summary of the data is as follows

We compare the new model with the following models:

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
43.00	82.75	102.50	141.80	149.20	598.00

Table 3. MLEs, LSE, the measures AIC, AIC_C, and BIC, and KS test to nicotine data for nested models.

Model	Parameter estimates	Standard error	Least square estimates	$-2LL$	AIC	AIC_C	BIC	KS
Kw-TEMW	$\hat{a} = 2.141139$	0.015711	5.1063669	210.2781	224.2781	224.3716	259.944	0.087
	$\hat{b} = 0.247417$	0.013630	0.3485265					
	$\hat{\alpha} = 0.70512$	0.0038388	0.4165127					
	$\hat{\beta} = 3.6802$	0.0058677	4.2257952					
	$\hat{\gamma} = 1.85414$	0.011762	1.2261089					
	$\hat{\theta} = 0.518697$	0.0056323	0.5157128					
	$\hat{\lambda} = 0.995983$	0.0012689	0.9769069					
TEMW	$\hat{\alpha} = 1.5003015$	0.4904903	1.1832182	217.0562	227.0562	227.1062	252.532	0.120
	$\hat{\beta} = 2.5918138$	0.2676644	2.9987029					
	$\hat{\gamma} = 1.1871988$	0.2615845	1.1725852					
	$\hat{\theta} = 0.6995263$	0.4976821	0.5008003					
	$\hat{\lambda} = -0.63255$	0.2303809	− 0.6175747					
TER	$\hat{\alpha} = 1.173716$	0.1410806	1.073245	224.886	230.886	230.906	246.172	0.122
	$\hat{\beta} = 1.31720$	0.0394503	1.12341					
	$\hat{\lambda} = -0.68126$	0.120449	− 0.5487					
TR	$\hat{\alpha} = 0.5555$	0.0135	0.8992271	242.448	246.448	246.458	256.638	0.124
	$\hat{\lambda} = -0.7718$	0.0728	− 0.64391					
R	$\hat{\alpha} = 0.6475$	0.0175	0.7735	284.714	286.714	286.717	291.809	0.184

Table 4. MLEs (standard errors in parentheses) and the measures AIC, AIC_C, and BIC to Greenwich data for non-nested models.

Model	Estimates					$-2LL$	AIC	AIC_C	BIC
Kw-TEMW	$\hat{a}$ **=12.3977 (9.5311)**	$\hat{b}$ **=12.3977 (0.03451)**	$\hat{\alpha}$ **=2.9777 (1.6921)**	$\hat{\beta}$ **=0.8314 (0.8115)**	$\hat{\gamma}$ **=0.01290 (0.07242)**	789.6	803.6	805.3	819.5
		$\hat{\theta}$ **= 0.02959 (0.02857)**	$\hat{\lambda}$ **= 0.9951 (0.008845)**						
Kw-MW	$\hat{a} = 1453.03$ (3277.42)	$\hat{b} = 0.5353$ (0.3477)	$\hat{\alpha} = 1.3013$ (1.3472)	$\hat{\gamma} = 0.3969$ (0.1914)	$\hat{\lambda} = 1E-8$ (.)	796.3	806.3	807.2	817.7
B-MW	$\hat{\alpha} = 0.3648$ (0.08804)	$\hat{\gamma} = 0.6054$ (0.06603)	$\hat{\lambda} = 1E-8$ (.)	$\hat{a} = 107.23$ (.)	$\hat{b} = 0.4221$ (0.1262)				
B-GE	$\hat{a} = 3.1133$ (1.2236)	$\hat{b} = 0.1985$ (0.02572)	$\hat{\lambda} = 0.06031$ (0.001259)	$\hat{\alpha} = 8.5670$ (3.6029)		803.0	811.0	811.6	820.1
W-P	$\hat{k} = 5.6793$ (.)	$\hat{c} = 2.1614$ (0.2336)	$\hat{\gamma} = 7.2290$ (.)	$\hat{\theta} = 38.4702$ (2.9693)		798.7	806.7	807.3	815.8
WG	$\hat{\alpha} = 3.3351$ (0.3371)	$\hat{\beta} = 0.001287$ (0.001550)	$\hat{p} = 0.9984$ (0.006394)			803.1	809.1	809.4	815.9
GL	$\hat{\alpha} = 10.5573$ (3.8060)	$\hat{\beta} = 9.4347$ (15.0265)	$\hat{a} = 27.5875$ (24.6720)			801.5	807.5	807.9	814.4
GG	$\hat{\alpha} = 28.4789$ (23.6912)	$\hat{\beta} = 0.3732$ (0.02504)	$\hat{\lambda} = 21.2331$ (3.5333)			812.1	818.1	818.5	825.0
EW	$\hat{\alpha} = 3.7224$ (4.0620)	$\hat{\beta} = 0.4982$ (0.09967)	$\hat{\upsilon} = 247.25$ (285.99)			821.9	827.9	828.3	834.8
QL	$\hat{\alpha} = 0.01410$ (0.001175)	$\hat{\alpha} = 1E-8$ (.)				828.3	832.3	832.5	836.9
L	$\hat{\alpha} = 0.01400$ (0.001167)					828.9	830.9	831.0	833.2

- The Kumaraswamy modified Weibull (Kw-MW) distribution introduced by Cordeiro et al. (2014). The pdf of Kw-MW distribution (with five parameters a, b, α, γ and λ) is

$$f(x) = ab\alpha x^{\gamma-1}(\gamma+\lambda x)\, e^{\left(\lambda x-\alpha x^{\gamma}\exp(\lambda x)\right)}[1-e^{-\alpha x^{\gamma}\exp(\lambda x)}]^{a-1} \times\{1-[1-e^{-\alpha x^{\gamma}\exp(\lambda x)}]^{a}\}^{b-1}, \quad x>0$$

where $\alpha > 0$ is scale parameter,γ, a, $b > 0$ are shape parameters, and $\lambda > 0$ is an acceleration parameter.

- The Beta modified Weibull (B-MW) distribution introduced by Silva et al. (2010). The pdf of B-MW distribution (with five parameters a, b, α, γ, and λ) is

$$f(x) = \frac{\alpha x^{\gamma-1}(\gamma+\lambda x)\, e^{\lambda x}}{B(a,b)} e^{\left(-\alpha b x^{\gamma}\exp(\lambda x)\right)}[1-e^{-\alpha x^{\gamma}\exp(\lambda x)}]^{a-1}, \quad x>0$$

where $\alpha > 0$ is scale parameter, γ, a, $b > 0$ are shape parameters, and $\lambda > 0$ is an acceleration parameter, and $B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$ is the complete beta function.

- The Beta generalized exponential (B-GE) distribution introduced by Barreto-Souza et al. (2010). The pdf of B-GE distribution (with four parameters α, θ, β, γ, and λ) is

$$f(x) = \frac{\alpha\lambda}{B(a,b)} e^{-\lambda x}(1-e^{-\lambda x})^{\alpha a-1}\{1-(1-e^{-\lambda x})^{\alpha}\}^{b-1}, \quad x>0$$

where $\lambda > 0$ is scale parameter and α, a, $b > 0$ are shape parameters.

- The Weibull-Pareto (W-P) distribution introduced by Alzaatreh et al. (2013). The pdf of W-P distribution (with four parameters k, c, γ, and θ) is

$$f(x) = \frac{kc}{\gamma x}\left\{\frac{k}{\gamma}\log\left(\frac{x}{\theta}\right)\right\}^{c-1} \exp\left\{\left[\frac{k}{\gamma}\log\left(\frac{x}{\theta}\right)\right]^{c}\right\}, \quad x>\theta$$

where $\theta > 0$ is scale parameter and k, c, $\gamma > 0$ are shape parameters.

- The Weibull-Geometric (W-G) distribution introduced by Barreto-Souza et al. (2011). The pdf of W-G distribution (with three parameters α, β, and p) is

$$f(x) = \alpha\beta^{\alpha}\left(1-p\right)x^{\alpha-1}e^{-(\beta x)^{\alpha}}\left\{1-pe^{-(\beta x)^{\alpha}}\right\}^{-2}, \quad x>0$$

where $\beta > 0$ is scale parameter, $\alpha > 0$ is shape parameter, and p is the probability of success in the geometric distribution.

- The Gamma Lomax (GL) distribution introduced by Cordeiro et al. (2015). The pdf of GL distribution (with three parameters α, β, and a) is

$$f(x) = \frac{\alpha\beta^{\alpha}}{\Gamma(a)}(\beta+x)^{-\alpha-1}\left\{-\alpha\log\left[\frac{\beta}{\beta+x}\right]\right\}^{a-1}, \quad x>0$$

where $\beta > 0$ is scale parameter and α, $a > 0$ are shape parameters.

- The Generalized Gamma (GG) distribution introduced by Stacy (1962). The pdf of GL distribution (with three parameters α, β, and λ) is

$$f(x) = \frac{\alpha\beta}{\Gamma(\lambda)}(\alpha x)^{\lambda\beta-1}e^{-(\alpha x)^{\beta}}, \quad x>0$$

where $\alpha > 0$ is scale parameter and β, $\lambda > 0$ are shape parameters.

- The extended Weibull (E-W) distribution introduced by Marshall and Olkin (1997). The pdf of E-W distribution (with three parameters α, β, and υ) is

$$f(x) = \frac{\upsilon\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\left[1-(1-\upsilon)\,e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{-2}, \quad x>0$$

where $\alpha > 0$ is scale parameter and $\beta > 0$ is shape parameter, and υ is tilt parameter.

- The Quasi Lindley (QL) distribution introduced by Shanker and Mishra (2013). The pdf of QL distribution (with two parameters α and θ) is

$$f(x) = \frac{\theta}{\alpha + 1}(\alpha + \theta x)\, e^{-\theta x}, \qquad x > 0$$

where $\theta > 0$ is scale parameter and $\alpha > -1$ is parameter.

- The Lindley (L) distribution introduced by Lindley (1958). The pdf of L distribution (with one parameter θ) is

$$f(x) = \frac{\theta^2}{\theta + 1}(1 + x)\, e^{-\theta x}, \qquad x > 0$$

where $\theta > 0$ is scale parameter.

These results indicate that the Kw-TEMW model has the lowest AIC, AIC_C, and BIC values among the fitted models. The values of these statistics indicate that the Kw-TEMW model provides the best fit to all of the data.

9. Concluding remarks

There has been a great interest among statisticians and applied researchers in constructing flexible lifetime models to facilitate better modeling of survival data. Consequently, a significant progress has been made towards the generalization of some well-known lifetime models and their successful application to problems in several areas. In this article, we introduce a new seven-parameter distribution obtained using the Kumaraswamy generalization technique. We refer to the new model as the Kw-TEMW distribution and study some of its mathematical and statistical properties. We provide the pdf, the cdf, and the hazard rate function of the new model, and explicit expressions for the moments. The model parameters are estimated by maximum likelihood and method of moment. The new model is compared with the nested and non-nested models and provides consistently better fit than other classical lifetime models. We hope that the proposed distribution will serve as an alternative model to other models available in the literature for modeling positive real data in many areas such as engineering, survival analysis, hydrology, and economics.

Acknowledgments

The authors are grateful to two anonymous referees for their valuable comments and suggestions that improved an earlier version of the article.

Funding

This project was supported by King Saud University, Deanship of Scientific Research, College of Sciences Research Center.

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