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Bayesian and Non-Bayesian Estimation of Composed Inverted Generalized Exponential – Exponential Distribution

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Abstract: In this paper, we introduce the composed- inverted generalized exponential- exponential (C-IGEE) distribution. The point and interval estimation based on maximum likelihood are proposed. We also obtain the Bayes estimates of the unknown parameters under the assumption of independent gamma priors. The Bayes estimates of the unknown parameters cannot be obtained in closed form. So Markov Chain Monte Carlo (MCMC) method has been used to compute the approximate Bayes estimates under the squared error loss function and also construct the highest posterior density (HPD) intervals. Further, a simulation study was conducted to compare the performances of Bayes estimators with corresponding maximum likelihood estimators.

Keywords: Star Shaped, Maximum Likelihood Estimation, Bayesian Estimation, Monte Carlo Markov Chain, Metropolis-Hastings Algorithm.

1 Introduction

In the last few years, several ways of generating new probability distributions from classic ones were developed and discussed. Some well-known generalized classes (or generators) are the beta generalized family of distributions introduced by Eugene et al. (2002), the general rank transmutation (GRT) defined by Shaw and Buckley (2007), Kumaraswamy-G family of distributions introduced by Cordeiro and de Castro (2011), Transformed-Transformer (T-X) by Alzaatreh et al. (2013), odd exponentiated half-logistic-G family of distributions by Afify et al. (2017), Fattah and Ahmed (2018) introduced a new method for generating a wide number of classes with known characteristics in the reliability theory based on the star shaped property and they refer to the new class as the composed$-G\ Q$ family or shortly ($C - G\ Q$) family and odd Lomax-G by Cordeiro et al. (2019).

Muhammed et al. (2019) introduced a new family using the new approach that introduced by Fattah and Ahmed (2018). This new family is the composed inverted generalized exponential family (C- IGEQ) and it enjoys the star-shaped property. They illustrated a set of important results in the theoretical reliability hold for a new family. These results make the family more important in applications.

Definition: Composed – G Q family

Let G and Q be two arbitrary continuous cumulative distribution functions of non-negative absolutely continuous random variables, G be strictly increasing on its support, and $G(0) = Q(0)$. Now define a cumulative distribution function (cdf), F, out of G and Q (called the composed- G Q family shortly (C- G Q)) as follows: [see Fattah and Ahmed (2018)]

The corresponding probability density function (pdf) is given by

Where g and q are the corresponding densities of G and Q respectively.

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These results extracted from Barlow and Proschan (1981) and mentioned as below.

Let F and G be continuous distributions, G be strictly increasing on its support, and $F(0) = G(0) = 0$. Then F is star-shaped with respect to G (written $\underset{*}{<} G$) if $G^{-1}(F(x))$ is star-shaped [that is, $\left(\frac{1}{x}\right) G^{-1}(F(x))$ is increasing for $x \geq 0$]. Then

a. $F \underset{c}{<} G$ implies $F \underset{*}{<} G$ (where $\underset{c}{<}$ implies the convex ordering) .
b. The relationship $F \underset{c}{<} G$is unaffected by a translation transformation of either F and G, assuming the random variables remain non-negative.
c. The relationship $F \underset{*}{<} G$ may be destroyed by a translation transformation of either F and G, assuming the random variables remain non-negative.
d. Let $G(x) = 1 - e^{-\lambda x}$, F be a continuous distribution function, with $F(0) = 0$. Then $F \underset{c}{<} G$ is equivalent to F increasing failure rate (IFR).
e. Let $G(x) = 1 - e^{-\lambda x}$, F be a continuous distribution function, with $F(0) = 0$. Then $F \underset{*}{<} G$ is equivalent to F increasing failure rate average (IFRA).

The Single Crossing Property. Let $F \underset{*}{<} G$, then

a. $\bar{F}(x)$ crosses $\bar{G}(\theta x)$ at most one, and from above, as x increases from 0 to ∞, for each $\theta > 0$.
b. If, in addition, F and G have the same mean, then a single crossing does occur, and F has smaller variance than G .
c. If we take G to be the exponential distribution, then F must be IFRA by the previous results.

The rest of the paper is organized as follows. In Section 2, the composed inverted generalized exponential Family and the composed- inverted generalized exponential- exponential (C-IGEE) distribution are proposed and demonstrate the graphs of the probability density function (pdf) and cumulative distribution function (cdf) of C-IGEE. In Section 3, the MLEs are obtained for the unknown parameters. In section 4, the Bayes estimators of the unknown parameters using MCMC are introduced. In Section 5, we compare the performance of MLE and Bayesian estimates based on simulation studies. Finally, in Section 6, we conclude the paper.

2 The Composed Inverted Generalized Exponential-Exponential Distribution

The cdf and pdf of the C- IGEQ family are given, respectively, by [Muhammed et al. (2019)] as follows

$$F(x) = 1 - (1 - e^{\frac{-\beta}{x\,Q(x)}})^{\alpha}, \quad x, \alpha, \beta > 0 \tag{1}$$

$$f(x) = \frac{\alpha\beta}{(x\,Q(x))^2} e^{-\frac{\beta}{x\,Q(x)}} (1 - e^{-\frac{\beta}{x\,Q(x)}})^{\alpha-1} \{x\,q(x) + Q(x)\} \tag{2}$$

Substituting $q(x) = \theta\ e^{-\theta x}$ and accordingly $Q(x) = 1 - e^{-\theta x}$ (where $x, \theta > 0$) into (1) and (2) one gets the composed- inverted generalized exponential- exponential (C-IGEE) distribution with cdf

$$f(x, \alpha, \beta, \theta) = \frac{\alpha\beta}{\left(x\left(1-e^{-\theta x}\right)\right)^2}\ e^{\frac{-\beta}{x\left(1-e^{-\theta x}\right)}} \left[1 - e^{\frac{-\beta}{x\left(1-e^{-\theta x}\right)}}\right]^{\alpha-1} \left\{x\,\theta\ e^{-\theta x} + 1 - e^{-\theta x}\right\}, \qquad \alpha, \beta, \theta, x > 0 \tag{3}$$

and its corresponding cdf

(4)

Figure (1) illustrates plots of the pdf and cdf of C-IGEE distribution for selected values of the parameters.

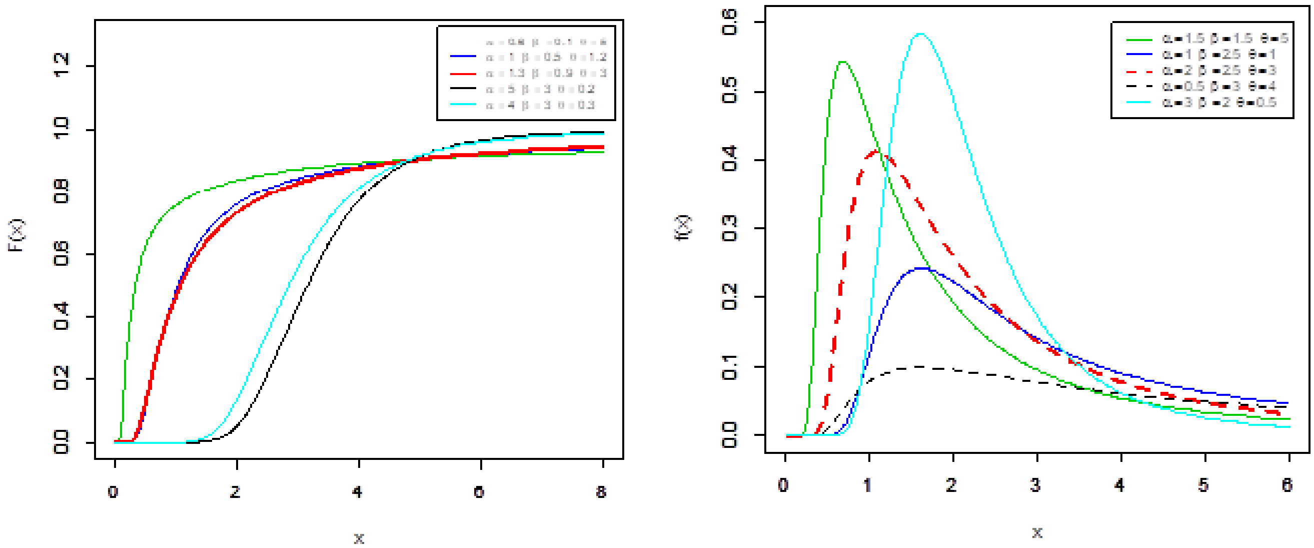

Fig. 1: Plots of pdf and cdf of C-IGEE model for some parameter values.

3 Maximum Likelihood Estimation

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the C-IGEE distribution from complete samples only.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from C-IGEE$(x; \boldsymbol{\phi}), \boldsymbol{\phi} \equiv (\alpha, \beta, \theta)$. The log likelihood function for the vector of parameters $\boldsymbol{\phi} \equiv (\alpha, \beta, \theta)$ can be written as

$$+(\alpha-1)\sum_{i=1}^{n} \ln\left(1 - e^{\frac{-\beta}{x_i\left(1-e^{-\theta x_i}\right)}}\right) + \sum_{i=1}^{n} \ln\left\{x_i\theta\ e^{-\theta x_i} + 1 - e^{-\theta x_i}\right\}. \quad (5)$$

$$ln\, L = n\, ln\, \alpha + n\, ln\, \beta - 2u(x_i,\theta) - v(x_i,\beta,\theta) + (\alpha-1)\psi(x_i,\beta,\theta) + q(x_i,\theta),$$

where,

$u(x_i,\theta) = \sum_{i=1}^{n} log\, w(x_i,\theta)$, $w(x_i,\theta) = x_i\left(1 - e^{-\theta x_i}\right), S(x_i,\beta,\theta) = \frac{\beta}{w(x_i,\theta)}$

$z(x_i,\beta,\theta) = 1 - e^{-S(x_i,\beta,\theta)}$, $v(x_i,\beta,\theta) = \sum_{i=1}^{n} S(x_i,\beta,\theta)$, $\psi(x_i,\beta,\theta) = \sum_{i=1}^{n} log\, z(x_i,\beta,\theta)$,

$q(x_i,\theta) = \sum_{i=1}^{n} log\, C(x_i,\theta)$ and $C(x_i,\theta) = x_i\theta e^{-\theta x_i} + 1 - e^{-\theta x_i}$.

By taking the partial derivatives of the log-likelihood function with respect to α, β and θ, we get

$$\frac{\partial\, ln\, L}{\partial\alpha} = \frac{n}{\alpha} + \psi(x_i,\beta,\theta).$$

$$\frac{\partial\, ln\, L}{\partial\beta} = \frac{n}{\beta} - v_\beta(x_i,\theta) + (\alpha-1)\psi_\beta(x_i,\beta,\theta).$$

$$\frac{\partial \ln L}{\partial \theta} = -2u_\theta(x_i,\theta) - v_\theta(x_i,\beta,\theta) + (\alpha-1)\psi_\theta(x_i,\beta,\theta) + q_\theta(x_i,\theta).$$

where,

$$v_\beta(x_i,\theta) = \frac{\partial v(x_i,\beta,\theta)}{\partial \beta} = \sum_{i=1}^{n} S_\beta(x_i,\theta),$$

$$S_\beta(x_i,\theta) = \frac{\partial S(x_i,\beta,\theta)}{\partial \beta} = \frac{1}{w(x_i,\theta)},$$

$$\psi_\beta(x_i,\beta,\theta) = \frac{\partial \psi(x_i,\beta,\theta)}{\partial \beta} = \sum_{i=1}^{n} \frac{z_\beta(x_i,\beta,\theta)}{z(x_i,\beta,\theta)},$$

$$z_\beta(x_i,\beta,\theta) = \frac{\partial z(x_i,\beta,\theta)}{\partial \beta} = e^{-S(x_i,\beta,\theta)} S_\beta(x_i,\theta),$$

$$u_\theta(x_i,\theta) = \frac{\partial u(x_i,\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{1}{w(x_i,\theta)} \frac{\partial w(x_i,\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{w_\theta(x_i,\theta)}{w(x_i,\theta)},$$

$$w_\theta(x_i,\theta) = \frac{\partial w(x_i,\theta)}{\partial \theta} = {x_i}^2 e^{-\theta x_i},$$

$$v_\theta(x_i,\beta,\theta) = \frac{\partial v(x_i,\beta,\theta)}{\partial \theta} = \sum_{i=1}^{n} S_\theta(x_i,\beta,\theta),$$

$$S_\theta(x_i,\beta,\theta) = \frac{\partial S(x_i,\beta,\theta)}{\partial \theta} = \frac{-\beta}{w^2(x_i,\theta)} \frac{\partial w(x_i,\theta)}{\partial \theta} = \frac{-\beta e^{-\theta x_i}}{(1-e^{-\theta x_i})^2},$$

$$\psi_\theta(x_i,\beta,\theta) = \frac{\partial \psi(x_i,\beta,\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{z_\theta(x_i,\beta,\theta)}{z(x_i,\beta,\theta)},$$

$$z_\theta(x_i,\beta,\theta) = \frac{\partial z(x_i,\beta,\theta)}{\partial \theta} = e^{-S(x_i,\beta,\theta)} S_\theta(x_i,\beta,\theta),$$

$$q_\theta(x_i,\theta) = \sum_{i=1}^{n} \frac{C_\theta(x_i,\theta)}{C(x_i,\theta)} \quad , \ and\ \ C_\theta(x_i,\theta) = \frac{\partial C(x_i,\theta)}{\partial \theta} = x_i e^{-\theta x_i}(2-\theta x_i).$$

The MLEs can be determined numerically from the solution of nonlinear system of equations; subsequently, these solutions will yield the MLE estimators $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$.

To construct asymptotic confidence interval, we need to obtain the observed Fisher information. Therefore, the Fisher information matrix is given by

(6)

where,

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = \frac{-n}{\alpha^2}.$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = \frac{-n}{\beta^2} + (\alpha-1)\psi_{\beta\beta}(x_i,\beta,\theta).$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -2u_{\theta\theta}(x_i,\theta) - v_{\theta\theta}(x_i,\beta,\theta) + (\alpha-1)\psi_{\theta\theta}(x_i,\beta,\theta) + q_{\theta\theta}(x_i,\theta).$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} = \psi_\beta(x_i,\beta,\theta).$$

$$\frac{\partial^2 \ln L}{\partial \theta \partial \alpha} = \psi_\theta(x_i,\beta,\theta).$$

$$\frac{\partial^2 \ln L}{\partial \theta \partial \beta} = -v_{\beta\theta}(x_i,\theta) + (\alpha-1)\psi_{\beta\theta}(x_i,\beta,\theta),$$

where,

$$\psi_{\beta\beta}(x_i,\beta,\theta) = \frac{\partial^2 \psi(x_i,\beta,\theta)}{\partial \beta^2} = \sum_{i=1}^{n} \frac{\partial}{\partial \beta}\left[\frac{z_\beta(x_i,\beta,\theta)}{z(x_i,\beta,\theta)}\right] = \sum_{i=1}^{n}\left[\frac{z_{\beta\beta}(x_i,\beta,\theta)}{z(x_i,\beta,\theta)} - \frac{z_\beta^2(x_i,\beta,\theta)}{z^2(x_i,\beta,\theta)}\right].$$

$$z_{\beta\beta} = \frac{\partial z_\beta(x_i,\beta,\theta)}{\partial \beta} = \frac{\partial}{\partial \beta}\left[e^{-S(x_i,\beta,\theta)} S_\beta(x_i,\theta)\right] = -e^{-S(x_i,\beta,\theta)} {S_\beta}^2(x_i,\theta).$$

$$v_{\beta\theta} = \sum_{i=1}^{n} S_{\beta\theta}(x_i,\theta), \quad S_{\beta\theta} = \frac{\partial}{\partial \theta}\left(\frac{1}{w(x_i,\theta)}\right) = \frac{-w_\theta(x_i,\theta)}{w^2(x_i,\theta)}.$$

$$\psi_{\beta\theta}(x_i,\beta,\theta) = \sum_{i=1}^{n} \frac{\partial}{\partial \theta}\left(\frac{z_\beta(x_i,\beta,\theta)}{z(x_i,\beta,\theta)}\right) = \sum_{i=1}^{n}\left[\frac{1}{z(x_i,\beta,\theta)} z_{\beta\theta}(x_i,\beta,\theta) - \frac{z_\theta(x_i,\beta,\theta)}{z^2(x_i,\beta,\theta)} z(x_i,\beta,\theta)\right].$$

$$z_{\beta\theta}(x_i,\beta,\theta) = \frac{\partial}{\partial \theta}\left(e^{-S(x_i,\beta,\theta)} S_\beta(x_i,\theta)\right) = e^{-S(x_i,\beta,\theta)} S_{\beta\theta}(x_i,\theta) - e^{-S(x_i,\beta,\theta)} S_\theta(x_i,\beta,\theta) S_\beta(x_i,\theta).$$

$$u_{\theta\theta}(x_i,\theta) = \frac{\partial^2 u(x_i,\theta)}{\partial \theta^2} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta}\left[\frac{1}{w(x_i,\theta)} \frac{\partial w(x_i,\theta)}{\partial \theta}\right]$$
$$= \sum_{i=1}^{n}\left[\frac{1}{w(x_i,\theta)} \frac{\partial^2 w(x_i,\theta)}{\partial \theta^2} - \frac{1}{w^2(x_i,\theta)}\left(\frac{\partial w(x_i,\theta)}{\partial \theta}\right)^2\right].$$

$$u_{\theta\theta}(x_i,\theta) = \sum_{i=1}^{n}\left[\frac{w_{\theta\theta}(x_i,\theta)}{w(x_i,\theta)} - \frac{{w_\theta}^2(x_i,\theta)}{w^2(x_i,\theta)}\right].$$

$$w_{\theta\theta}(x_i,\theta) = \frac{\partial^2 w(x_i,\theta)}{\partial \theta^2} = -{x_i}^3 e^{-\theta x_i}.$$

$$v_{\theta\theta}(x_i,\beta,\theta) = \frac{\partial^2 v(x_i,\beta,\theta)}{\partial \theta^2} = \sum_{i=1}^{n} S_{\theta\theta}(x_i,\beta,\theta).$$

$$S_{\theta\theta}(x_i,\beta,\theta) = \frac{\partial}{\partial \theta}\left[\frac{-\beta}{w^2(x_i,\theta)} \frac{\partial w(x_i,\theta)}{\partial \theta}\right]$$
$$= \frac{-\beta}{w^2(x_i,\theta)} \frac{\partial^2 w(x_i,\theta)}{\partial \theta^2} + \frac{2\beta}{w^3(x_i,\theta)}\left(\frac{\partial w(x_i,\theta)}{\partial \theta}\right)^2$$
$$= \frac{-\beta w_{\theta\theta}(x_i,\theta)}{w^2(x_i,\theta)} + \frac{2\beta {w_\theta}^2(x_i,\theta)}{w^3(x_i,\theta)}.$$

$$\psi_{\theta\theta}(x_i,\beta,\theta) = \sum_{i=1}^{n} \frac{\partial}{\partial \theta}\left[\frac{1}{z(x_i,\beta,\theta)} \frac{\partial z(x_i,\beta,\theta)}{\partial \theta}\right]$$
$$= \sum_{i=1}^{n}\left[\frac{1}{z(x_i,\beta,\theta)} \frac{\partial^2 z(x_i,\beta,\theta)}{\partial \theta^2} - \frac{1}{z^2(x_i,\beta,\theta)}\left(\frac{\partial z(x_i,\beta,\theta)}{\partial \theta}\right)^2\right]$$
$$= \sum_{i=1}^{n}\left[\frac{z_{\theta\theta}(x_i,\beta,\theta)}{z(x_i,\beta,\theta)} - \frac{z_\theta^2(x_i,\beta,\theta)}{z^2(x_i,\beta,\theta)}\right].$$

$$z_{\theta\theta}(x_i,\beta,\theta) = \frac{\partial^2 z(x_i,\beta,\theta)}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left[e^{-S(x_i,\beta,\theta)}\frac{\partial S(x_i,\beta,\theta)}{\partial\theta}\right]$$
$$= e^{-S(x_i,\beta,\theta)}\frac{\partial^2 S(x_i,\beta,\theta)}{\partial\theta^2} - e^{-S(x_i,\beta,\theta)}\left(\frac{\partial S(x_i,\beta,\theta)}{\partial\theta}\right)^2$$
$$= e^{-S(x_i,\beta,\theta)}\left[S_{\theta\theta}(x_i,\beta,\theta) - S_\theta^{\,2}(x_i,\beta,\theta)\right].$$
$$q_{\theta\theta}(x_i,\theta) = \frac{\partial^2 q(x_i,\theta)}{\partial\theta^2} = \sum_{i=1}^{n}\frac{\partial}{\partial\theta}\left[\frac{1}{C(x_i,\theta)}\frac{\partial C(x_i,\theta)}{\partial\theta}\right]$$
$$= \sum_{i=1}^{n}\left[\frac{C_{\theta\theta}(x_i,\theta)}{C(x_i,\theta)} - \frac{C_\theta^{\,2}(x_i,\theta)}{C^2(x_i,\theta)}\right].$$
$$C_{\theta\theta}(x_i,\theta) = \frac{\partial^2 C(x_i,\theta)}{\partial\theta^2} = -x_i^{\,2}e^{-\theta x_i}(3-\theta x_i).$$

Obtaining the inverse of the matrix $\hat{F}$, which we dented by $\hat{V}$ provides the asymptotic variance co-variances matrix for $\boldsymbol{\phi} \equiv (\alpha,\beta,\theta)$. Assume that the regularity condition satisfied using (6) to get the $100(1-\gamma)\%$ confidence intervals for the parameters α,β,θ.

where, $Z_{\frac{\gamma}{2}}$ is the upper γ^{th} percentile of the standard normal distribution.

4 Bayesian Estimation

In this section, the MCMC algorithm for computing the Bayes estimates of the parameters α,β and $\theta$is used. MCMC is one of the best techniques for obtaining the Bayes estimates, For more details about the MCMC methods see, e.g., Smith and Gelfand (1990), Robert and Casella (2005) and Upadhyaya and Gupta (2010). The Metropolis-Hastings algorithm (MH) to generate samples from the conditional posterior distributions is used and then compute the Bayes estimates. Assume that α,β and θ are independent and have prior distributions π_1,π_2 and π_3 respectively. Prior for each α,β and θ is assumed to follow a gamma distribution as in (Lee et al., 2017):

$$\pi_1(\alpha) = \frac{m_1^{\,b_1}\alpha^{b_1-1}e^{-m_1\alpha}}{\Gamma(b_1)} \quad \alpha > 0, m_1, b_1 > 0,$$
$$\pi_2(\beta) = \frac{m_2^{\,b_2}\beta^{b_2-1}e^{-m_2\beta}}{\Gamma(b_2)} \quad \beta > 0, m_2, b_2 > 0,$$

and

$$\pi_3(\theta) = \frac{m_3^{\,b_3}\theta^{b_3-1}e^{-m_3\theta}}{\Gamma(b_3)} \quad \theta > 0, m_3, b_3 > 0.$$

Here, the hyper-parameters $m_1, b_1, m_2, b_2, m_3, b_3$ are chosen to reflect the prior knowledge about the unknown parameters.

Suppose that we have n number of samples available from C-IGEE distribution and the maximum likelihood estimates of (α,β,θ) are $(\hat{\alpha}^j,\hat{\beta}^j,\hat{\theta}^j), j = 1,2,\ldots,n$. Then equating the mean and variance of $(\hat{\alpha}^j,\hat{\beta}^j,\hat{\theta}^j)$with the mean and variance of the suggested priors (gamma priors), we can get [see Dey et al. (2016)]:

$$\frac{1}{n}\sum_{j=1}^{n}\hat{\alpha}^j = \frac{b_1}{m_1} \quad , \quad \frac{1}{n-1}\sum_{j=1}^{n}\left(\hat{\alpha}^j - \frac{1}{n}\sum_{j=1}^{n}\hat{\alpha}^j\right)^2 = \frac{b_1}{m_1^{\,2}}, \tag{7}$$

$$\frac{1}{n}\sum_{j=1}^{n}\hat{\beta}^j = \frac{b_2}{m_2} \quad , \quad \frac{1}{n-1}\sum_{j=1}^{n}\left(\hat{\beta}^j - \frac{1}{n}\sum_{j=1}^{n}\hat{\beta}^j\right)^2 = \frac{b_2}{m_2^{\,2}}, and \tag{8}$$

$$\frac{1}{n}\sum_{j=1}^{n}\hat{\theta}^{j}=\frac{b_3}{m_3} \quad , \quad \frac{1}{n-1}\sum_{j=1}^{n}\left(\hat{\theta}^{j}-\frac{1}{n}\sum_{j=1}^{n}\hat{\theta}^{j}\right)^{2}=\frac{b_3}{{m_3}^{2}}. \tag{9}$$

Solving equations (7), (8) and (9), we get the estimated hyper-parameters as follows:

$$b_1=\frac{\left(\frac{1}{n}\sum_{j=1}^{n}\hat{\alpha}^{j}\right)^{2}}{\frac{1}{n-1}\sum_{j=1}^{n}\left(\hat{\alpha}^{j}-\frac{1}{n}\sum_{j=1}^{n}\hat{\alpha}^{j}\right)^{2}} \quad , \quad m_1=\frac{\frac{1}{n}\sum_{j=1}^{n}\hat{\alpha}^{j}}{\frac{1}{n-1}\sum_{j=1}^{n}\left(\hat{\alpha}^{j}-\frac{1}{n}\sum_{j=1}^{n}\hat{\alpha}^{j}\right)^{2}},$$

$$b_2=\frac{\left(\frac{1}{n}\sum_{j=1}^{n}\hat{\beta}^{j}\right)^{2}}{\frac{1}{n-1}\sum_{j=1}^{n}\left(\hat{\beta}^{j}-\frac{1}{n}\sum_{j=1}^{n}\hat{\beta}^{j}\right)^{2}} \quad , \quad m_2=\frac{\frac{1}{n}\sum_{j=1}^{n}\hat{\beta}^{j}}{\frac{1}{n-1}\sum_{j=1}^{n}\left(\hat{\beta}^{j}-\frac{1}{n}\sum_{j=1}^{n}\hat{\beta}^{j}\right)^{2}},$$

and

$$b_3=\frac{\left(\frac{1}{n}\sum_{j=1}^{n}\hat{\theta}^{j}\right)^{2}}{\frac{1}{n-1}\sum_{j=1}^{n}\left(\hat{\theta}^{j}-\frac{1}{n}\sum_{j=1}^{n}\hat{\theta}^{j}\right)^{2}} \quad \& \quad m_3=\frac{\frac{1}{n}\sum_{j=1}^{n}\hat{\theta}^{j}}{\frac{1}{n-1}\sum_{j=1}^{n}\left(\hat{\theta}^{j}-\frac{1}{n}\sum_{j=1}^{n}\hat{\theta}^{j}\right)^{2}}.$$

Therefore, the joint prior distribution of α, β and θ can be written as

$$\pi(\alpha,\beta,\theta)=\frac{{m_1}^{b_1}\alpha^{b_1-1}e^{-m_1\alpha}}{\Gamma(b_1)}\frac{{m_2}^{b_2}\beta^{b_2-1}e^{-m_2\beta}}{\Gamma(b_2)}\frac{{m_3}^{b_3}\theta^{b_3-1}e^{-m_3\theta}}{\Gamma(b_3)}$$

$$\alpha,\beta,\theta>0, m_i, b_i>0, \tag{10}$$

where, $i=1,2,3$.

The joint posterior density function of α, β and θ can be written as

$$\pi(\alpha,\beta,\theta|x)=k\frac{{m_1}^{b_1}\alpha^{n+b_1-1}e^{-m_1\alpha}}{\Gamma(b_1)}\frac{{m_2}^{b_2}\beta^{n+b_2-1}e^{-m_2\beta}}{\Gamma(b_2)}\frac{{m_3}^{b_3}\theta^{b_3-1}e^{-m_3\theta}}{\Gamma(b_3)}$$
$$\prod_{i=1}^{n}\left(1-e^{\frac{-\beta}{x_i\left(1-e^{-\theta x_i}\right)}}\right)^{\alpha-1}\prod_{i=1}^{n}\left\{x_i\,\theta\;e^{-\theta x_i}+1-e^{-\theta x_i}\right\}$$
$$\prod_{i=1}^{n}\left(x_i\left(1-e^{-\theta x_i}\right)\right)^{-2}\;e^{\sum_{i=1}^{n}\frac{-\beta}{x_i\left(1-e^{-\theta x_i}\right)}}, \tag{11}$$

where, k is the normalizing constant which given as

$$.\prod_{i=1}^{n}\left(1-e^{\frac{-\beta}{x_i\left(1-e^{-\theta x_i}\right)}}\right)^{\alpha-1}\prod_{i=1}^{n}\left\{x_i.\theta\;e^{-\theta x_i}+1-e^{-\theta x_i}\right\}\,d\alpha\,d\beta\,d\theta$$
$$.\prod_{i=1}^{n}\left(x_i\left(1-e^{-\theta x_i}\right)\right)^{-2}.e^{\sum_{i=1}^{n}\frac{-\beta}{x_i\left(1-e^{-\theta x_i}\right)}}.$$

Therefore, the Bayes estimate of any function of α, β and θ say $g(\alpha,\beta,\theta)$, under the squared error loss function, is given by

$$\tilde{g}(\alpha,\beta,\theta) = E_{\alpha,\beta,\theta|x}\big(g(\alpha,\beta,\theta)\big) = \int_0^\infty \int_0^\infty \int_0^\infty g(\alpha,\beta,\theta)\, \pi(\alpha,\beta,\theta|x) d\alpha d\beta d\theta.$$

It is clear from the equation (11) that there is no closed form for the estimators, and hence MCMC procedure is suggested to compute the Bayes estimates. We consider the Metropolis-Hastings algorithm with a normal proposal distribution to generate samples from the conditional posterior distributions. The follows is the used code to generate the required samples

M H algorithm:

(1): Set initial value of $\boldsymbol{\phi}$ as $\boldsymbol{\phi} = \boldsymbol{\phi}^{(0)}$ and set $i = 1$, where $\boldsymbol{\phi} = (\alpha,\beta,\theta)$.

(2): Set $\boldsymbol{\phi} = \boldsymbol{\phi}^{(i-1)}$.

(3): Generate a proposal $\boldsymbol{\phi}^*$ following a multivariate normal $N(\boldsymbol{\phi}, S_{\boldsymbol{\phi}})$, where $S_{\boldsymbol{\phi}}$ is Standard deviation (we suggest S_{ϕ}=(0.001,0.003,0.002).

(4): Calculate the acceptance probability $\tau = \min\left(1, \frac{\pi(\boldsymbol{\phi}^*|x)}{\pi(\boldsymbol{\phi}|x)}\right)$.

(5) Generate $U \sim U(0,1)$.

(6) If $U \le \tau$ set $\boldsymbol{\phi}^{(i)} = \boldsymbol{\phi}^*$, otherwise set $\boldsymbol{\phi}^{(i)} = \boldsymbol{\phi}^{(i-1)}$.

(7) Set $i = i + 1$.

(8) Repeat steps 2 to 7, N times and obtain $\boldsymbol{\phi}^{(j)}, j = 1,2, \dots, N$.

After getting MCMC samples from the posterior distribution, we can find the Bayes estimate for the parameters in the following way

$$\tilde{\alpha} = \frac{1}{N-B}\sum_{i=B+1}^{N} \alpha^{(i)},$$

$$\tilde{\beta} = \frac{1}{N-B}\sum_{i=B+1}^{N} \beta^{(i)},$$

$$\tilde{\theta} = \frac{1}{N-B}\sum_{i=B+1}^{N} \theta^{(i)},$$

where B is the number of burn-in samples. Then we calculated the highest posterior density (HPD) intervals for the unknown parameters of the C-IGEE distribution using the method of Chen and Shao (1999).

One can also refer to Kundu and Pradhan (2009) and Dey and Dey (2014) for a review on this method. We will use the samples drawn using the proposed MH algorithm to construct the interval estimates. Let us assume that $\boldsymbol{\Pi}(\boldsymbol{\phi}|X)$ denotes the posterior distribution function of $\boldsymbol{\phi}$. Let us further suppose that $\boldsymbol{\phi}^{(p)}$ be the p^th^ quantile of $\boldsymbol{\phi}$, that is, $\boldsymbol{\phi}^{(p)} = inf\{\lambda : \boldsymbol{\Pi}(\boldsymbol{\phi}|X) \ge p\}$, where $0 < p < 1$. Notice that for a given $\boldsymbol{\phi}^*$, a simulation consistent estimator of $\boldsymbol{\Pi}(\boldsymbol{\phi}^*|X)$ can be estimated as

$$\boldsymbol{\Pi}(\boldsymbol{\phi}^*|X) = \frac{1}{N-B}\sum_{i=B+1}^{N} I_{\boldsymbol{\phi} \le \boldsymbol{\phi}^*}.$$

Here $I_{\phi \le \phi^*}$ is the indicator function. Then the corresponding estimate is obtained as

$$\widehat{\boldsymbol{\Pi}}(\boldsymbol{\phi}^*|X) = \begin{cases} 0 & if\ \boldsymbol{\phi}^* < \boldsymbol{\phi}_{(B)} \\ \sum_{j=B}^{i} w_j & if\ \boldsymbol{\phi}_{(i)} < \boldsymbol{\phi}^* < \boldsymbol{\phi}_{(i+1)} \\ 1 & if\ \boldsymbol{\phi}^* > \boldsymbol{\phi}_{(N)} \end{cases}$$

where $w_j = \frac{1}{N-B}$ and $\boldsymbol{\phi}_{(j)}$ are the ordered values of $\boldsymbol{\phi}_j$.

Now, for $i = B, \dots, N, \phi^{(p)}$ can be approximated by

$$\hat{\phi}^{(p)} = \begin{cases} \phi_{(B)} & if\ p=0 \\ \phi_{(i)} & if \sum_{j=B}^{i-1} w_j < p < \sum_{j=B}^{i} w_j. \end{cases}$$

Now to obtain a $100(1-p)\%$ HPD credible interval for ϕ,let $R_j = \left(\hat{\phi}^{\left(\frac{j}{N}\right)}, \hat{\phi}^{\left(\frac{j+(1-p)N}{N}\right)}\right)$ for $j = B, \dots, [pN]$, here [a] denotes the largest integer less than or equal to a. Then choose R_{j^*} among all the $\hat{R}_j$s such that it has the smallest width.

5 Simulation Study

In this section, we compare the performances of MLEs and Bayesian estimates using the MCMC method. Sample of size $\{n = 15, 25,35,45, 75,100\}$ are used to generate observations from a C-IGEE distribution with different values of initial. We assume that the number of repetition is 1000, then we calculate their means, means square errors (MSE) and associated 95% confidence interval (CI) estimates of MLEs of each parameter. For Bayesian estimations the MCMC method will be used according to MH Algorithm.

The number of iteration for this algorithm is N = 10000 with the burn-in period B = 2000. Each prior for α, β and θ is assumed to be gamma distribution with hyper parameters $(b_1 = 12, m_1 = 5.8, b_2 = 14,\ m_2 = 5.6,\ b_3 = 1.5,\ m_3 = 0.41)$ .Then the bayes estimates and HPD interval estimates are obtained using the technique of Chen and Shao (1999) .The performances of the estimators for both methods using MSE and average interval lengths (AIL) with coverage percentages (CP) are reported in Tables (1-6). From Tables (1-6) we observed that;

- The MSE of all estimators decreases when the sample size n increases.
- The MSE of the Bayes estimators are smaller than their corresponding MSE of MLEs.
- The 95% Bayes intervals are much narrower than the asymptotic confidence intervals of MLEs.

Table 1: Estimate Values and MSEs (With $\alpha = 2, \beta = 2.5,\ \theta = 1.5$).

N	Parameters	MLE			Bayesian		
		AE	Bias	MSE	AE	Bias	MSE
15	α	2.651	0.651	3.236	1.9999	-0.001	0.00199
	β	2.658	0.158	2.843	2.494	-0.006	0.0177
	θ	2.27	0.77	5.21	1.50042	0.00042	0.00847
25	α	2.405	0.405	1.627	1.99975	-0.00025	0.00197
	β	2.649	0.149	1.079	2.4967	-0.0033	0.0167
	θ	2.23	0.73	4.96	1.495	-0.005	0.0082
35	α	2.256	0.256	0.87	2.0061	0.0061	0.00194
	β	2.572	0.072	0.862	2.4965	-0.0035	0.0161
	θ	2.146	0.646	4.737	1.50117	0.00117	0.00789

45	α	2.168	0.168	0.682	1.99617	-0.00383	0.00191
	β	2.489	-0.011	0.830	2.4955	-0.0045	0.0166
	θ	2.018	0.518	3.755	1.49883	-0.00117	0.00779
75	α	2.067	0.067	0.379	2.00049	0.00049	0.00189
	β	2.442	-0.058	0.619	2.498	-0.002	0.015
	θ	1.891	0.391	3.420	1.49542	-0.00458	0.00734
100	α	2.051	0.051	0.274	2.00002	0.00002	0.00183
	β	2.445	-0.055	0.512	2.501	0.001	0.016
	θ	1.1817	-0.318	2.698	1.4987	-0.0013	0.0078

Table 2: CI, HPD interval, AILs and CPs (With Initial $\alpha = 2, \beta = 2.5, \ \theta = 1.5$).

N	Parameters	MLE			Bayesian		
		AE	Bias	MSE	AE	Bias	MSE
15	α	2.651	0.651	3.236	1.9999	-0.001	0.00199
	β	2.658	0.158	2.843	2.494	-0.006	0.0177
	θ	2.27	0.77	5.21	1.50042	0.00042	0.00847
25	α	2.405	0.405	1.627	1.99975	-0.00025	0.00197
	β	2.649	0.149	1.079	2.4967	-0.0033	0.0167
	θ	2.23	0.73	4.96	1.495	-0.005	0.0082
35	α	2.256	0.256	0.87	2.0061	0.0061	0.00194
	β	2.572	0.072	0.862	2.4965	-0.0035	0.0161
	θ	2.146	0.646	4.737	1.50117	0.00117	0.00789
45	α	2.168	0.168	0.682	1.99617	-0.00383	0.00191
	β	2.489	-0.011	0.830	2.4955	-0.0045	0.0166
	θ	2.018	0.518	3.755	1.49883	-0.00117	0.00779
75	α	2.067	0.067	0.379	2.00049	0.00049	0.00189
	β	2.442	-0.058	0.619	2.498	-0.002	0.015
	θ	1.891	0.391	3.420	1.49542	-0.00458	0.00734
100	α	2.051	0.051	0.274	2.00002	0.00002	0.00183
	β	2.445	-0.055	0.512	2.501	0.001	0.016
	θ	1.1817	-0.318	2.698	1.4987	-0.0013	0.0078

Table 3: Estimate Values and MSEs (With $\alpha = 2, \beta = 2, \ \theta = 2$).

N	Parameters	MLE			Bayesian		
		AE	Bias	MSE	AE	Bias	MSE
15	α	2.651	0.651	3.236	1.9999	-0.001	0.00199

	β	2.658	0.158	2.843	2.494	-0.006	0.0177
	θ	2.27	0.77	5.21	1.50042	0.00042	0.00847
25	α	2.405	0.405	1.627	1.99975	-0.00025	0.00197
	β	2.649	0.149	1.079	2.4967	-0.0033	0.0167
	θ	2.23	0.73	4.96	1.495	-0.005	0.0082
35	α	2.256	0.256	0.87	2.0061	0.0061	0.00194
	β	2.572	0.072	0.862	2.4965	-0.0035	0.0161
	θ	2.146	0.646	4.737	1.50117	0.00117	0.00789
45	α	2.168	0.168	0.682	1.99617	-0.00383	0.00191
	β	2.489	-0.011	0.830	2.4955	-0.0045	0.0166
	θ	2.018	0.518	3.755	1.49883	-0.00117	0.00779
	α	2.067	0.067	0.379	2.00049	0.00049	0.00189
75	β	2.442	-0.058	0.619	2.498	-0.002	0.015
	θ	1.891	0.391	3.420	1.49542	-0.00458	0.00734
	α	2.051	0.051	0.274	2.00002	0.00002	0.00183
100	β	2.445	-0.055	0.512	2.501	0.001	0.016
	θ	1.1817	-0.318	2.698	1.4987	-0.0013	0.0078

Table 4: CI, HPD interval, AILs and CPs (With Initial $\alpha = 2, \beta = 2, \theta = 2$).

N	Parameters	MLE			Bayesian		
		AE	Bias	MSE	AE	Bias	MSE
15	α	2.651	0.651	3.236	1.9999	-0.001	0.00199
	β	2.658	0.158	2.843	2.494	-0.006	0.0177
	θ	2.27	0.77	5.21	1.50042	0.00042	0.00847
25	α	2.405	0.405	1.627	1.99975	-0.00025	0.00197
	β	2.649	0.149	1.079	2.4967	-0.0033	0.0167
	θ	2.23	0.73	4.96	1.495	-0.005	0.0082
35	α	2.256	0.256	0.87	2.0061	0.0061	0.00194
	β	2.572	0.072	0.862	2.4965	-0.0035	0.0161
	θ	2.146	0.646	4.737	1.50117	0.00117	0.00789
45	α	2.168	0.168	0.682	1.99617	-0.00383	0.00191
	β	2.489	-0.011	0.830	2.4955	-0.0045	0.0166
	θ	2.018	0.518	3.755	1.49883	-0.00117	0.00779
	α	2.067	0.067	0.379	2.00049	0.00049	0.00189
75	β	2.442	-0.058	0.619	2.498	-0.002	0.015
	θ	1.891	0.391	3.420	1.49542	-0.00458	0.00734

100	α	2.051	0.051	0.274	2.00002	0.00002	0.00183
	β	2.445	-0.055	0.512	2.501	0.001	0.016
	θ	1.1817	-0.318	2.698	1.4987	-0.0013	0.0078

Table 5: Estimate Values and MSEs (With $\alpha = 5, \beta = 20,\ \theta = 5$).

N	Parameters	MLE			Bayesian		
		AE	Bias	MSE	AE	Bias	MSE
15	α	6.77	1.77	34.24	4.9976	-0.002	0.00199
	β	20.5	0.5	49.7	19.9513	-0.049	0.0209
	θ	3.7665	-1.2335	17.2991	4.99828	-0.002	0.00784
25	α	5.78	0.78	8.02	4.99673	-0.003	0.00202
	β	20.3	0.3	19	19.9569	-0.043	0.0207
	θ	3.928	-1.072	13.702	4.99889	-0.001	0.00876
35	α	5.48	0.48	4.2	4.99653	-0.003	0.00196
	β	20.1	0.1	14.6	19.9559	-0.044	0.0188
	θ	3.841	-1.159	9.951	4.99858	-0.001	0.00816
45	α	5.29	0.29	2.6	4.99615	-0.004	0.00186
	β	19.95	-0.05	9.65	19.95	-0.050	0.02
	θ	4.01	-0.99	8.489	5.00318	0.003	0.00765
75	α	5.22	0.22	1.63	4.99697	-0.003	0.00184
	β	20.11	0.11	6.24	19.9586	-0.041	0.0189
	θ	4.069	-0.931	7.402	4.99778	-0.002	0.00714
100	α	5.11	0.11	1.01	5.00162	0.002	0.00109
	β	20.03	0.03	4.11	20.0047	0.005	0.00873
	θ	4.018	-0.982	6.449	5.00139	0.001	0.00392

Table 6: CI, HPD interval, AILs and CPs (With Initial $\alpha = 5\ , \beta = 20\ , \theta = 5$).

N	Parameters	MLE			Bayesian		
		CI	AIL	CP	HPD interval	AIL	CP
15	α	(2.48, 20.05)	17.58	97.5	(4.91104, 5.0809)	0.1698	97.6
	β	(9.1, 34.8)	25.7	97.5	(19.6762, 20.2037)	0.5274	97.2
	θ	(0.0894, 11.5434)	11.454	97.5	(4.83344, 5.1825)	0.3491	98
25	α	(2.69, 12.35)	9.66	97.5	(4.9118, 5.0843)	0.1725	97.4
	β	(11.9, 28.9)	16.9	97.5	(19.696, 20.2198)	0.5237	97.8
	θ	(0.155, 10.34)	10.184	97.5	(4.81361, 5.17324)	0.3596	97.3

35	α	(2.85, 10.49)	7.63	97.5	(4.9014, 5.08229)	0.1809	97.2
	β	(12.1, 27.9)	15.7	97.5	(19.7108, 20.21)	0.4992	97.6
	θ	(0.183,9.674)	9.490	97.5	(4.81974,5.17147)	0.3517	97.7
45	α	(3.03, 9.07)	6.04	97.5	(4.91536,5.07816)	0.1628	98
	β	(14.1, 25.98)	11.88	97.5	(19.708, 20.223)	0.515	98.1
	θ	(0.244,9.461)	9.217	97.5	(4.83204, 5.17255)	0.3405	97.3
75	α	(3.36, 8.41)	5.05	97.5	(4.91814, 5.07905)	0.1609	98.1
	β	(15.36, 25.46)	10.1	97.5	(19.6825, 20.2057)	0.5233	97.5
	θ	(0.318, 8.593)	8.275	97.5	(4.82826, 5.1593)	0.331	97.4
	α	(3.55, 7.51)	3.96	97.5	(4.94.94, 5.06984)	0.1289	97.9
100	β	(16.41, 24.19)	7.79	97.5	(19.8051, 20.1784)	0.3734	96.2
	θ	(0.357, 7.858)	7.501	97.5	(4.87494, 5.11756)	0.2426	97.3

6 Conclusions

In this paper, we proposed the Bayes estimation of the unknown parameters of C-IGEE distribution. The proposed estimators are compared with maximum likelihood estimators using Monte Carlo simulations. Under the assumptions of independent gamma priors of all parameters the Bayes estimates are obtained using MCMC under squared error loss function. The methods are compared by computing the MSE and AIL. From Tables (1-6), it can be noticed that the Bayes estimates are better than the MLE estimates.

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