Visible Light Communications Localization Error Enhancement using Parameter Relaxation

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***Abstract*—In this paper, we propose applying a parameter relaxation technique to the location estimation algorithm that is based on the Received Signal Strength (RSS) of Visible Light Communications (VLC). A hybrid system of localization balancing is introduced, where the localization algorithm is developed with and without this efficient parameter relaxation. The results show that applying the parameter relaxation reduces the localization Root Mean Square (RMS) error by 43% of that without relaxation; and the processing time is reduced by 18% of that without relaxation. Moreover, the parameter relaxation approach is compared with previous work that is based on Artificial Neural Networks (ANN), where the performance of the proposed relaxation technique is shown to be over 4 times better than that of the ANN-based approach.**

***Keywords*—Localization, LED, VLC, Dog-leg, Trust-Region, Relaxation, Reduction of Unknowns, Localization Balancing, Artificial Neural Networks.**

I. Introduction

Light Emitting Diode (LED) technology has become a promising tool in Visible Light Communications (VLC) systems. Based on [1], such a communications system outperforms the current RF systems that were proved to have other health side effects in the long run. However, based on [2], LED communications systems have much to do with enhancements; to perform as that of RF systems. That is due to the factor of communications distance that limits the VLC transmit bit rate. It was found that the Bit Error Rate (BER) reached reasonable values at a distance of 1.5 m between LED transmitter and Photo Diode Receiver (PDR). Furthermore, this enhancement opens the gate for using double of the room's height as in our proposed system model.

Based on [3], it is possible to use VLC to localize items through a Local Dynamic Map (LDM) with four layers. The first layer is Permanent Static Data (as the map, which can never change as long as the geographical conditions are the same); the second layer is Transient Static Data (as traffic sign, landmarks, subways which can be re-allocated but on the long term); the third layer is Transient Dynamic Data (as weather situation, traffic information, signal phase that change but with high correlation); and the fourth layer is Highly Dynamic Data (where only dynamic users are localized, which is applied by this paper work).

For a larger scale, the localized items can be connected as Internet of things (IoT) with VLC IDs. Actually, the International Electrotechnical Commission (IEC) standardized the VLC ID under the name of Visible Light Beacon System for Multimedia Applications [4].

Moreover, in the case of Universal Traffic Management Systems (UMTS), Infra-red beacon systems - that reached 55,000 beacon units - are used to provide larger scale information with only a 1 Mbps infrared network system data rate. With introducing the LED transmitters and PDR receivers supplied with the gain lens, the gain increased with decreasing the field of view (FOV). In that event, the LED and PDR systems yield a data rate equals to 3.1 Mbps, in addition to the edge of having a small FOV that reduces the background visible noise such as the sun's light [5].

Despite the introduction of novel localization methodologies, which focus on two parallel aspects: adaptation to the uncertain noise of the Micro Electro Mechanical System-based Inertial Navigation Systems (MEMS-INS); and accuracy predicting errors of the Inertial Navigation System (INS) as in [6], localization using GPS suffers a kind of error. Accordingly, chance of having less localization error under VLC becomes considerable.

Concerning the first aspect, which is the adaptation of the uncertainty noise in MEMS-INS in such a proposed algorithm, three consecutive steps are taken: first, a two-stage Kalman filter is used to fuse information from MEMS-INS; GPS and in-vehicle sensors; secondly, three bias filters with different covariance matrices are used to cover a wide range of noise characteristics; Thirdly, an Interactive Multiple Model (IMM) algorithm is used to smoothly switch between those three filters to adapt this MEMS-INS uncertainty noise.

In order to accomplish this prediction task, an elaborate predictor is used that includes two trained models: online and offline. The online trained model - called the Auto Regressive Integrated Moving Average (ARIMA) - is designed to predict the baseline accumulation of MEMS-INS errors; because error spreads not only to the current time state but also overwhelms the system for the next stages. The offline model - called the Extreme Learning Machine (ELM) - is designed to correct errors caused by a change in noise characteristics. However, referring to Yamazatu et al. [5], efforts are being made to replace the GPS localization system with VLCs with better accuracy. That is because GPS has a vast range of errors in addition to the fact that it is grabbed into outage states based on [6].

Two main points appeared per the previous review: the offline aid; and the attempt to substitute GPS for position prediction [3][6]. According to this, the VLC offline localization is proposed. Also, the parameter relaxation is one of the tools that improves the localization accuracy.

The main contribution of this work is to provide a location estimation through a practical offline method. Such a tool forms a load balancing between the original RSS-based localization and the proposed localization algorithm based on parameter relaxation radius of convergence.

The measurement error effect on the location estimation's accuracy in RSS-based localization was concluded in [7]. Where a new term called Dilution of Precision (DOP) was introduced. Moreover, there were five types of this DOP: Horizontal; Vertical; Positional; Time and Geometric DOP [8]. On that account, it is extremely important to model the error of location estimation as a function of measurement error.

Localization load balancing is done in this work when the relaxation parameter swings around $1\pm, 0.025$m. As a result, when the relaxation parameter is in this convergence radius, the localization estimation process is loaded with our proposed algorithm, adding a localization accuracy of 48% of that without applying for the proposed work. Otherwise, the localization process is loaded with the ordinary RSS-based location estimation.

In this paper, the system model is presented in Section-II. Section-III shows this paper's contribution to the localization algorithm. Section-IV shows a localization error at every receiver point in the indoor environment without applying a relaxation parameter. Section-V shows applying an old method called "Reduction of unknowns", that proved to have a huge localization error as a verification for the efficiency of our proposed algorithm. Section-VI presents the results of application of our proposed type-I and type-II parameter relaxation, followed by numerical improvements in both Root Mean Square Error and processing time. Finally, a conclusion is set out in Section-VII.

II. System Model

In this section, the system model is presented, and the transmitter characteristics are obtained in order to minimize co-channel interference in addition to having zero dead-zones. A scenario consists of a room whose dimensions $(L_x \times L_y \times L_z)$ are $16m \times 16m \times 3$ m; 64 transmitters arranged in an 8×8 square matrix; and one PDR whose coordinates are $(x_r, y_r, 1)$ m as shown in Fig. 1.

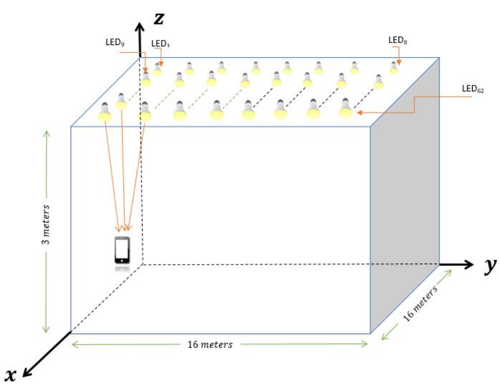

Fig. 1. LOS 3-D system model

The critical angle of irradiance Θ_c is defined as the cone angle at the transmitter, i.e., the critical angle between the normal to the transmitter and the line connecting the transmitter to the receiver. To have coverage for the transmitter, θ should be $\leq \Theta_C$. Therefore, we have two opposing criteria:

- Minimizing Θ_C to minimize the massive Co-Channel Interference (CCI) caused by the overlaying of coverage of LED sources;
- Maximizing θ_c to guarantee no dead zones the receiver may suffer as in Fig. 2.

As a result, the coverage cone circles are modelled and numerically calculated to be $\Theta_C = 17.62^0$ based on the current dimensions of the room, as shown in Fig. 2.

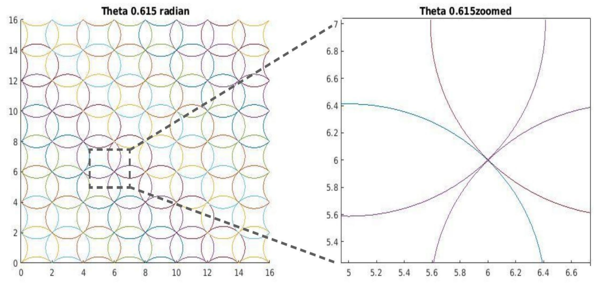

Fig. 2. Θ_{critical} with zero dead-zones

III. Contribution to Localization Algorithm

This section presents the equations used for location estimation and the proposed algorithm.

A. Used Equations

Reviewing the explicit relationship for the equation Signal to Interference Noise Ratio (SINR) in [9], the power obtained due to transmitter$_i$ is given as follows:

$$\because P_{r_i} = \frac{R \cdot P_t \cdot A_{\text{PD}} \cdot (L_z - h)^2}{\pi \cdot d_0^4} \cdot \frac{n_c^2}{\sin^2 \psi_c}$$

$$\therefore d_0^4 = \frac{R \cdot P_t \cdot A_{\text{PD}} \cdot (L_z - h)^2}{\pi \cdot P_{r_i}} \cdot \frac{n_c^2}{\sin^2 \psi_c} \tag{1}$$

$$\because d_0 = \sqrt{(x_r - x_t)^2 + (y_r - y_t)^2 + (h - Lz)^2}$$

Substituting by d_0 in (1) yields:

$$\begin{aligned} &((x - x_{t_i})^2 + (y - y_{t_i})^2 + (h - L_z)^2)^2 \\ &= \frac{R \cdot A_{\text{PD}} \cdot P_t \cdot (L_z - h)^2 \cdot n_c^2}{\pi \cdot P_{r_i} \cdot \sin^2 \psi_c} \end{aligned} \tag{2}$$

$\forall i \in \{1, 2, 3, 4, ..., N\}$, where N is total number of transmitters; x, y and h are the required variables representing the location; R is the responsivity of the PDR; and the rest are the system parameters as given in Table I.

Using the fact in [10]: That the best RSS signals - among all our 64 signals - each has a unique ID used to have three distinguishable equations in three unknowns: $\tilde{x_r}, \tilde{y_r}, \tilde{h}$ where the tilde accent stands for the estimated parameters. The distinguishable IDs of the best three transmitters are optimized using the tool in [11], where the handovers are optimized alongside the parameters errors among the network. Thus the given system is feasible to be solved in an offline manner.

B. Proposed Algorithm

The proposed localization algorithm is based on applying the parameter relaxation to $\tilde{h}$ the measured RSS-based vertical distance between the transmitter and the receiver (2). The Dog-leg numerical solver algorithm takes fewer iterations which ultimately decreases the processing time.

In this work, the concept of localization balancing is applied. Based on [12], optimization could be applied to the load balancing in LiFi/RF hybrid networks to achieve optimum unified service efficiency. Additionally, with the proposed relaxation algorithm, a similar idea of localization balancing could be applied, resulting in better overall location estimation performance.

The localization process is loaded with the proposed algorithm when the relaxation parameter hRel. is approximate (1 ± 0.025)m, which is the average height that the user can hold the PDR. Therefore, the average localization error throughout the room is enhanced by 48%, which is more than that without the relaxation. Note that this relaxation parameter might well be designed to be more dynamic based on the user's probabilistic behavior. The flowchart of the algorithm balancing is shown as in Fig. 3.

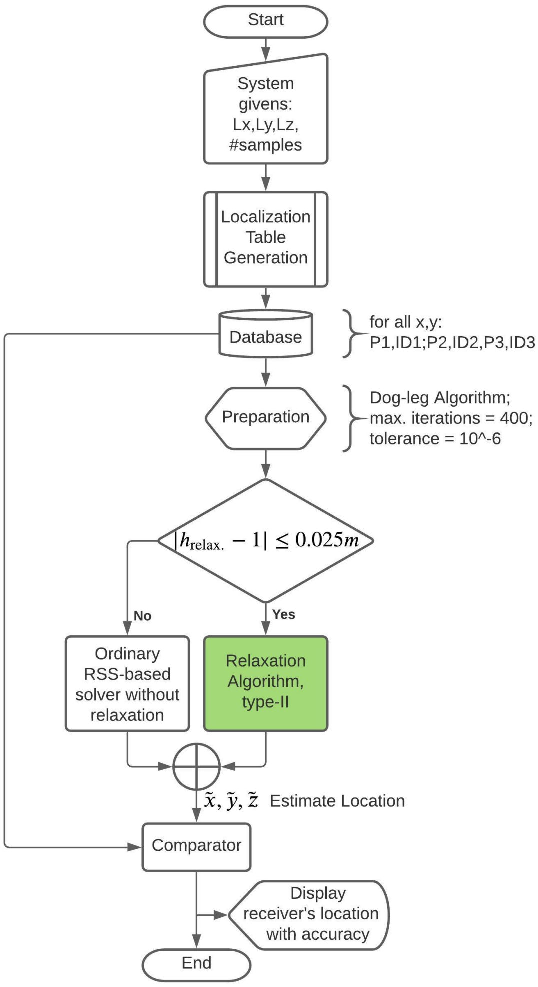

Fig. 3. Proposed Relaxation Algorithm flowchart

Based on the above flowchart, the enhanced localization algorithm works as follows:

1) System givens: Input givens are introduced, as room dimensions and number of spatial samples (which is here $\simeq$ 23k samples;
2) Generation phase: It receives system inputs as well as the angle of half power, and outputs a localization table. Such a table contains all the x,y receiver coordinates, as well as the supposed received powers from the highest radiating transmitters (not necessarily to be nearest);
3) Database: The Localization table is stored in a Database in order to be compared later with the estimated location;
4) Preparation phase: It sets the parameters for the used equation solver algorithm. Here, it is Dog-leg with 400 maximum iterations per estimated receiver location, and $10^{-}6$ tolerance;
5) Decision stage: As long as the relaxed parameter is in the neighborhood of 1 m -which makes sense that the time average of the receiver's height lies in this range- with the shown radius of convergence, the proposed enhanced localization algorithm is used to estimate the receiver's location, otherwise, the ordinary problem solving algorithms are used;
6) Comparator: receives the estimate location and compares it with the already stored localization table in the Database;
7) Display: An interactive phase to show the localization error.

C. System Parameters

The system parameters are shown in the Table I.

TABLE I
PROPOSED SYSTEM MODEL PARAMETERS

$Parameter$	$Value$	$Unit$
L_x	15	m
L_y	15	m
L_z	3	m
P_{t_i}	1000	W
n_c	1.47	Dimensionless
Ψ_c	60	Degrees
θ_c	17.62	Degrees
$Area_{PD}$	10^{-4}	m^2
Responsivity	1	Dimensionless

IV. Results without Reduction of Unknowns and Parameter Relaxation Technique

In this section, we show the Root Mean Square (RMS) error without the parameter relaxation, and the used equation solving algorithm which is chosen to be the "Dog-leg".

A. Root Mean Square Error Definition

The RMS error is the selected parameter to be the criteria of the efficiency of the algorithm per user's location, i.e. at once according to (3)

$$\text{RMS Error} = \sqrt{(\tilde{x}-x)^2 + (\tilde{y}-y)^2 + (\tilde{z}-z)^2} \qquad (3)$$

where: $\tilde{x}, \tilde{y}$ and $\tilde{z}$ are the estimate location coordinates obtained from the following five algorithms, and the RMS error is calculated per the location of a receiver as mentioned earlier.

We have two pathways to solve optimization problems: line search methods and trust-region non-linear optimization algorithms. Trust region algorithms are robust and can be applied to unconditioned [13] problems. This is the reason beyond the use of the trust-region technique to resolve the location of the receiver, given the non-linear RSS-based equation system.

B. Dog-leg Algorithm

Based on latest Mathworks documentation for the equation solving algorithms [14], *fsolve* attempts to solve a set of N independent non-linear equations:

$$F_i(\mathbf{x}) = 0$$

where i=1,2,3,...,N and **x** is a vector whose components are our N unknowns.

The equation solving function -*fsolve*- starts with initial vector x_0 from which it undergoes multiple numerical iterations to reach the final solution that satisfies all the equations to be equal to zero. The idea of numerical solution is that it reaches the required solution **x** with a tolerance that is predetermined by the user where all $F_i(\mathbf{x}) \simeq 0, \forall i \in \{1, 2, 3, ..., N\}$

fsolve has three algorithms:

1) Trust-region
2) Trust-region-dogleg
3) Levenberg-Marquardt

The Dog-leg algorithm was chosen for this paper since its complexity is less than that of the Levenberg-Marquardt algorithm [15] . We could thus guarantee less processing requirements. Moreover, it is more efficient as it requires only one linear solve per iteration (to calculate the Gauss-Newton step). In addition to this, the Dog-leg algorithm is more robust than the Gauss-Newton method with a line search [14].

Building on the above mentioned derivation in [14], the algorithm minimizes the function based on the directional argument d as in the following equation:

$$\min_{d} f(d) = \frac{1}{2} F(x_k, d)^T \cdot F(x_k, d), \tag{4}$$

where k is the number of the current iteration, the algorithm continues updating the iteration point x_{k+1} as long as minimizes our target function.

The key question is the determination of the direction d in (4) to minimize this target function. Here it comes the role of Dog-leg algorithm that uses Powell Dog-leg procedure [16].

Based on [14], the algorithm makes convex combination between two steps in order to reach the direction of steepest descent in (4):

1) Cauchy step;
2) Gauss-Newton step.

In contrast, a new Dog-leg method was reached in [17] to solve the trust-region sub-problems. Such a new method converges in fewer iterations and is both efficient and accurate. Furthermore, this gives the green light for mathematics to enhance such non-linear systems, and therefore we can do more with location estimation. After solving the previous system of equations using the Dog-leg iterative method, the spatial RMS error for the location estimation is calculated and shown in Fig. 4.

It was of great importance to introduce the average root mean square ($\overline{\text{RMS}}$) error for all the receiver's estimate locations that are almost 26,000 estimate location points. The results show that $\overline{\text{RMS}}$ error for location estimation without applying the relaxation parameter $\simeq$ 7 cm.

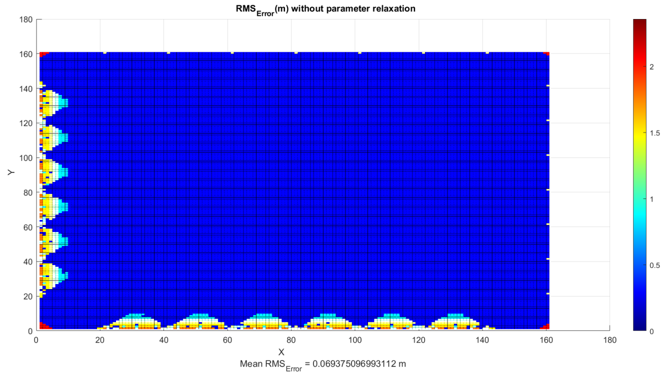

Fig. 4. Top-view: RMS Error without Parameter Relaxation

V. Results with Reduction of Unknowns Technique

D. E. Simpson et al. [18] showed that applying the method of reduction of unknowns is an effective solution approach for a constrained minimization problem. Thus applying such an approach converts the problem as mentioned earlier (4) into an unconstrained one with fewer variables. Degrading the number of unknowns, from three to two, through eliminating the estimate height $\tilde{h}$ from (2), by substituting by its mean value, i.e. $\tilde{h} = \overline{h} = 1$ m. As a consequence, the equations are optimized to two equations in two unknowns, i.e. $\tilde{x}, \tilde{y}$. The $\overline{\text{RMS}}$ error for the mentioned method has shown less performance with respect to $\overline{\text{RMS}}$ error, as shown in Fig. 5.

Although the unknowns reduction approach is efficient, it shows less performance concerning $\overline{\text{RMS}}$ error, in the range of mean value of real receiver's height, i.e., $\overline{h}$ = 1 m. Since the approach mentioned earlier becomes out of the sense of the Trilateration localization technique. However, when we apply the reduction of unknowns method, the estimated location $\tilde{x}, \tilde{y}$ lies on the straight line joining the best two transmitters, only the distance is in ration with the received powers.

VI. Results with Parameter Relaxation

It is quite challenging to reach a simpler system - that gives information about the original problem - when solving a system of non-linear equations, and this is the key point for using parameter relaxation. Parameter relaxation is applied to the system of equations by giving approximate values of some parameters. As a consequence, the system becomes much simpler from the point of view of the required iterations and the radius of convergence for the unknowns. In this context, a new notion was derived in [19] from the waveform relaxation iterative method. Such a new notion coupled with different numerical methods to solve numerical solutions. And this opens the way for proposing mathematical ways - one of which is the relaxation iterative methods - to solve much more complex systems. We study the error declination after applying the parameter relaxation outside and inside the bulk of the parameters of (2).

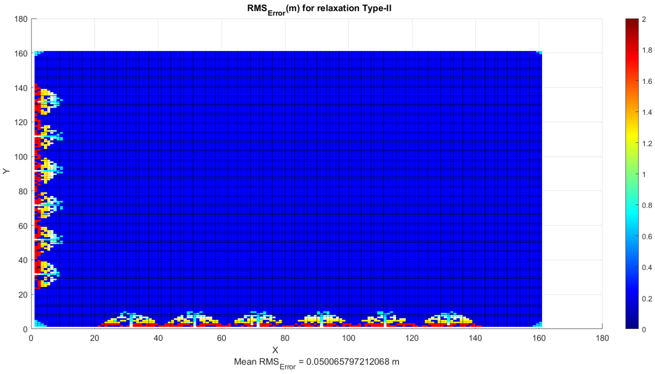

Fig. 7. Top-view: RMS Error with Parameter Relaxation Type-II

A. Applying Relaxation Type-I

In order to apply the parameter relaxation for the receiver's estimate height, it was proposed to perform the relaxation outside the bulk of (2). After applying this first type of parameter relaxation, $\overline{\text{RMS}}$ error becomes 8 cm, where spatial $\text{RMS}_{\text{Error}}$ is shown as in Fig. 6.

Applying relaxation type-I resulted in $\overline{\text{RMS}}$ error worse than that without relaxation, i.e. $\overline{\text{RMS}}$ error type-I $=$ 8cm $>$ $\overline{\text{RMS}}$ error without-relaxation $=$ 7 cm. Thus, searching for the second type for parameter relaxation was inevitable.

B. Applying Relaxation Type-II

On the other hand, the second type of parameter relaxation is implemented by relaxing $\tilde{h}$ inside the bulk of (2). Applying this second type of relaxation resulted in $\overline{\text{RMS}}$ error $\simeq 0.0497$ m.
Numerically speaking, the $\overline{\text{RMS}}$ error is reduced by $\simeq 43\%$ compared to the non-parameter relaxation case, as in Section-IV. The $\overline{\text{RMS}}$ error as function of the room's X-Y plane is shown in Fig. 7.

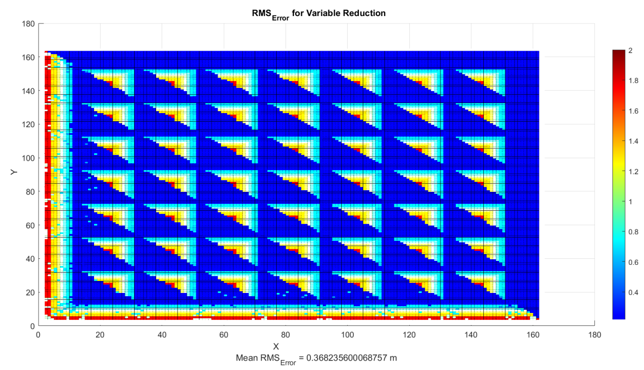

Fig. 5. Top-view: $\overline{\text{RMS}}$ Error with the Reduction of Unknowns Method

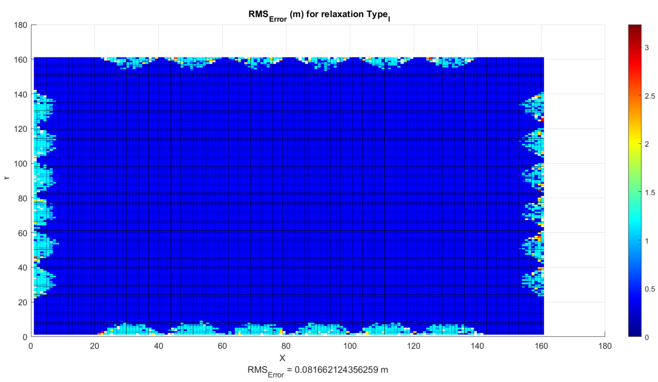

Fig. 6. Top-view: RMS Error with Parameter Relaxation Type-I

Not only localization enhancements were in $\overline{\text{RMS}}$ error, but also the localization processing time decreased by almost 18% of the non-parameter relaxation processing time.

It is obvious that the spatial $\overline{\text{RMS}}$ error at the walls is slightly different from that for the rest of the room. It was shown in [20] that the behavior for the VLC system is classified into three main categories per attocell: corner, edge, and center. That is the reason behind the variation in results for the walls and the entire parts of the room as in Fig. 7.

Accuracy of each of the five algorithms based on $\overline{\text{RMS}}$ error is shown as in Fig. 8. It can be seen that the $\overline{\text{RMS}}$ error for type-II relaxation reaches to the minimum value when $\tilde{h}$ = $\overline{h}$ = 1 m. Moreover, the reduction of unknowns algorithm performs the worst accuracy because it suppresses the least received power equation from the system. Accordingly, the receiver's estimate location becomes a feasible value only among the line joining between the best two transmitters, which is almost wrong. The type-II with variable guess is a more complex way of equations that is used to verify the performance of the type-II under certain simplifications, and this is obvious in the mentioned figure.

C. Time Performance

The execution time required for localization is measured without and with parameter relaxation. Self Time is the criterion of execution performance which is defined as total time in seconds spent in a function, excluding time spent in any child functions [15].

For the location estimation performance without relaxation, self-time for the solving function is 43.229 seconds, the solving function was called 742601 times. Meanwhile, after applying parameter relaxation, the self-time declined to only 35.658 seconds. Accordingly, self-time is enhancement by $\simeq$ 18% of that elapsed in case of without parameter relaxation.

Comparing the number of function calls, it is decreased by $\simeq$ 20% of that without parameter relaxation; that is because of the parameter relaxation lead to the convergence in fewer iterations. In addition to this enhancement, our solving function was only called 597177 times.

The performance evaluation is shown in Table II. We could observe that the localization type-II algorithm is better for both the number of iterations and self-time. This comparison further presented in Fig. 8

Finally, we compared the results with previous work [21] that uses the ANN in positioning and location prediction. As a result, the neural network had a mean error of 16 cm by using testing - untrained - data, while it had a mean error of 14cm after using the training data. On the other hand, our algorithm based on the relaxation parameter method has accomplished a mean error of only 4 cm using more than 25,000 samples in a room whose dimensions are more significant than that of the case of the neural network.

TABLE II

TIME PERFORMANCE TABLE FOR RELAXATION EFFECT

	Without Relaxation			Rel. Type-I			Rel. Type-II		
Function	Calls	Total Time (s)	Self Time (s)	Calls	Total Time (s)	Self Time (s)	Calls	Total Time (s)	Self Time (s)
fsolve	25921	329.08	19.78	same	390.83	20.87	same	331.60	18.25
myfun	742601	43.23	43.23	732073	48.45	48.45	597177	35.03	35.66
mylocation	1	333.81	2.07	1	400.6	3.67	1	336.01	1.93

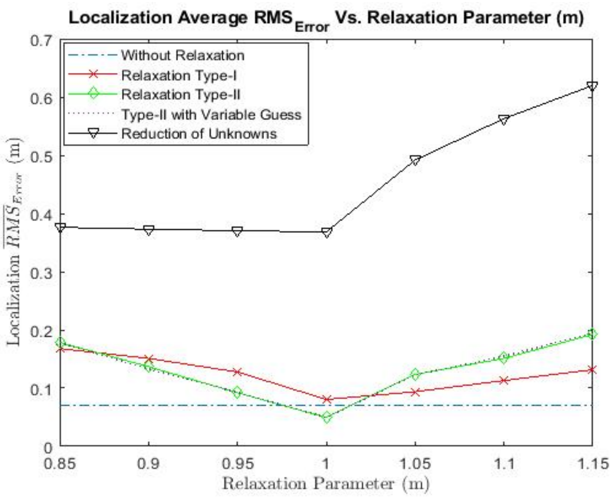

Fig. 8. Zoomed performance comparison of the five algorithms

VII. CONCLUSION

Applying the parameter relaxation opens the gate for using offline location estimation. In this paper, there are two localization enhancements: The Localization error and the processing time decrease. When it comes to the first localization enhancement, we have the localization error decreased by $\simeq 43\%$ after applying the parameter relaxation of type-II.

When it comes to the second localization enhancement, we have the localization processing time decreased by $\simeq 18\%$ than the case without relaxation. Also, the number of calling times for the location-solving function decreased by $\simeq 20\%$ than its number of calls in the case without relaxation. The location estimation numerical solutions converged in fewer iterations due to applying our proposed parameter relaxation algorithm to the dog-leg for trust-region problems, which reduced the localization order of complexity. As a consequence, the localization error and processing time were enhanced.

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