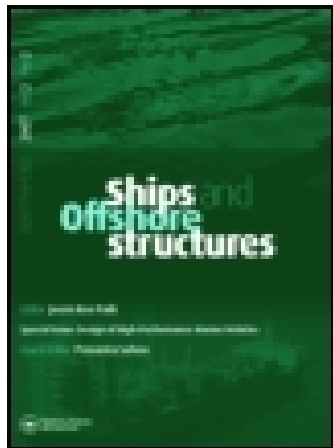


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Recursive differentiation method: application to the dynamics of beams on two parameter foundations

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Recursive differentiation method: application to the dynamics of beams on two parameter foundations

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The recursive differentiation method has been suggested and utilised to obtain analytical solutions for differential equations describe many boundary value problems. The method has been used to investigate an axially loaded beam resting on a two parameter foundation subjected to nonlinear lateral excitation. The obtained analytical solution expressions are similar to those obtained from other analytical techniques but with relatively less mathematical effort. Several examples are solved to describe the method and the obtained results reveal that the method is convenient for solving linear, nonlinear and higher order ordinary differential equations. It is found that in the case of beams on elastic foundations, the critical load corresponding to the first buckling is not always the smallest critical load. However, as the stiffness of the foundations increases relative to the beam, a critical load corresponding to one of the higher buckling modes may be smaller than the critical load corresponding to the first buckling mode which must be considered to avoid the buckling instability. Further, a similar phenomenon has been found for the natural frequencies of stressed beams on elastic foundations.

Keywords: boundary value problems; recursive differentiation method; differential equations; natural frequencies; critical loads; beams on elastic foundation

1. Introduction

In the last decades, numerous analytical and numerical methods have been developed to obtain approximate solutions for boundary value problem (BVP) that involve nonuniform and/or nonlinear characteristics. The most commonly used analytical techniques are as follows: Adomian decomposition method (ADM) was proposed by Adomian (1994) and used by Taha and Nassar (2014a), Bahnasawi et al. (2004) and Wazwas (2001); Variational iteration (VIM) used by He (2007) and Noor and Mohyuddin (2008); Homotopy perturbation (HPM) which was used by Tan and Abbasbandy (2008) and Jin (2008); Differential transform (DTM) was used by Ali (2012), perturbation techniques used by Nayfeh and Nayfeh (1994) and Maccari (1999) and Green's functions has been used by Fotiu et al (1987) to study the behaviour of elastoplastic vibrating beams including damage accumulations. On the other hand, numerical methods such as finite element method (FEM) used by Naidu and Rao (1996), finite difference method used by Doedel (1979), differential quadrature method DQM used by Taha and Nassar (2014b) and Chen (2002), and quasi-static second order transfer matrix used by Brunner and Irschik (1994) offer tractable alternative for solving many BVPs which have complicated formulations. In addition, a functional iterative method appropriate for the static analysis of nonuniform beams has been proposed by Jang and Sung (2012).

Most of the analytical methods construct the solution of BVP as a polynomial such that the coefficients of these polynomials are obtained to satisfy both the governing differential equation and the boundary conditions. However, numerical techniques transform the differential equation into a system of algebraic equations either on the boundary of the BVP domain or at discrete points in the BVP domain and solve it to obtain the required results. A method success is measured by its simplicity in mathematical manipulation and the time required for achieving results with fair accuracy.

In the present work, a new method named "the recursive differentiation method" (RDM) is suggested and employed to solve differential equations governing various types of BVPs. The method constructs solutions for differential equations based on Taylor expansion in a form of recursive functions. The coefficients of the terms in the recursive functions are obtained using a simple recurrence formulae obtained from the recursive differentiations of the given differential equation. It will be illustrated that the method yields exact solutions for BVPs governed by linear differential equations. However, it will be indicated that, for nonlinear differential equations, the method yields, with relatively less effort and high efficiency, similar analytical expressions obtained from other techniques (ADM and DTM).

Furthermore, the RDM method is used to obtain analytical solutions for the forced vibration of an axially loaded

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beam (beam-column) resting on a two parameter foundations and subjected to a nonlinear lateral excitation taking into account the rotational inertia. Critical loads and natural frequencies will be investigated. Also, the influences of the different parameters on the amplitude of the maximum lateral response due to nonuniform lateral excitation acting on the beam are highlighted.

2. Recursive differentiation method

Consider the nonlinear n th-order differential equation in the form

$$y^{(n)}(x) = F(x, y, y^{(1)}, y^{(2)}, \dots, y^{(n-1)}), \quad 0 \leq x \leq a \quad (1)$$

with the boundary conditions

$$B_i(y, y^{(1)}, \dots, y^{(n-1)}) = b_i, \quad i = n \quad (2)$$

where $y^{(n)}$ is the n th-derivative and b_i are constants.

In the RDM, the solution of Equation. (1) is assumed as a polynomial in the form

$$y(x) = \sum_{m=0}^{\infty} T_m \frac{(x - x_o)^m}{m!}, \quad (3)$$

where T_m are coefficients obtained to satisfy both the governing equation and the boundary conditions, x_o is the boundary coordinates.

The coefficients T_m are related to the governing differential equation on the boundary as

$$T_m = y^{(m)}|_{x=x_o} \quad (4)$$

It is found that the transformation of the solution domain to $[0, 1]$ has great effect on enhancing the convergence of the obtained solutions. In addition, the coefficients T_m may have been obtained by an algorithm derived from the recursive differentiations of the governing equation as it will be illustrated in the following examples. Actually, in practical applications, the series in Equation. (3) is truncated to a certain terms depending on the required accuracy.

Example 1: Consider the following nonlinear quadratic Riccati differential equation:

$$y^{(1)}(t) = 2y - y^2 + 1, \quad 0 \leq t \leq 1 \quad (5)$$

Subject to the boundary condition: $y(0) = 0$

Let the solution of Equation. (5) be in the form

$$y(t) = \sum_{m=0}^N T_m \frac{t^m}{m!} \quad (6)$$

Substitution of the boundary condition into the recursive equations derived from the differentiation of Equation (5) at $t = 0$, the coefficients T_m are obtained as

$$T_0 = 0, \quad T_1 = 1, \quad T_2 = 2, \quad T_3 = 2, \quad T_4 = -8, \quad T_5 = -56, \\ T_6 = -112, \quad T_7 = 848, \quad T_8 = 9088, \quad T_9 = 25216 \quad (7)$$

Substitution of Equation (7) into Equation (6), the solution for $N = 9$ is

$$y(t) = t + t^2 + \frac{2}{3!}t^3 - \frac{8}{4!}t^4 - \frac{56}{5!}t^5 - \frac{112}{6!}t^6 + \frac{848}{7!}t^7 \\ + \frac{9088}{8!}t^8 + \frac{25216}{9!}t^9 + O(t^{10}). \quad (8)$$

The exact solution to this problem is (Bahnasawi, et al 2004)

$$y(t) = 1 + \sqrt{2} \tanh \left(\sqrt{2} t + \frac{1}{2} \ln \left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right) \right) \quad (9)$$

It is found that the numerical results of RDM method are in close agreement of those obtained from the exact solution.

Example 2: Consider the nonlinear fifth order boundary value problem which arises in the mathematical modelling of viscoelastic flow (Wazwas 2001)

$$y^{(5)}(x) = e^{-x} y^2, \quad 0 \leq x \leq 1 \quad (10)$$

Subject to the boundary conditions $y(0) = y^{(1)}(0) = y^{(2)}(0) = 1$ and $y(1) = y^{(1)}(1) = e$

Let the solution of Equation (10) in the form

$$y(x) = \sum_{m=0}^N T_m \frac{x^m}{m!} \quad (11)$$

Using Equation (4), recursive equations obtained from the higher differentiation of Equation (10) and boundary conditions at $x = 0$, coefficients T_m are obtained as

$$T_0 = 1, \quad T_1 = 1, \quad T_2 = 1, \quad T_3 = 1, \quad T_6 = 1, \\ T_7 = 1, \quad T_8 = -1 + 2T_3, \quad T_9 = -1 + 4T_3 + 2T_4. \quad (12)$$

Substitution of Equation (12) into Equation (11) yields the solution of Equation (10) as

$$y(x) = 1 + x + \frac{1}{2!}x^2 + \frac{T_3}{3!}x^3 + \frac{T_4}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 \\ + \frac{1}{7!}x^7 + \frac{T_8}{8!}x^8 + \frac{T_9}{9!}x^9 + \dots \quad (13)$$

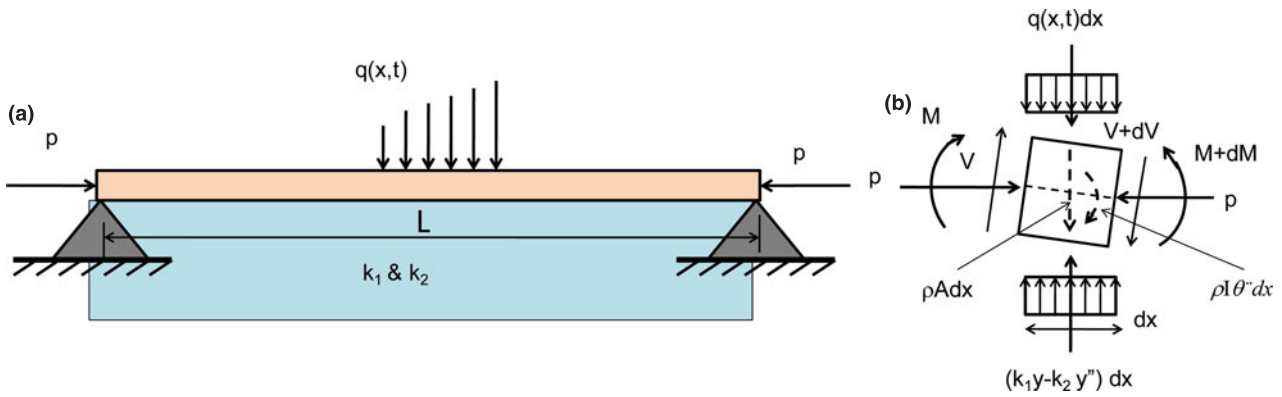


Figure 1. (a) Beam-columns on elastic foundation. (b) Element forces. This figure is available in colour online.

Using the boundary conditions at \$x = 1\$ with Equation (13), the coefficients \$T_3\$ and \$T_4\$ for truncation index \$N = 9\$ are obtained as

$$T_3 = 1, \quad T_4 = 1 \quad (14)$$

Then, the complete solution of Equation (10) is obtained as

$$y(x) = e^x \quad (15)$$

which is the exact solution.

3. Solution for beams on elastic foundations

3.1. The dynamic equation

The equations of motion of an infinitesimal element of the axially loaded beam-column subjected to lateral excitation resting on two parameter foundation, shown in Figure 1, considering rotational inertia are

$$\frac{\partial V}{\partial x} + q(x, t) - k_1 y(x, t) + k_2 \frac{\partial^2 y}{\partial x^2} = \rho A \frac{\partial^2 y}{\partial t^2}, \quad (16)$$

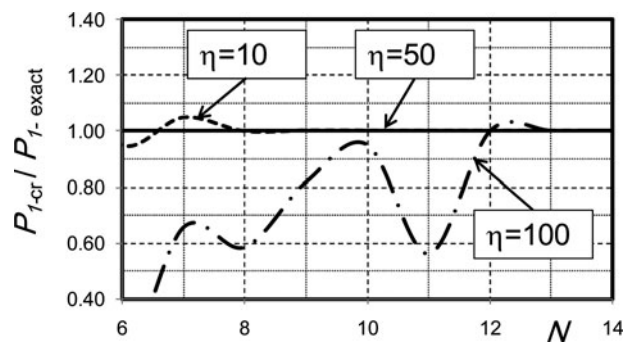


Figure 2. The effects of truncation index \$N\$ on the solution accuracy.

$$V(x, t) + p \frac{\partial y}{\partial x} - \frac{\partial M}{\partial x} = \rho I \frac{\partial^2 \theta}{\partial t^2}, \quad (17)$$

where the elastic foundation reaction is represented by two parameters; \$k_1\$ for spring (Winkler) effect and \$k_2\$ to represent the shear interaction between adjacent springs. This model has been proposed by Pasternak (1954) and used by Kerr (1965). In the present work, the foundation parameters (\$k_1, k_2\$) are calculated using the expressions proposed by Zhaohua and Cook (1983).

The slope-deflection and force-displacement relations are

$$\theta = \frac{\partial y}{\partial x} \text{ and } M(x, t) = -EI \frac{\partial^2 y}{\partial x^2}. \quad (18)$$

where \$EI\$ is the flexural stiffness of the beam, \$\rho\$ is the density, \$A\$ is the area of the cross section, \$p\$ is the axial applied load, \$q(x, t)\$ is the lateral excitation, \$E\$ is the modulus of elasticity, \$I\$ is the moment of inertia, \$\theta(x, t)\$ is the rotation, \$V(x, t)\$ is the shear force, \$M(x, t)\$ is the bending moment, \$y(x, t)\$ is the lateral response of the beam, \$x\$ is the coordinate along the beam and \$t\$ is the time.

Substitution of Equations (17) and Equation (18) into Equation (16), the equation of the beam lateral response is obtained as

$$EI \frac{\partial^4 y}{\partial x^4} + (p - k_2) \frac{\partial^2 y}{\partial x^2} - \rho I \frac{\partial^4 y}{\partial x^2 \partial t^2} + k_1 y(x) + \rho A \frac{\partial^2 y}{\partial t^2} = q(x, t). \quad (19)$$

For harmonic excitation, the beam response is harmonic, i.e.

$$q(x, t) = q(x) e^{i\Omega t} \text{ and } y(x, t) = y(x) e^{i\Omega t}. \quad (20)$$

To simulate most cases of nonuniform distributions of continuous lateral excitation, the amplitude is assumed

parabolic

$$q(x) = q_0 + q_1x + q_2x^2 \quad (21)$$

where Ω is the excitation frequency and q_0 , q_1 and q_2 are coefficients obtained to closely fit the given nonuniform distributed excitation.

Introducing the dimensionless variables, $\xi = x/L$ and $w = y/L$, where L is the beam length and substituting Equations (20) and Equation (21) into Equation (19), then the equation of the amplitude of the lateral response in a dimensionless form is obtained as

$$\begin{aligned} \frac{d^4w}{d\xi^4} + \left(P_1 - K_2 + \frac{\lambda^4}{\eta^2} \right) \frac{d^2w}{d\xi^2} + (K_1 - \lambda^4)w(\xi) \\ = Q_0 + Q_1\xi + Q_2\xi^2 \end{aligned} \quad (22)$$

where the following dimensionless parameters are introduced:

$$\begin{aligned} K_1 = \frac{k_1L^4}{EI}, \quad K_2 = \frac{k_2L^2}{EI}, \quad P_1 = \frac{pL^2}{EI} \\ \lambda^4 = \frac{\rho A \Omega^2 L^4}{EI}, \quad \eta = \frac{L}{r} \text{ and } r = \sqrt{\frac{I}{A}} \end{aligned} \quad (23.a)$$

$$Q_0 = \frac{q_0L^3}{EI}, \quad Q_1 = \frac{q_1L^4}{EI} \text{ and } Q_2 = \frac{q_2L^5}{EI}, \quad (23.b)$$

where K_1 and K_2 are the foundation linear and shear stiffness parameters, respectively; P_1 is the axial load parameter; λ is the frequency parameter; η is the slenderness ratio; r is the radius of gyration of the beam cross section and Q_0 , Q_1 and Q_2 are the amplitude of the lateral excitation parameters.

3.2. Boundary conditions

For the case of the pinned beam at both ends (P-P), the boundary conditions in dimensionless forms may be expressed as

$$\begin{aligned} \text{at } \xi = 0, \quad w(0) = w''(0) = 0 \quad \text{and at} \\ \xi = 1, \quad w(1) = w''(1) = 0. \end{aligned} \quad (24)$$

3.3. Application of recursive differentiation method

To use the RDM, Equation (22) is rewritten in the recursive form

$$w^{(4)}(\xi) = A_1w^{(0)}(\xi) + B_1w^{(2)}(\xi) + C_1 + D_1\xi + E_2\xi^2 \quad (25)$$

where

$$\begin{aligned} A_1 = -K_1 + \lambda^4, \quad B_1 = -P_1 + K_2 - \frac{\lambda^4}{\eta^2}, \\ C_1 = Q_0, \quad D_1 = Q_1, \quad E_2 = Q_2. \end{aligned} \quad (26)$$

Let the solution of Equation (25) be in the form

$$w(x) = \sum_{m=0}^N T_m \frac{\xi^m}{m!}. \quad (27)$$

The recursive differentiations of Equation (25) yield the recurrence formula for $n > 2$ as

$$w^{(n+4)}(\xi) = A_1w^{(n)} + B_1w^{(n+2)}. \quad (28)$$

Using Equation (4) and the recurrence formula (28), recurrence formulae for calculations of T_{2n+2} , $n \geq 2$ may be obtained as

$$T_{2n+2} = A_nT_0 + B_nT_2 + C_n + 2E_n, \quad (29.a)$$

$$T_{2n+3} = A_nT_1 + B_nT_3 + D_n. \quad (29.b)$$

In addition, recurrence formula for the constants A_n, B_n, C_n, D_n, E_n for $n \geq 2$ can be obtained as

$$A_{n+1} = A_1A_{n-1} + B_1A_n, \quad (30.a)$$

$$B_{n+1} = B_1B_{n-1} + B_1B_n, \quad (30.b)$$

$$C_{n+1} = A_1C_{n-1} + B_1C_n, \quad (30.c)$$

$$D_{n+1} = A_1D_{n-1} + B_1D_n, \quad (30.d)$$

$$E_{n+1} = A_1E_{n-1} + B_1E_n, \quad (30.e)$$

where

$$\begin{aligned} A_2 = A_1B_1, \quad B_2 = A_1 + B_1B_1, \quad C_2 = B_1C_1, \\ D_2 = A_1D_1 \text{ and } E_1 = 0. \end{aligned} \quad (31)$$

Then, the amplitude $w(\xi)$ is obtained as

$$\begin{aligned} w(\xi) = T_0R_0(\xi) + T_1R_1(\xi) + T_2R_2(\xi) \\ + T_3R_3(\xi) + F(\xi), \end{aligned} \quad (32)$$

where the recursive functions $R_i(\xi)$, $i = 0, 1, 2, 3$ and $F(\xi)$ are

$$R_0(\xi) = 1 + A_1 \frac{\xi^4}{4!} + \sum_{n=2}^N A_n \frac{\xi^{2n+2}}{(2n+2)!}, \quad (33.a)$$

$$R_1(\xi) = \xi + A_1 \frac{\xi^5}{5!} + \sum_{n=2}^N A_n \frac{\xi^{2n+3}}{(2n+3)!}, \quad (33.b)$$

$$R_2(\xi) = \frac{\xi^2}{2!} + B_1 \frac{\xi^4}{4!} + \sum_{n=2}^N B_n \frac{\xi^{2n+2}}{(2n+2)!}, \quad (33.c)$$

$$R_3(\xi) = \frac{\xi^3}{3!} + B_1 \frac{\xi^5}{5!} + \sum_{n=2}^N B_n \frac{\xi^{2n+3}}{(2n+3)!}, \quad (33.d)$$

$$F(\xi) = \frac{\xi^4}{4!} C_1 + \frac{\xi^5}{5!} D_1 + \sum_{n=2}^N \left((C_n + 2E_n) \frac{\xi^{2n+2}}{(2n+2)!} + D_n \frac{\xi^{2n+3}}{(2n+3)!} \right). \quad (33.e)$$

Using the boundary conditions at (Equation 24), then the unknown coefficients are obtained as

$$T_0 = T_2 = 0, \quad (34.a)$$

$$T_1 = \frac{R_3 F_2 - R_{32} F}{R_1 R_{32} - R_{12} R_3}, \quad (34.b)$$

$$T_3 = \frac{R_{12} F - R_1 F_2}{R_1 R_{32} - R_{12} R_3}, \quad (34.c)$$

where $R_i = R_i(1)$, $R_{i2} = R_i''(1)$, $i = 1, 3$, $F = F(1)$ and $F_2 = F''(1)$.

The bending moment distribution $M(\xi)$ and the shearing force distribution $V(\xi)$ may be obtained as

$$M(\xi) = -\frac{EI}{L} \frac{d^2 w}{d\xi^2} \quad \text{and} \quad V(\xi) = -\frac{EI}{L^2} \left(\left(\frac{d^3 w}{d\xi^3} + \left(P_1 + \frac{\lambda^4}{\eta^2} \right) \frac{dw}{d\xi} \right) \right). \quad (35)$$

Using Equation (34), the coefficients T_1 and T_3 can be calculated, and then substituted into Equations (32), (33) and Equation (35) to obtain the distributions of the lateral response amplitude, the bending moment and the shear force.

3.4. Critical loads and natural frequencies

Equation (32) indicates that the amplitude of the lateral response becomes unbounded when

$$R_1 R_{32} - R_{12} R_3 = 0. \quad (36)$$

This equation involves two parameters (P_1 and Ω), and the unbounded lateral response represents the instability of the system at certain critical values of P_1 for a given value of Ω and vice versa. Actually, for a given value of P_1 , Equation (36) simulates the resonance condition of the

system when the excitation frequency coincides with the natural frequency of the system ($\Omega = \omega_n$). In addition, for a given value of $\Omega = 0$, Equation (36) simulates the instability of the system due to buckling when the axial applied load equal the system critical load.

Furthermore, the RDM solution of the homogenous version of Equation (22) requires Equation (36) as a condition of non-trivial solution; either for static buckling modes or for free vibration mode shapes.

Mathematically, Equation (36) represents the eigen value problem of the system in two parameters P_{1-cr} and ω_n . It is an algebraic equation of degree N and may be solved by a simple iterative algorithm for critical loads P_{1-cr} for static case ($\omega_n = 0$) and for natural frequencies ω_n for free vibration of the beam-foundation system (for $P < P_{cr}$).

On the other hand, the closed form solution of the homogenous version of Equation (22) yields an exact expression for the aforementioned instability condition in the form

$$A_{1n} = n^2 \pi^2 + \frac{A_{2n}}{n^2 \pi^2}, \quad n = 1, 2, \dots \quad (37)$$

$$A_{1n} = P_1 - K_2 + \frac{\lambda_n^4}{\eta^2} \quad \text{and} \quad A_{2n} = K_1 - \lambda_n^4 \quad (38)$$

Substitution Equation (38) into Equation (37), the natural frequencies of a P-P beam-column on a two parameter foundation is obtained as

$$\lambda_n^4 = \frac{\eta^2 (n^4 \pi^4 - n^2 \pi^2 (P_1 - K_2) + K_1)}{(\eta^2 + n^2 \pi^2)}, \quad n = 1, 2, \dots \quad (39)$$

For buckling mode ($\lambda_n = 0$) Equation (39) yields the critical loads as

$$P_{1-cr} = n^2 \pi^2 + K_2 + \frac{K_1}{n^2 \pi^2}, \quad n = 1, 2, \dots \quad (40)$$

3.5. Verification

To verify the analytical expressions obtained from the RDM, the natural frequency parameter of a beam-column resting on a two parameter foundation calculated from the RDM (Equation 36) and those obtained from the exact solution (Equation 39) are presented in Table 1. It is clear that the RDM results are in close agreement with those calculated from the exact solution.

4. Numerical Results

4.1. Convergence analysis

To investigate the accuracy of the RDM, several attempts have been conducted to determine the effect of the truncation index N (number of terms considered in the se-

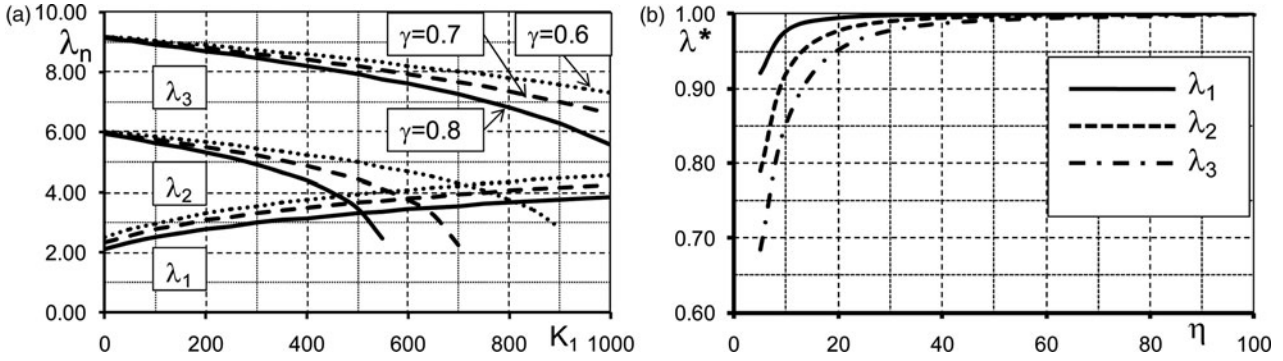


Figure 3. (a) The influence of linear foundation stiffness on natural frequencies ($K_2 = 0$). (b) Effect of rotational inertia on natural frequencies.

ries) on the accuracy of the obtained results. The critical load calculated using RDM (Equation 36) divided by the exact value obtained from Equation (40) for different N are illustrated in Figure (2). It is found that the solution fluctuates and loses its accuracy at small N and the convergence increases as N increases. As the beam slenderness ratio increases, the coefficients of the governing differential equation increase and more terms in the series need to be considered. However, it is obvious that using $N = 15$ is adequate to obtain fair accuracy for the critical loads in the practice range of beam configurations. However, it is expected that a greater N is required in the case of differential equations with larger coefficients. It is common in series solutions that the truncation error increases as the coefficients of the differential equation increase and more terms need to be considered to control the accuracy. In the present work, the obtained results are calculated considering $N = 25$.

4.2. The influence of nonlinear lateral excitation

Although the solution expressions are obtained in dimensionless forms to be valid for any specific case, the properties of the beam-column considered in the present parametric study are concrete beam, $b = 0.2$ m, $h = 0.5$ m, $\eta = 50$ m, $E = 2.1E10$ Pa, Poisson ratio μ is 0.15 and the average lateral excitation amplitude acting on the beam $q_{av} = 6E4$ N/m. However, a simple MTLAB code or even Excel spreadsheet may be used to calculate the required parameters.

Investigation of Equation (22) indicates that the effect of K_2 is inversely proportional to the influence of P_1 , hence the effect of K_2 may be obtained from the influence of P_1 .

The natural frequencies are calculated for different system parameters and illustrated in Figure 3. Figure 3 a indicates that the first natural frequency λ_1 is not always the smaller natural frequency. However, it is a common practice to use the first natural frequency λ_1 to avoid the dynamic instability of the system. It is clear that, for slender beams on elastic foundation, λ_2 may be smaller than λ_1 which may

lead to dynamic instability of the system before approaching λ_1 . Also, the same result is observed for the critical loads. For example, for $K_1 = 550$, $P_{cr-1} = 65.6$ and $P_{cr-2} = 53.41$, while for $K_1 = 550$ and $P_1 = 52.5$ the values of $\lambda_1 = 3.37$ and $\lambda_2 = 2.45$ which must be considered in the stability analysis. Actually, for beams without foundation, the first critical load and the first natural frequency (P_{cr-1} and λ_1) are always the smaller (critical) values. Furthermore, for the beams without axial loads, the natural frequency accompanied the first vibration mode is always the smaller natural frequency.

Figure 3b shows the effect of the rotational inertia on the natural frequencies where $\lambda^* = \lambda(\eta)/\lambda(\eta = 200)$. It is found that the inertia effect is significant for lower values of slenderness ratio ($\eta < 20$) and should be considered in the dynamic stability analysis and the effect decreases as the slenderness ratio increases. Also, the effect of rotational inertia is more pronounced for higher modes.

The influence of α on w^* is shown in Figure 4, where it is observed that the system loses its dynamic stability as the excitation frequency approaches the natural frequency. However, in real systems the system damping reduces the response amplitude in the resonance neighbourhood. Also, it is clear from Figure 4 that the amplitude decreases as the foundation stiffness increases and as the applied axial load decreases.

The RDM expressions are used to investigate the influence of the loading type on the maximum value of the lateral response amplitude parameter w^*_{max} and the results are presented in Figure 5 where the following parameters are defined:

$$\gamma = \frac{P_1}{P_{cr}}, \quad \alpha = \frac{\Omega}{\omega_1}, \quad \text{and} \quad w^*_{max} = \frac{w_{max}}{w_U} \quad (41)$$

It is clear that the amplitude increases as the axial load increases and as the foundation stiffness decreases. The maximum amplitude due to nonlinear loading ($q_0 = q_1 = 0$) is greater than the maximum amplitude due to linear loading

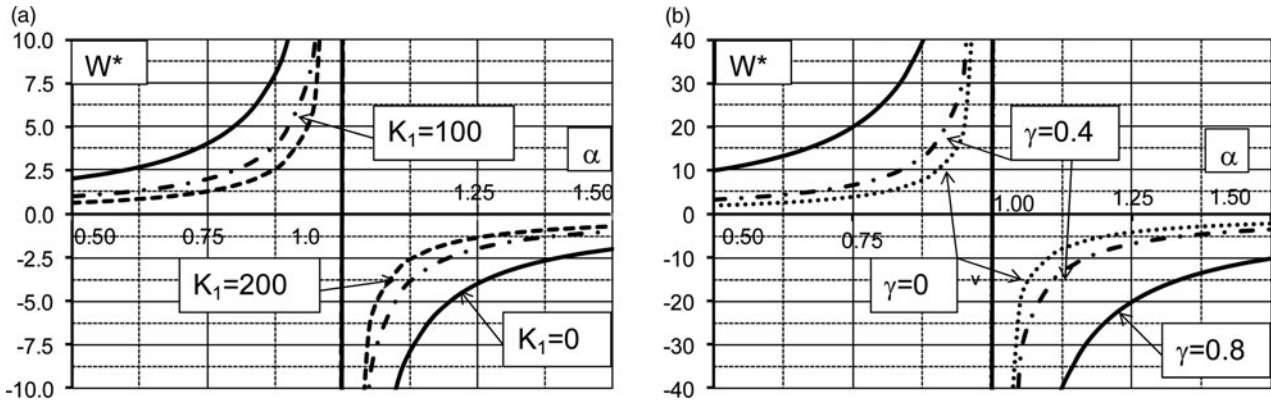


Figure 4. The effects of excitation frequency on the amplitude of the mid-span response ($K_2 = 0$). (a) Different foundation linear stiffness K_1 . (b) Different loading ratio γ .

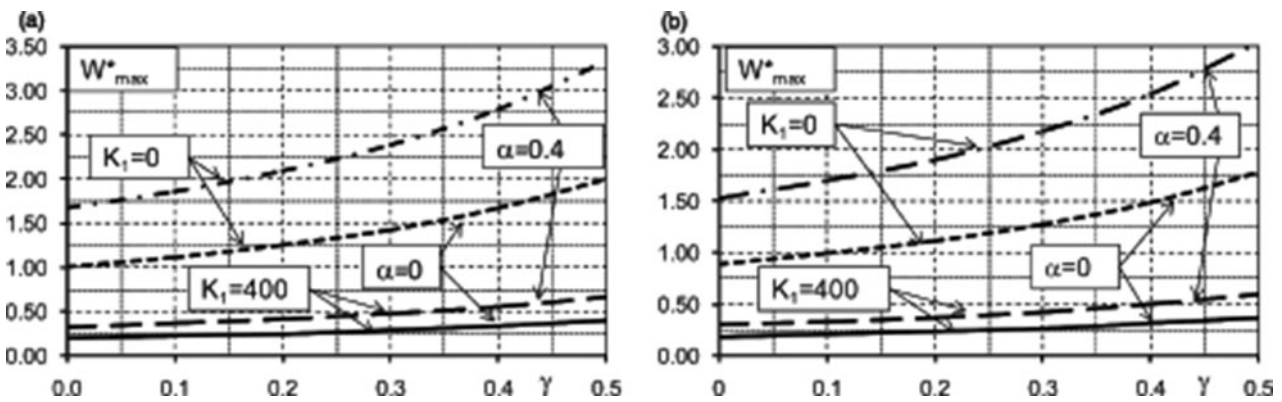


Figure 5. The effects of axial load on the maximum response amplitude of the beam-column ($K_2 = 0$). (a) Linear loading. (b) Nonlinear loading.

($q_0 = q_2 = 0$). Also, it is observed that the effect of the excitation frequency on the lateral excitation amplitude in the case of a beam without foundation ($K_1 = 0$) is greater than its effect in the case of beam resting on foundation. Further, the effect of loading ratio γ is more noticeable in the case of a beam without foundation.

Table 1. The frequency parameters for P-P beam on elastic foundation ($\eta = 20$).

K_1	P_{cr}	P/P_{cr}	λ_1		λ_2	
			Exact	RDM	Exact	RDM
0	9.87	0	3.12	3.12	6.14	6.13
		0.50	2.63	2.62	5.94	5.93
100	20.00	0	3.73	3.72	6.23	6.23
		0.50	3.13	3.13	5.82	5.82
200	30.13	0	4.13	4.12	6.33	6.32
		0.50	3.47	3.47	5.70	5.7
500	60.53	0	4.91	4.91	6.58	6.57
		0.50	4.13	4.13	5.30	5.29

5. Conclusions

A new analytical method named the recursive differentiation method (RDM) is proposed to tackle numerous types of boundary value problems in finite domain. The obtained results reveal that the proposed method is simple, straightforward and requires a relatively less mathematical effort compared with the available analytical techniques.

However, it is found that, the accuracy of the obtained RDM expressions is greatly enhanced when the solution domain is transformed to the domain $[0, 1]$. In addition, the mathematical effort required to obtain the solution expressions is highly decreased if recurrence formulae for calculating the coefficients are derived from recursive differentiations of the corresponding differential equation as illustrated in the given examples.

Furthermore, the RDM has been used to derive expressions for the dynamic behaviour of beams resting on two parameter foundations subjected to axial static load and lateral nonuniform excitation in terms of recursive functions. It is found that the influence of rotational inertia on the natural frequencies is significant for short beams (beams with slenderness ratio $\eta < 20$) and should be considered in

the dynamic stability analysis and it may be neglected in the calculations of the response amplitude. It is concluded that the maximum values of the amplitude of the lateral response for nonlinear loading type is greater than its value for linear loading type with the same average excitation amplitude.

In contradiction to beams without foundation, it is found that for beams resting on elastic foundation, the critical loads and natural frequencies accompanied higher modes may be smaller than those for the first mode which must be considered to avoid static and dynamic instability.

Notes on contributor



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