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# General analysis of stressed beams on elastic foundation subjected to discontinuous loads

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Analytical solutions describing the static and dynamic behaviour of axially loaded beams with elastic end restrains resting on two parameter foundation subjected to multiple discontinuous lateral loads have been obtained using the Adomian decomposition method (ADM). Fourier expansion (FE) is used to transform discontinuous loads to continuous loading functions. The effects of the truncation index of both the ADM and FE on the accuracy of the obtained solutions are studied to assess the validity of the given solutions. The influences of end restraints, foundation stiffness, loading type, axial stressing and excitation frequency on the beam deflection and straining actions are investigated. It has been concluded that as the foundation stiffness increases relative to the beam stiffness, the effects of the end conditions on the beam behaviour decrease.

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Discontinuous loads; beam on elastic foundation; Adomian decomposition method

## 1. Introduction

The computational model of a beam on elastic foundation is used widely to describe the behaviour of many engineering applications in geotechnical, road, railroad and marine structures. Numerous analytical and numerical techniques were being developed to investigate the behaviour of such model with different configurations. Analytical methods developed recently to deal with boundary value problems (BVPs) have been used to study the cases of beams with simple configurations. The Adomian decomposition method (ADM) proposed by Adomian (1994) was used by Bahnasawi et al. (2004), Wazwas (2001) and Taha and Nassar (2015) to solve differential equations governing some BVPs. Variational iteration method (VIM) developed by He (2007) was used by Noor and Mohyud-Din (2008) to solve heat and wave-like equations. Homotopy perturbation method (HPM) was used by Tan and Abbasbandy (2008) and Jin (2008) and differential transform (DTM) was used by Ali (2012) to investigate beams with different configurations.

Recursive differentiation method (RDM), which is based on Taylor series, is a simple, efficient and straightforward method proposed by Taha (2014) to obtain analytical solutions for BVPs and was used by Taha and Doha (2015) to study the dynamics of beams on elastic foundation.

Numerical methods, on the other hand, represent a tractable alternative to obtain approximate solutions for cases of beams that involve complicated configurations. Finite element method (FEM) was used by Mullapudi and Ayoub (2010) and Naidu and Rao (1996) to deal with beams on elastic foundations. Also, differential quadrature method (DQM) was used by Chen (2002) and Taha and Nassar (2014) to study nonuniform beams on two-parameter foundation. Ahmed et al. (2014) used a numerical iterative algorithm based on Green's functions to investigate static deflection of an infinite beam on nonlinear foundations. Lokshin et al. (2008) studied the buckling of a simply supported beam on an elastic foundation supported within its span with elastic supports using a finite difference algorithm based on the Euler's equation for elastic line of the beam.

Few analytical solutions that deal with beams involve nonuniform configurations in beam geometry or loading functions due to its mathematical complexity. In case of multiple discontinuous loads, analytical solution necessitates dividing the beam into continuous regions. Afterwards; the conditions of continuity and equilibrium are applied at the ends of these regions to obtain a system of equations solved for deflections and straining actions along the beam. Yavari et al. (2001) proposed a generalised solution for beams with jump discontinuities on elastic foundations. Analytical solution of a beam on an elastic foundation using singularity functions was introduced by Dinev (2012). These analytical solutions involve a lot of mathematical complications.

In the present work, the ADM is utilised to deal with the cases of beams subjected to multiple discontinuous loads. The discontinuous loading functions are transformed to continuous loading functions using Fourier expansion (FE). The effects of axial stressing, excitation frequency, loading type and stiffness of elastic foundations on the beam behaviour will be investigated.

The paper is organised as follows: In Section 2, the governing equations describing the beam behaviour are derived; then the ADM is used to solve the governing equation in Section 3. Discussion of the obtained results is presented in Section 4 and some concluding remarks are given in Section 5.

## 2. Beam dynamic equation

Figure 1a shows a prismatic beam of length $L$ resting on a two-parameter foundation and connected at its ends with translational springs of stiffness ($k_{T1}$ and $k_{T2}$) and rotational springs of stiffness ($k_{R1}$ and $k_{R2}$). The beam is subjected to a static axial load $p$, a lateral concentrated dynamic load $q_1(x, t)$ and

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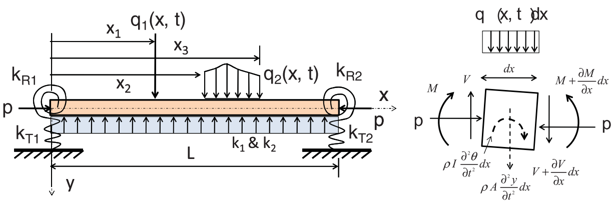


Figure 1. (a) Stressed beam on elastic foundations. (b) Forces acting on a beam element.

a lateral partially distributed load $q_2(x,\ t)$. The equations of motion of the beam element shown in Figure 1b, considering the rotational inertia, may be expressed as

$$\frac{\partial V}{\partial x} + q(x,t) - k_1 y(x,t) + k_2\frac{\partial^2 y}{\partial x^2} = \rho A\frac{\partial^2 y}{\partial t^2}, \tag{1}$$

$$V(x,t) + p\frac{\partial y}{\partial x} - \frac{\partial M}{\partial x} = \rho I\frac{\partial^2 \theta}{\partial t^2}, \tag{2}$$

The slope–deflection and force–displacement relations are

$$\theta = \frac{\partial y}{\partial x} \quad \text{and} \quad M(x,t) = -EI\frac{\partial^2 y}{\partial x^2}, \tag{3}$$

where $EI$ is the flexural stiffness of the beam, $\rho$ is the density, $A$ is the area of the cross section, $k_1$ and $k_2$ are the linear and shear foundation stiffness per unit length of the beam, $q(x,\ t)$ is the lateral excitation, $E$ is the modulus of elasticity, $I$ is the moment of inertia, $\theta(x,\ t)$ is the rotation, $V(x,\ t)$ is the shear force, $M(x,\ t)$ is the bending moment, $y(x,\ t)$ is the lateral response of the beam, $x$ is the coordinate along the beam and $t$ is time.

Substituting Equations (2) and (3) into Equation (1), the lateral response equation is

$$EI\frac{\partial^4 y}{\partial x^4} + (p-k_2)\frac{\partial^2 y}{\partial x^2} - \rho I\frac{\partial^4 y}{\partial x^2 \partial t^2} + k_1 y(x,t) + \rho A\frac{\partial^2 y}{\partial t^2} = q(x,t). \tag{4}$$

Under the assumption of harmonic excitation $q(x,t) = q(x)e^{i\Omega t}$, the response will be harmonic, i.e. $y(x,t) = y(x)e^{i\Omega t}$, where $\Omega$ is the excitation frequency. Using dimensionless variables, $\xi = x/L$ and $w(x) = y(x)/L$, $w(x)$ is the lateral response dimensionless amplitude, then, the equation governing $w(x)$ may be expressed as

$$\frac{d^4 w}{d\xi^4} + \left(P - K_2 + \frac{\lambda^4}{\eta^2}\right)\frac{d^2 w}{d\xi^2} + (K_1 - \lambda^4)w(\xi) = \frac{L^3 q(\xi)}{EI}, \tag{5}$$

where

$$K_1 = \frac{k_1 L^4}{EI}, \quad K_2 = \frac{k_2 L^2}{EI}, \quad P = \frac{pL^2}{EI},$$
$$\lambda^4 = \frac{\rho\Omega^2 L^2 \eta^2}{E}, \quad \eta = \frac{L}{r} \quad \text{and} \quad r = \sqrt{\frac{I}{A}}. \tag{6}$$

$K_1$ and $K_2$ are the foundation linear and shear stiffness parameters, $P$ is the axial load parameter, $\lambda$ is the frequency parameter, $\eta$ is the slenderness parameter and $r$ is the radius of gyration of the beam cross section.

### 2.1. Expansion of discontinuous loads

To overcome the discontinuity of both concentrated load $q_1(x,t)$ and partially distributed load $q_2(x,t)$ shown in Figure 1, FE may be used to expand these loads as a continuous function. Let a discontinuous load $q(x)$ is to be transformed to a continuous function in the form

$$q(x) = \sum_{m=1}^{M_1} q_m \sin\left(\frac{m\pi x}{L}\right), \tag{7}$$

where $M_1$ is the truncation index determined to control the solution accuracy. Multiply both sides of Equation (7) by $\sin(\frac{n\pi x}{L})$ and integrate, then

$$\int_o^L q(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_o^L q_m \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx. \tag{8}$$

Applying the trigonometric functions orthogonality property,

$$\int_o^1 \sin(n\pi\xi)\sin(m\pi\xi)d\xi = 0 \quad \text{for } m \neq n$$
$$\int_o^1 \sin(n\pi\xi)\sin(m\pi\xi)d\xi = \frac{L}{2} \quad \text{for } m = n \tag{9}$$

Then, the continuous load amplitude $q_m$ may be obtained as

$$q_m = \frac{2}{L}\int_o^1 q(\xi)\sin(m\pi\xi)d\xi. \tag{10}$$

In case of concentrated load $q_1(x)$, assuming the load is distributed over an infinitesimal interval $\delta$, then

$$q_m = \frac{2q_1}{\delta L}\int_{\xi_1-\frac{\delta}{2}}^{\xi_1+\frac{\delta}{2}} \sin(m\pi\xi)\,d\xi = \frac{4q_1}{m\pi\delta L}\left(\sin(m\pi\xi_1)\sin\left(\frac{m\pi\delta}{2}\right)\right). \tag{11}$$

For $\delta \to 0$, $\sin\left(\frac{m\pi\delta}{2}\right) \cong \frac{m\pi\delta}{2}$, then

$$q_m = \frac{2q_1}{L}\sin(m\pi\xi_1). \tag{12}$$

On the other hand, for the case of partially distributed load $q_2(\xi)$, then

$$q_m = \frac{2q_2}{L}\int_{\xi_2}^{\xi_3} \sin(m\pi\xi)\,d\xi = \frac{2q_2}{m\pi}\left(\cos(m\pi\xi_2) - \cos(m\pi\xi_3)\right). \tag{13}$$

Then, for both concentrated load $q_1$ and partially distributed load $q_2$, $q_m$ may be expressed as

$$q_m = \frac{2q_1}{L}\sin(m\pi\xi_1) + \frac{2q_2}{m\pi}\left(\cos(m\pi\xi_2) - \cos(m\pi\xi_3)\right). \tag{14}$$

### 2.2. Boundary conditions

For the beam shown in Figure 1, boundary conditions due to elastic restraints at the beam ends are given by

$$V(0,t) = -k_{T1}y(0,t) \quad \text{and} \quad M(0,t) = -k_{R1}\frac{\partial y}{\partial x}(0,t) \quad \text{at } x = 0, \tag{15}$$

$$V(L,t) = k_{T2}y(L,t) \quad \text{and} \quad M(L,t) = k_{R2}\frac{\partial y}{\partial x}(L,t) \quad \text{at } x = L. \tag{16}$$

Using Equations (2) and (3), the boundary conditions in dimensionless forms for harmonic excitation become

$$w^{(3)}(0) + \left(P + \frac{\lambda_F^4}{\eta^2}\right)w^{(1)}(0) - K_{T1}w(0) = 0 \quad \text{and} \quad w^{(2)}(0) - K_{R1}w^{(1)}(0) = 0 \quad \text{at } \xi = 0, \tag{17}$$

$$w^{(3)}(1) + \left(P + \frac{\lambda_F^4}{\eta^2}\right)w^{(1)}(1) + K_{T2}w(1) = 0 \quad \text{and} \quad w^{(2)}(1) + K_{R2}w^{(1)}(1) = 0 \quad \text{at } \xi = 1, \tag{18}$$

where $w^{(m)}(\xi)$ is the $m$-derivative of $w(\xi)$, and

$$K_{T1} = \frac{k_{T1}L^3}{EI}, \quad K_{T2} = \frac{k_{T2}L^3}{EI}, \quad K_{R1} = \frac{k_{R1}L}{EI}, \quad K_{R2} = \frac{k_{R2}L}{EI}.$$

## 3. Solution of the amplitude equation

### 3.1. Applications of the ADM

To implement the ADM, Equation (5) is rewritten in the form

$$D^{(4)}w(\xi) = E_1 w(\xi) + E_2 D^{(2)}w(\xi) + F(\xi), \tag{19}$$

where

$D^{(j)} = \frac{d^j}{d\xi^j}$, $E_1 = -(K_1 - \lambda^4)$, $E_2 = -(P - K_2 + \frac{\lambda^4}{\eta^2})$ and $F(\xi) = \sum_{m=1}^{M_1} Q_m \sin(m\pi\xi)$

$Q_m = 2Q_1\sin(m\pi\xi_1) + 2Q_2\,(\cos(m\pi\xi_2) - \cos(m\pi\xi_3))$

$Q_1 = \frac{q_1 L^2}{EI}$ and $Q_2 = \frac{q_2 L^3}{m\pi EI}$.

Apply the inverse operator $D^{(-4)}$ on both sides of Equation (19), one obtains

$$D^{(-4)}D^{(4)}w = E_1 D^{(-4)}w + E_2 D^{(-4)}D^{(2)}w + D^{(-4)}F(\xi), \tag{20}$$

where the inverse operator $D^{(-4)}$ is defined as

$$D^{(-4)}w(\xi) = \int_0^{\xi}\int_0^{\xi}\int_0^{\xi}\int_0^{\xi} w(\xi)d\xi = w(\xi) - w(0) - \xi w^{(1)}(0) - \frac{1}{2}\xi^2 w^{(2)}(0) - \frac{1}{6}\xi^3 w^{(1)}(0). \tag{21}$$

Then, Equation (20) may be expressed as

$$w(\xi) = \delta_1 + \xi\delta_2 + \frac{\xi^2}{2!}\delta_3 + \frac{\xi^3}{3!}\delta_4 + E_1 D^{(-4)}w + E_2 D^{(-4)}D^{(2)}w + \sum_{m=1}^{M_1} Q_m\left(\frac{\sin(m\pi\xi)}{(m\pi)^4} - \frac{\xi}{(m\pi)^3} + \frac{\xi^3}{3!(m\pi)}\right), \tag{22}$$

where $\delta_i$, $i = 1{:}4$ are the amplitude $w(\xi)$ and its derivatives at $\xi = 0$.

According to the ADM, $w(\xi)$ may be decomposed from $N$ components as

$$w(\xi) = \sum_{n=1}^{N} w_n(\xi), \tag{23}$$

where $N$ is the truncation index selected to control the solution accuracy. Substituting Equation (23) into Equation (22) and let the first component taken as

$$w_1(\xi) = \delta_1 + \xi\delta_2 + \frac{\xi^2}{2!}\delta_3 + \frac{\xi^3}{3!}\delta_4 + \sum_{m=1}^{M_1} Q_m\left(\frac{\sin(m\pi\xi)}{(m\pi)^4} - \frac{\xi}{(m\pi)^3} + \frac{\xi^3}{3!(m\pi)}\right). \tag{24}$$

Then, Equation (22) yields

$$\sum_{n=2}^{N} w_n(\xi) = E_1\sum_{n=1}^{N} D^{(-4)}w_n(\xi) + E_2\sum_{n=1}^{N} D^{(-4)}D^{(2)}w_n(\xi). \tag{25}$$

According to the ADM, the recurrence formula for $w_n(\xi)$ is

$$w_{n+1}(\xi) = E_1 D^{(-4)} w_n(\xi) + E_2 D^{(-4)} D^{(2)} w_n(\xi). \quad (26)$$

The $n$-component of the amplitude $w(\xi)$ may be expressed as

$$w_n(\xi) = \sum_{i=1}^{4} a_{n,i}(\xi)\delta_i + a_n(\xi), \quad (27)$$

where the coefficients $a_{1,i}(\xi)$ and $a_1(\xi)$ for the first component are

$$a_{1,i}(\xi) = \frac{\xi^{i-1}}{(i-1)!} \quad \text{and}$$
$$a_1(\xi) = \sum_{m=1}^{M_1} Q_m \left( \frac{\sin(m\pi\xi)}{(m\pi)^4} - \frac{\xi}{(m\pi)^3} + \frac{\xi^3}{3!(m\pi)} \right). \quad (28)$$

Inserting Equation (27) into Equation (23), then

$$w(\xi) = \sum_{n=1}^{N} \left( \sum_{i=1}^{4} a_{n,i}(\xi)\delta_i + a_n(\xi) \right). \quad (29)$$

The recurrence formulae for the coefficients $a_{n,\,\mathrm{i}}(\xi)$ and $a_n(\xi)$ are

$$a_{n+1,i}(\xi) = E_1 D^{(-4)} a_{n,i}(\xi) + E_2 D^{(-4)} D^{(2)} a_{n,i}(\xi) \quad \text{and}$$
$$a_{n+1}(\xi) = E_1 D^{(-4)} a_n(\xi) + E_2 D^{(-4)} D^{(2)} a_n(\xi). \quad (30)$$

The higher derivatives of the amplitude $w(\xi)$ may be expressed as

$$D^{(1)} w(\xi) = \sum_{n=1}^{N} \left( \sum_{i=1}^{4} b_{n,i}(\xi)\delta_i + b_n(\xi) \right), \quad (31)$$
$$D^{(2)} w(\xi) = \sum_{n=1}^{N} \left( \sum_{i=1}^{4} c_{n,i}(\xi)\delta_i + c_n(\xi) \right), \quad (32)$$
$$D^{(3)} w(\xi) = \sum_{n=1}^{N} \left( \sum_{i=1}^{4} d_{n,i}(\xi)\delta_i + d_n(\xi) \right), \quad (33)$$

where

$$b_{n,i}(\xi) = Da_{n,i}(\xi),\ c_{n,i}(\xi) = Db_{n,i}(\xi) \quad \text{and} \quad d_{n,i}(\xi) = Dc_{n,i}(\xi),$$
$$b_n(\xi) = Da_n(\xi),\ c_n(\xi) = Db_n(\xi) \quad \text{and} \quad d_n(\xi) = Dc_n(\xi). \quad (33)$$

However, Equation (29) may be expressed in a more compact form as

$$w(\xi) = \sum_{i=1}^{4} \left( a_{t,i}(\xi)\delta_i \right) + a_t(\xi), \quad (34)$$

where the coefficients $a_{t,i}(\xi)$ are given by

$$a_{t,\,i}(\xi) = \sum_{n=1}^{N} a_{n,i}(\xi) \quad \text{and} \quad a_t(\xi) = \sum_{n=1}^{N} a_n(\xi). \quad (35)$$

### 3.2. Applications of the boundary conditions

Substituting Equation (34) into the boundary conditions, Equations (17) and (18), a system of four-algebraic equations in four-unknowns, $\delta_i$,$i = 1{:}4$ is obtained as follows:

At $\xi = 0$:

$$K_{T1}\delta_1 - \left( P + \frac{\lambda^4}{\eta^2} \right)\delta_2 - \delta_4 = 0, \quad (36)$$
$$K_{R1}\delta_2 - \delta_3 = 0. \quad (37)$$

At $\xi = 1$:

$$\sum_{i=1}^{4} \left( K_{T2} a_{t,i}(\xi) + \left( P + \frac{\lambda^4}{\eta^2} \right) b_{t,i} + d_{t,i} \right)\delta_i = -\left( K_{T2} a_t(\xi) + \left( P + \frac{\lambda^4}{\eta^2} \right) b_t + d_t \right), \quad (38)$$

$$\sum_{i=1}^{4} \left( K_{R2} b_{t,i}(\xi) + c_{t,i} \right)\delta_i = -\left( K_{R2} b_t(\xi) + c_t \right). \quad (39)$$

Using matrix notations, Equations (36): (39) take the form

$$[A_T]_{4x4}[\delta_i]_{4x1} = [B_T]_{4x1}. \quad (40)$$

Then, the response dimensionless amplitude $w(\xi)$ and its derivatives at $\xi = 0$, $\delta_i$,$i = 1{:}4$ become

$$[\delta_i]_{4x1} = [A_T]_{4x4}^{-1}[B_T]_{4x1}. \quad (41)$$

In static analysis of axially loaded beams ($\Omega = 0$), the lateral displacement becomes unbounded when the determinant of the coefficient matrix $|A_T|$ vanishes. This case represents the buckling conditions of the beam, and the expansion of the determinant $|A_T|$ yields the critical loads, while the corresponding eigenvectors represent the buckling modes.

On the other hand, in free vibration analysis of axially loaded beams with given axial load $P \le P_{cr}$, the lateral response becomes unbounded when the determinant of the coefficient matrix $|A_T|$ vanishes; also, this condition represents the resonance condition of the beam, and the expansion of the determinant yields the natural frequencies of the beam, while the corresponding eigen vectors represent the vibration mode shapes.

### 3.3. Accuracy analysis

Results of the present solution are verified against both the FEM solution (Naidu and Rao 1996) and the RDM solutions (Taha

**Table 1.** Effect of the FE truncation index ($M_1$) on the deflection calculations accuracy.

| Error % of the ADM solution against exact solution for beam without foundation | | | | | | | |
|---|---|---|---|---|---|---|---|
| $M_1$ | | 3 | 5 | 10 | 20 | 100 | 1000 |
| Distributed load | $w(0.5)$ | 0.7763 | 0.3113 | 0.0081 | 0.0081 | 0.0001 | 0.0000 |
| | $M(0.5)$ | 1.4124 | 0.7629 | 0.0150 | 0.0150 | 0.0001 | 0.0000 |
| | $V(0)$ | 11.0330 | 7.1748 | 2.0666 | 2.0666 | 0.4069 | 0.0405 |
| Concentrated load | $w(0.5)$ | 0.9244 | 0.9216 | 0.2114 | 0.0109 | 0.0001 | 0.0000 |
| | $M(0.5)$ | 23.8481 | 15.7368 | 8.8469 | 4.2135 | 0.8171 | 0.0811 |
| | $V(0)$ | 17.8097 | 9.3771 | 5.9314 | 3.2794 | 0.6406 | 0.0637 |

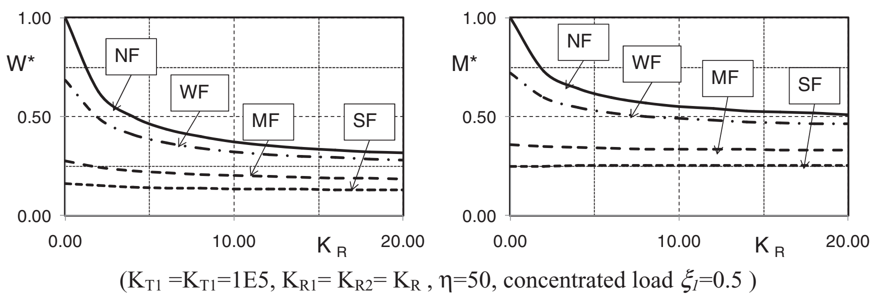


**Figure 2.** Influence of end rotation restraints on $w^*$ and $M^*$ in case of concentrated load ($K_{T1} = K_{T1} = 1E5$, $K_{R1} = K_{R2} = K_R$, $\eta = 50$, concentrated load $\xi_1 = 0.5$).

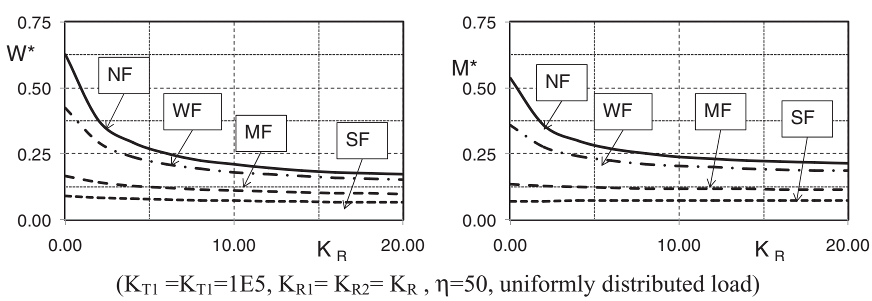


**Figure 3.** Influence of end rotation restraints on $w^*$ and $M^*$ in the case of distributed load ($K_{T1} = K_{T1} = 1E5$, $K_{R1} = K_{R2} = K_R$, $\eta = 50$, uniformly distributed load).

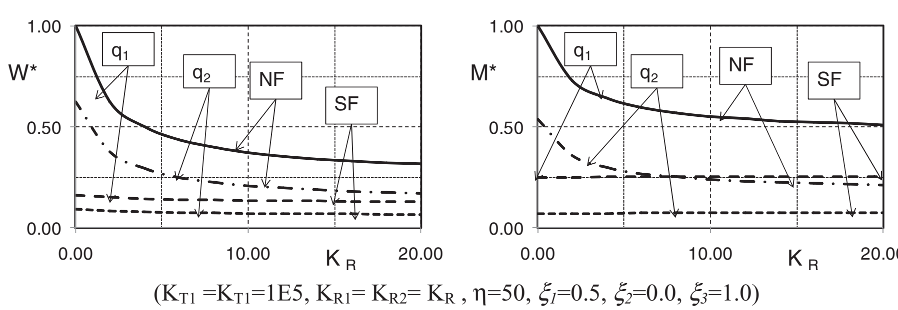


**Figure 4.** Influence of end rotation restraints, foundation stiffness and loading type on $w^*$ and $M^*$ ($K_{T1} = K_{T1} = 1E5$, $K_{R1} = K_{R2} = K_R$, $\eta = 50$, $\xi_1 = 0.5$, $\xi_2 = 0.0$, $\xi_3 = 1.0$).

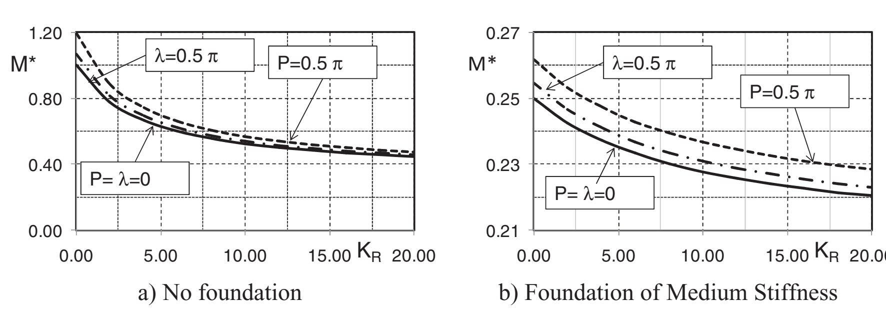


**Figure 5.** Influence of excitation frequency and axial stressing on $M^*$. (a) No foundation, (b) foundation of medium stiffness.

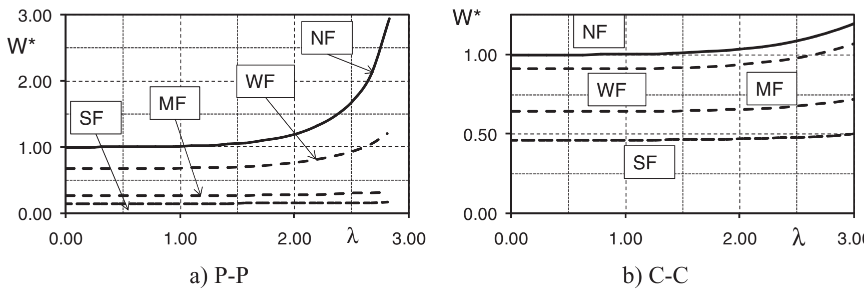


**Figure 6.** Influence of excitation frequency and foundation stiffness on $w^*$. (a) P-P, (b) C-C.

2014) for the critical loads and natural frequencies values and found compatible.

In addition, the influence of the FE truncation index $M_1$ on the accuracy of the obtained solution is shown in Table 1. The error percentage (error %) of the present solution compared with the exact solutions for a beam without foundation is shown. It is clear that taking $M_1 = 100$ yields results with accuracy fair for practical applications.

On the other hand, taking the truncation index of the ADM $N = 3$ yields a solution with expansion up to ($\xi/9!$). As the value of the dimensionless coordinate $\xi$ is ($0 < \xi < 1$), the accuracy of the solution will be of the order (1/9!).

## 4. Numerical results and discussion

The obtained solutions are used to investigate the effects of different parameters on the beam midspan deflections $w(0.5)$ and on the bending moment acting on the beam midspan cross section $M(0.5)$. To highlight the influences of the different parameters, normalised deflection $w^*(0.5)$ and normalised bending moments $M^*(0.5)$ are defined as

$$w*(0.5) = w(0.5)/w_o(0.5), \tag{42}$$

$$M*(0.5) = M(0.5)/M_o(0.5), \tag{43}$$

where $w_o$ (0.5) and $M_o$ (0.5) are the beam midspan deflection and bending moments, respectively, due to a concentrated load acting at the midspan of a simply supported beam. The foundations are defined as no foundation (NF), weak foundations (WF), medium foundations (MF) and stiff foundation (SF). The properties and parameter values of such types of foundations are indicated in Taha and Doha (2015).

The influence of the end elastic restraints against rotation $K_R$ on the midspan deflection for different types of foundation stiffness due to lateral concentrated static load acting at the midspan is indicated in Figure 2a. It is clear that as the foundation stiffness increases, the effect of the end restraints on the beam deflection decreases. However, the effects of the same parameters on the midspan bending moments indicate the same conclusions as depicted in Figure 2b.

The effects of $K_R$ on the midspan deflection of a uniformly distributed load covering the whole beam span are shown in Figure 3. It should be noted that, the values in Figure 3 are normalised by corresponding values of the resultant concentrated load acting at the midspan. Figure 4 compares between the influences of both loading types and foundation stiffness on the midspan deflection and bending moments normalised by the corresponding values due to resultant concentrated load acting at the midpoint.

The behaviour of the beam due to dynamic loading is indicated in Figures 5 and 6. Figures 5a and 5b show the influence of the end rotation restraints, the axial stressing and the excitation frequencies on the midspan bending moment normalised by the corresponding static ones. It is obvious that the influence of the lower levels of axial stressing and excitation frequencies becomes more pronounced as the foundation stiffness increases. The effects of the excitation frequency on the midspan lateral response amplitude normalised by its corresponding static value due to resultant concentrated load acting at the midspan are indicated in Figure 6a for simply supported beam (*P-P*), and in Figure 6b for clamped–clamped beam (*C-C*). Furthermore, it is found that the vibration amplitude decreases apparently due to the increase in the stiffness of the underneath foundation.

## 5. Conclusions

The ADM was used to obtain analytical solutions describing the behaviour of axially stressed beams with elastic end restraints resting on a two-parameter foundation and subjected to multiple discontinuous dynamic loads. The effects of loading type, end restraints stiffness and stiffness of the underneath foundation were investigated. It is found that as the beam becomes slender or the foundation becomes stiff, the end restraints have minor significance on the beam behaviour. However, the effects of the excitation frequency and axial loading on the lateral deflection of the beam become more significant in the case of stiff foundation. The proposed solutions may be extended to analyse nonuniform beams subjected to multiple loads with different configurations.

## Disclosure statement

No potential conflict of interest was reported by the author.

## Notes on contributor

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