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Stability behavior and free vibration of tapered columns with elastic end restraints using the DQM method

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Abstract The stability behavior and free vibration of axially loaded tapered columns with rotational and/or translational end restraints are studied using the differential quadrature method (DQM). The governing differential equations are derived and transformed into a homogeneous system of algebraic equations using the DQM technique. The boundary conditions are discretized and substituted into the governing differential equations, then the problem is transformed into a two parameter eigenvalue problem, namely the critical load and the natural frequency. The solution of the eigenvalue problem yields the critical load for the static case ($\omega = 0$) and yields the natural frequencies for the dynamic case with a prescribed value of axial load ($P_o < P_{cr}$). The obtained solutions were verified against those obtained from FEM and found in close agreement.

1. Introduction

Many practical engineering applications are very sensitive to the weight of the structural elements for different reasons. In space shuttles, the weight of different structural element is optimized as functional requirements, while in ordinary structures the weight optimization is needed for architectural and/or economical issues. To optimize the weight of structural elements in such applications, elements with non-prismatic configurations are commonly used. It is very difficult to obtain closed form solutions representing the behavior of such non-uniform elements under the effect of static and dynamic loads. Often, to obtain closed-form solutions, many idealizations are introduced to simplify the mathematical treatments yielding mathematical models that misrepresent physical models. In addition, the conventional idealization of the end conditions (fixed–hinged–clamped–free) may not represent most situations where support movements are expected and need to be considered in the analysis. Near optimum configurations are studied by many researchers to obtain both stability and vibration behavior of structural elements. Analytical solutions for simple cases of prismatic and non-prismatic elements with elastic end restraints are found in literature [1–3].

Taha and Abohadima [4] studied the vibration of non-uniform shear beam resting on elastic foundation. Semi-analytical methods such as series solutions are suggested to obtain analytic expressions for frequencies and mode shapes of

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non-uniform beams resting on elastic foundation [5]. Numerical methods such as the perturbation method [6], Ritz method [7], the finite element method [8–10] and the differential quadrature method [11,12] are used to study certain configurations of such models.

On the other hand, the great expansion in the power of the personal computers and availability of solving algorithms add many advantages to numerical techniques. The differential quadrature method (DQM) is a very efficient numerical method with simple straightforward formulation that needs very limited memory storage and computational time to obtain results with accuracy fair to practical applications.

In the present work, the stability and vibration behavior of axially-loaded tapered columns with translational and/or rotational elastic end restraints are studied using the DQM. The addressed problem was previously solved using the FEM for free vibration of beams (no axial load) [9]. The main differences between the present work and the previous one are the method of solution, the implementation of translational elastic end restraints as well as the presence of the axial load into present analysis. The boundary conditions are discretized and substituted into the governing differential equations. The obtained results are verified against the FEM results and are found to be in close agreement. The effects of numerous tapered configurations for different boundary conditions on the load and frequency parameters are investigated.

2. Formulation of the problem

2.1. Vibration equation

The free vibration equation of a non-prismatic column loaded by an axial force P_o, shown in Fig. 1, is given as:

$$\frac{\partial^2}{\partial X^2}\left[EI(X)\frac{\partial^2 Y}{\partial X^2}\right]+P_o\frac{\partial^2 Y}{\partial X^2}+\rho A(X)\frac{\partial^2 Y}{\partial t^2}=0. \quad (1)$$

where $I(X)$ is the moment of inertia of the column cross section at X, ρ is the mass density per unit volume, E is the modulus of elasticity, $A(X)$ is the area of cross section at X, $Y(X,t)$ is the lateral displacement, P_o is the axial load acting on the column, X is the distance along the column and t is time.

Using dimensionless parameters $x = X/L$ and $y = Y/L$, Eq. (1) is transformed to:

$$\frac{\partial^2}{\partial x^2}\left[\frac{EI(x)}{L^3}\frac{\partial^2 y}{\partial x^2}\right]+\frac{P_o}{L}\frac{\partial^2 y}{\partial x^2}+\rho A(x)\frac{\partial^2 y}{\partial t^2}=0. \quad (2)$$

The solution of linear partial differential Eq. (2) is obtained by employing both the separation of variables and the differential quadrature methods. The first step consists in representing the distribution of the lateral displacement by two independent functions, one describing the spatial variation (mode shape function) and the other represents the time variation. The second step consists in using the DQM method to transform the governing differential equation into a homogeneous system of N algebraic equations solved numerically according to the boundary conditions.

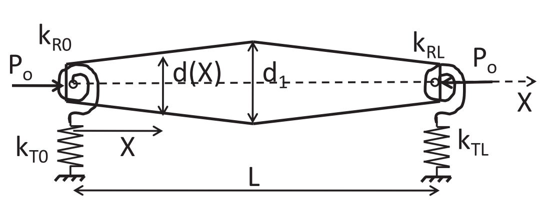

Figure 1 Axially-loaded tapered column with elastic end restraints.

2.2. Boundary conditions

The dimensionless elastic end restraints at $x = 0$ are related to the derivatives of lateral displacement at the column ends as:

$$k_{T0}y(0,t)=-\frac{\partial}{\partial x}\left(\frac{EI_o}{L^3}\frac{\partial^2 y(0,t)}{\partial x^2}\right), \quad (3a)$$

$$k_{R0}\frac{\partial y(0,t)}{\partial x}=\frac{EI_o}{L}\frac{\partial^2 y(0,t)}{\partial x^2}. \quad (3b)$$

Also, the dimensionless elastic end restraints at $x = 1$ are expressed as:

$$k_{TL}y(1,t)=\frac{\partial}{\partial x}\left(\frac{EI_o}{L^3}\frac{\partial^2 y(1,t)}{\partial x^2}\right), \quad (3c)$$

$$k_{RL}\frac{\partial y(1,t)}{\partial x}=-\frac{EI_o}{L}\frac{\partial^2 y(1,t)}{\partial x^2}. \quad (3d)$$

where k_{T0} and k_{TL} are the lateral elastic stiffnesses at $x = 0, 1$ respectively, I_o is the moment of inertia of the column cross section at $x = 0$ and k_{R0} and k_{RL} are the rotational elastic stiffnesses at $x = 0, 1$ respectively.

Following the separation of variables analogy, the solution of Eq. (2) may be assumed as:

$$y(x,t)=y_o\phi(x)\psi(t), \quad (4)$$

where $\phi(x)$ is the linear mode function, $\psi(t)$ is a function representing the time variation and y_o is the dimensionless vibration amplitude (obtained from the initial conditions). Substituting Eq. (4) into Eq. (2), Eq. (2) is separated into:

$$\frac{d^2}{dx^2}\left[\frac{EI(x)}{L^3}\frac{d^2\phi}{dx^2}\right]+\frac{P_o}{L}\frac{d^2\phi}{dx^2}+\rho A(x)\omega^2\phi=0, \quad (5)$$

$$\frac{d^2\psi}{dt^2}+\omega^2\psi(t)=0, \quad (6)$$

where ω is the separation constant which represents the natural frequency.

The solution of Eq. (6), assuming at $t = 0$, $\psi = 1.0$ and $d\psi/dt = 0$ is:

$$\psi(t)=\cos(\omega t). \quad (7)$$

The general solution of Eq. (5) depends on the distribution of the section geometry along the column. Fig. 1 shows the case of a symmetric tapered column, where the depth of the column increases lineally from d_o at $x = 0$ to d_1 at $x = 0.5$, then decreases linearly form d_1 at $x = 0.5$ to d_o at $x = 1$, while the width of the column b is assumed constant, then:

$$d(x)=d_o\eta(x), \quad (8)$$

where

$$\eta(x)=\begin{cases}1-2x(1-\alpha) & \text{for } 0.0\leqslant x\leqslant 0.5\\ 2\alpha+2x(1-\alpha)-1 & \text{for } 0.5\leqslant x\leqslant 1.0\end{cases}$$

and $\alpha = d_1/d_o$ is the tapering ratio.

Using the distribution of section geometry expressed in Eq. (8), the distribution of area and moment of inertia of the column cross section are given as:

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$$I(x) = I_o\eta^3(x),$$
$$A(x) = A_o\eta(x), \tag{9}$$

where A_o is the area of the column cross section at $x = 0$

Substitution Eqs. (8) and (9) into Eq. (5) yields:

$$\frac{d^4\phi}{dx^4} + \frac{6\eta'}{\eta}\frac{d^3\phi}{dx^3} + \left[\frac{3\eta''}{\eta} + \frac{6\eta'^2}{\eta^2} + \frac{P_oL^2}{EI_o\eta^3}\right]\frac{d^2\phi}{dx^2} - \frac{\rho A_o\omega^2L^4}{EI_o\eta^2}\phi(x) = 0, \tag{10}$$

where the prime is used to denote the derivative w.r. to x.

Eq. (10) is a fourth-order differential equation with variable coefficients, which is difficult to solve analytically. The solution of Eq. (10) considering $\omega = 0$ yields the critical (buckling) loads, while solving the equation with a prescribed value of P_0 (less than critical load) yields the natural frequencies of axially-loaded tapered columns. The boundary conditions at $x = 0$ in terms of dimensionless parameters can be rewritten as:

$$\bar{k}_{T0}\phi(0) = -\frac{d^3\phi(0)}{dx^3} + 6(1-\alpha)\frac{d^2\phi}{dx^2}, \tag{11a}$$

$$\bar{k}_{R0}\frac{d\phi(0)}{dx} = \frac{d^2\phi(0)}{dx^2}, \tag{11b}$$

and at $x = 1$ can be rewritten as:

$$\bar{k}_{TL}\phi(0) = \frac{d^3\phi(0)}{dx^3} + 6(1-\alpha)\frac{d^2\phi}{dx^2}, \tag{11c}$$

$$\bar{k}_{RL}\frac{d\phi(1)}{dx} = -\frac{d^2\phi(1)}{dx^2}, \tag{11d}$$

where

$$\bar{k}_{T0} = \frac{L^3k_{T0}}{EI_o},\ \bar{k}_{TL} = \frac{L^3k_{TL}}{EI_o},\ \bar{k}_{R0} = \frac{Lk_{R0}}{EI_o} \text{ and } \bar{k}_{RL} = \frac{Lk_{RL}}{EI_o}. \tag{11e}$$

are the dimensionless elastic restraints stiffness parameters at the column ends.

3. Solution of the problem

3.1. The differential quadrature method (DQM)

The solution of Eq. (10) is obtained using the DQM method, where the solution domain is discretized into N sampling points and the derivatives at any point are approximated by a weighted linear summation of all the functional values at all other points as follow [10]:

$$\left[\frac{d^mf(x)}{dx^m}\right]_{X_i} \approx \sum_{j=1}^{N} C_{i,j}^{(m)}f(x_j), \quad (i = 1, N), (m = 1, M), \tag{12}$$

where M is the order of the highest derivative in the governing equation, $f(x_j)$ is the functional value at point of $x = x_j$ and C_{ij}^m is the weighting coefficient relating the derivative of order m at $x = x_i$ to the functional value at $x = x_j$. For obtaining the weighting coefficients, many polynomials with different base functions can be used. Lagrange interpolation formula is the most common one, where the functional value at a point x is approximated by all the functional values $f(x_k)$, $(k = 1,N)$ as:

$$f(x) \approx \sum_{k=1}^{N} \frac{L(x)}{(x - x_k)L_1(x_k)}f(x_k), \tag{13}$$

where
$L(x) = \prod_{j=1}^{N}(x - x_j), L_1(x_k) = \prod_{k=1}^{N}(x_i - x_k), (i, k = 1, N)$

Substitution of Eq. (13) into Eq. (12) yields the weighting coefficients of the first derivative as [10]:

$$C_{i,j}^{(1)} = \frac{L_1(x_i)}{(x_i - x_j)L_1(x_j)} \quad \text{for } (i \neq j) \text{ and } (i, j = 1, N), \tag{14a}$$

$$C_{i,j}^{(1)} = -\sum_{j=1, j\neq i}^{N} C_{i,j}^{1} \text{ for } (i = j) \text{ and } (i, j = 1, N). \tag{14b}$$

Applying the chain rule to Eq. (12), the weighting coefficients of the derivative of order m can be related to the weighting coefficients of the derivative of order $(m - 1)$ as:

$$C_{i,k}^{(m)} = \sum_{k=1}^{N} C_{i,k}^{(1)}C_{i,k}^{(m-1)}, \quad (i, k = 1, N), (m = 1, M). \tag{15}$$

As the DQM is a numerical method, its accuracy is affected by both the number and distribution of discretization points. In boundary value problems, it is found that the irregular distribution of the discretization points with smaller mesh spaces near the boundary, to cope the steep variation near the boundaries, is more efficient. One of the frequently used distributions for mesh points is the normalized Gauss–Chebychev–Lobatto distribution given as:

$$x_i = \frac{1}{2}\left[1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right], \quad (i = 1, N). \tag{16}$$

3.2. Implementation of the boundary conditions

The boundary conditions can be expressed in discretized form as:

$$\bar{k}_{T0}\phi_1 = -\sum_{j=1}^{N} C_{i,j}^{(3)}\phi(x_j) + 6(1-\alpha)\sum_{j=1}^{N} C_{i,j}^{(2)}\phi(x_j), \tag{17a}$$

$$\bar{k}_{R0}\phi_1 = \sum_{j=1}^{N} C_{i,j}^{(2)}\phi(x_j), \tag{17b}$$

$$\bar{k}_{TL}\phi_N = \sum_{j=1}^{N} C_{i,j}^{(3)}\phi(x_j) + 6(1-\alpha)\sum_{j=1}^{N} C_{i,j}^{(2)}\phi(x_j), \tag{17c}$$

$$\bar{k}_{RL}\phi_N = -\sum_{j=1}^{N} C_{i,j}^{(2)}\phi(x_j). \tag{17d}$$

To implement the boundary conditions in the system of N algebraic equations, the unknowns ϕ_1, ϕ_2, ϕ_{N-1} and ϕ_N can be expressed in terms of the other $(N - 4)$ unknowns (ϕ_i, $i = 3, N - 2$). However, discretizing the governing differential equations at N − 4 sampling points x_i ($i = 3, N - 2$) and substituting the boundary conditions in the governing equation leads to a system of $N - 4$ homogenous algebraic equations in $N - 4$ unknowns (ϕ_i, $i = 3, N - 2$) in addition to the two parameters P_o and ω.

Using Eq. (17) the unknowns ϕ_1, ϕ_2, ϕ_{N-1} and ϕ_N can be expressed as:

$$\phi_1 = \sum_{i=3}^{N-2}\xi_{1,i}\phi_i \quad \phi_2 = \sum_{i=3}^{N-2}\xi_{2,i}\phi_i \quad \phi_{N-1} = \sum_{i=3}^{N-2}\xi_{N-1,i}\phi_i \quad \phi_N = \sum_{i=3}^{N-2}\xi_{N,i}\phi_i, \tag{18}$$

where ξ's are numerical coefficients and ϕ's are required unknowns.

3.3. Discretization of the governing equation

The mode shape differential Eq. (10) may be rewritten as:

$$\frac{d^4\phi}{dx^4} + \Omega_1(x)\frac{d^3\phi}{dx^3} + \Omega_2(x)\frac{d^2\phi}{dx^2} + \Omega_3(x)\omega^2\phi(x) = 0. \quad (19)$$

where

$$\Omega_1(x) = \frac{6\eta'}{\eta^3}, \quad \Omega_2(x) = \left[\frac{3\eta''}{\eta^2} + \frac{6\eta'^2}{\eta^2} + \frac{P_oL^2}{EI_o\eta^3}\right] \quad \text{and } \Omega_3(x) = \frac{\rho A_o L^4}{EI_o\eta^2}, \quad (20)$$

Using the DQM, Eq. (19) can be discretized at sampling point x_i as:

$$\sum_{j=3}^{N-2} C_{i,j}^{(4)}\phi_j + \sum_{j=3}^{N-2}\Omega_{1,i}C_{i,j}^{(3)}\phi_j + \sum_{j=3}^{N-2}\Omega_{2,i}C_{i,j}^{(2)}\phi_j = \Omega_{3,i}\omega^2\phi_i, \quad (i = 3, N-2). \quad (20)$$

Using Eqs. (18) and (19), the governing differential Eq. (20) is transformed into a system of algebraic equations:

$$\sum_{j=3}^{N-2}\left[\gamma_{1,i}\xi_{1,i} + \gamma_{2,i}\xi_{2,i} + \gamma_{N-1,i}\xi_{N-1,i} + \gamma_{N,i}\xi_{N,i} + \gamma_{i,j}\right]\phi_i = 0, \quad (i = 3, N-2). \quad (21)$$

Eq. (21) represents a system homogeneous algebraic equations in N − 4 unknowns in addition to P_o and ω, then the condition of nontrivial solution leads to the eigenvalue problem. The solution of such eigenvalue problem yields the critical loads in the static case ($\omega = 0$) and yields the natural frequency ω_n in the dynamic case (for a given value of $P_o < P_{cr}$), then the mode shapes can be obtained. A MATLAB code has been designed to solve the above system to obtain critical loads or fundamental natural frequencies. Furthermore, knowing the natural frequencies of the column, the functional values ϕ_i, $i = 1, N$ can be obtained and mode shapes can be investigated.

3.4. Verification of the present solution

The calculated results of the frequency parameter λ for prismatic columns ($\alpha = 1$) using the obtained expressions are compared with those calculated from closed form solutions [13] and are found to be the same, which validates the present work. The frequency parameter λ is defined as:

$$\lambda^4 = \frac{\rho A_o L^2\omega^2}{EI_o}. \quad (23)$$

In addition, the values of the frequency parameter λ for tapered columns obtained from the present solution are compared against those obtained from the FEM [9] for the different values of tapering ratio α in Table 1. The values shown in the table indicate close agreement between the results of the present analysis and those obtained using the FEM for low tapering ratio α. There are some deviations for values of large α, as in the finite element results, the beam is divided into 10 prismatic elements [9] of equal length. For higher values of tapering ratio, the limited number of prismatic elements in FEM may yield less accurate values.

The stability parameter λ_b is defined as:

$$\lambda_b^2 = \frac{P_{cr}L^2}{EI_o}, \quad (24)$$

where P_{cr} (also called buckling or Euler's load) is the critical value of the axial load after which the column loses its stability. The values of stability parameters λ_b calculated from closed form solution for the case of prismatic columns with Pinned–Pinned (P–P) end conditions $\lambda_b = \pi$, for Clamped–Clamped (C–C) case $\lambda_b = 2\pi$ and for Clamped–Pinned (C–P) $\lambda_b = 1.431\pi$ which are the same obtained from the present analysis.

4. Numerical results

The accuracy of numerical methods depends on the mesh distribution, the number of mesh points and the type of the analyzed problem. In the DQM, using a limited number of sampling points compared to other numerical techniques (FEM, FD or BEM) yielding results of acceptable accuracy for practical applications. In the present study, the use of 10–25 sampling points is investigated. It is observed that the gain in accuracy for $N > 15$ is negligible, therefore, the present results are obtained using $N = 15$.

The influences of the elastic restraints stiffness parameters ($\bar{k}_T, \bar{k}_R$) at both ends and tapering ratio $\alpha = d_1/d_o$ on the stability parameter are investigated. Table 2 indicates the variation of the stability parameter λ_b with tapering ratio and end-rotation stiffness parameters. The stability parameter increases as the tapering ratio increases and as the rotation stiff-

Table 1 Values of the frequency parameter for the tapered columns.

End restraints stiffness				Tapering ratio: $\alpha = d_1/d_o$						Analysis
$\bar{k}_{T0}$	$\bar{k}_{TL}$	$\bar{k}_{R0}$	$\bar{k}_{RL}$	1.0	1.1	1.2	1.3	1.4	1.5	
1E5	1E5	0	0	3.141	3.248	3.349	3.449	3.534	3.620	FEM
				3.141	3.2831	3.392	3.496	3.540	3.588	Present
1E5	1E5	0.1	0.1	3.173	3.276	3.373	3.466	3.554	3.638	FEM
				3.1687	3.3105	3.420	3.505	3.571	3.621	Present
1E5	1E5	1	1	3.399	3.479	3.557	3.634	3.708	3.780	FEM
				3.373	3.5173	3.634	3.729	3.807	3.872	Present
1E5	1E5	10	10	4.156	4.201	4.247	4.294	4.341	4.388	FEM
				4.109	4.274	4.419	4.549	4.667	4.776	Present

Table 2 Influence of the rotation stiffness on the stability parameter.

$\bar{k}_{T1}$	$\bar{k}_{T2}$	$\bar{k}_{R1}$	$\bar{k}_{R2}$	Tapering ratio: $\alpha = d_1/d_o$					
				1	1.1	1.2	1.3	1.4	1.5
10^5	10^5	0	0	3.1413	3.4631	3.768	4.0369	4.292	4.5309
10^5	10^5	0	10	4.1319	4.8795	5.5935	6.3099	7.0373	7.7829
10^5	10^5	0	100	4.4504	5.3063	6.1068	6.868	7.5913	8.2776
10^5	10^5	0	10^5	4.4938	5.3617	6.1664	6.9243	7.6399	8.3143
10^5	10^5	1	0	3.4053	3.8295	4.231	4.61	4.9868	5.3535
10^5	10^5	1	10	4.4226	5.1434	5.8644	6.6055	7.3703	8.1855
10^5	10^5	1	100	4.7476	5.6061	6.431	7.2326	8.0068	8.764
10^5	10^5	1	10^5	4.792	5.664	6.4924	7.2921	8.0628	8.7992
10^5	10^5	10	0	4.1319	4.8795	5.5935	6.3099	7.0373	7.7829
10^5	10^5	10	10	5.3071	5.9909	6.7194	7.5095	8.3887	9.4374
10^5	10^5	10	100	5.7044	6.6069	7.5593	8.581	8.7085	9.759
10^5	10^5	10	10^5	5.7578	6.6885	7.6578	8.6913	9.8041	11.157
10^5	10^5	100	0	4.4504	5.3063	6.1068	6.868	7.5913	8.2776
10^5	10^5	100	10	5.7044	6.6069	7.5593	8.581	8.7085	9.759
10^5	10^5	100	100	6.1578	7.086	8.0825	9.1913	10.287	12.2438
10^5	10^5	100	10^5	6.2205	7.1763	8.2064	9.3407	10.624	12.4934
10^5	10^5	10^3	0	4.4898	5.3617	6.1578	6.9185	7.6353	8.311
10^5	10^5	10^3	10	5.7528	6.6812	7.6486	8.6811	9.8041	11.1507
10^5	10^5	10^3	100	6.2148	7.1701	8.1941	9.3265	10.624	12.4934
10^5	10^5	10^3	10^5	6.2713	7.2314	8.2616	9.3906	10.707	12.5286

ness parameter increases. Table 3 indicates the variation of the stability parameter λ_b with tapering ratio and end-translation stiffness parameters. The stability parameter increases as the tapering ratio increases and as the translation stiffness parameter increases. For the same end restraints stiffness, the gain in stability parameter is 44% due to increasing d_1 by 50% relative to d_o. Also, it is clear that when the rotation end restraints at one end increases from simply supported to clamped one, the stability parameter increases by 43% for prismatic column ($\alpha = 1$), while increases by 83% for tapered column with $\alpha = 1.5$.

The effects of the restraints stiffness parameter at one support ($\bar{k}_{RL}$) on the frequency parameter (λ) for the different values of tapering ratio (α) and load parameter $\overline{P}_o$ are shown in Fig. 2, where the dimensionless load parameter is defined as:

$$\overline{P}_o = \frac{P_o L^2}{\pi^2 E I_o}. \tag{25}$$

It is observed that as the restraint stiffness parameter at one end increases, the frequency parameter increases. This is due to the increase in the total column stiffness. It is observed that, the increase in the frequency parameter λ due to the increase in the restraint stiffness parameter $\bar{k}_{RL}$ at one end is more pronounced for large values of $\bar{k}_{RL}$. Fig. 3 is a modified version of Fig. 2, but with equal rotational stiffness parameters at both ends and for $\overline{P}_o = 0$.

In Fig. 4, the influences of lateral translation stiffness parameter $\bar{k}_{RL}$ at one end on the frequency parameter are investigated. It is clear that, the frequency parameter increases

Table 3 Influence of the translation stiffness on stability parameter.

$\bar{k}_{T1}$	$\bar{k}_{T2}$	$\bar{k}_{R1}$	$\bar{k}_{R2}$	Tapering ratio: $\alpha = d_1/d_o$					
				1	1.1	1.2	1.3	1.4	1.5
10^5	0	0	0	3.1413	3.1427	3.1665	3.28	3.3567	3.3858
10^5	10	0	0	3.1413	3.1581	3.1859	3.3465	3.4118	3.4631
10^5	100	0	0	3.1413	3.1859	3.2394	3.4247	3.4758	3.5136
10^5	10^5	0	0	3.1413	3.4631	3.768	4.0369	4.292	4.5309
10^5	0	10	10	3.1426	5.3945	5.5157	5.5237	5.5261	5.5556
10^5	1	10	10	3.1426	5.3945	5.5157	5.5237	5.5261	5.5556
10^5	10	10	10	3.9841	5.4756	5.5397	5.5468	5.5635	5.603
10^5	100	10	10	5.3071	5.5556	5.6344	5.7121	5.7429	5.7888
10^5	10^4	10	10	5.3071	5.9391	6.5734	7.2739	8.0233	8.8591
10^5	10^5	10	10	5.3071	5.9909	6.7194	7.5095	8.3887	9.4374
10^5	0	10^5	10^5	3.1427	6.3966	6.5263	6.56	6.5734	6.5801
10^5	1	10^5	10^5	3.2665	6.3966	6.5263	6.56	6.5734	6.5801
10^5	10	10^5	10^5	4.1996	6.431	6.56	6.5801	6.5935	6.6069
10^5	100	10^5	10^5	6.2643	6.6402	6.8106	6.8686	6.8815	6.8879
10^5	10^4	10^5	10^5	6.2643	7.2069	8.1865	9.22	10.3297	11.6534
10^5	10^5	10^5	10^5	6.2643	7.2253	8.2509	9.3892	10.7069	12.5286

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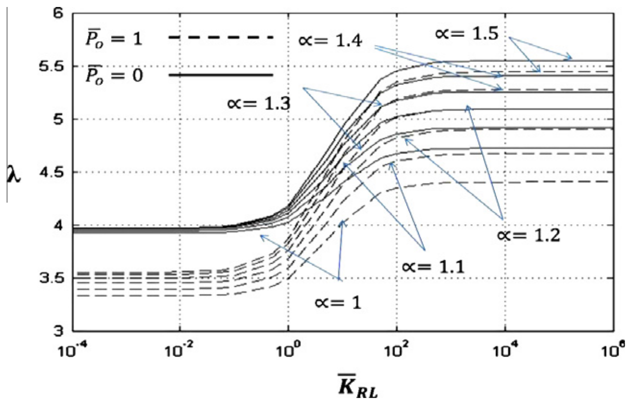

Figure 2 Influence of the rotational stiffness parameter $\bar{k}_{RL}$ at one end on the frequency parameter λ ($\bar{k}_{T0} = \bar{k}_{TL} = 10^5$ and $\bar{k}_{R0} = 10^3$).

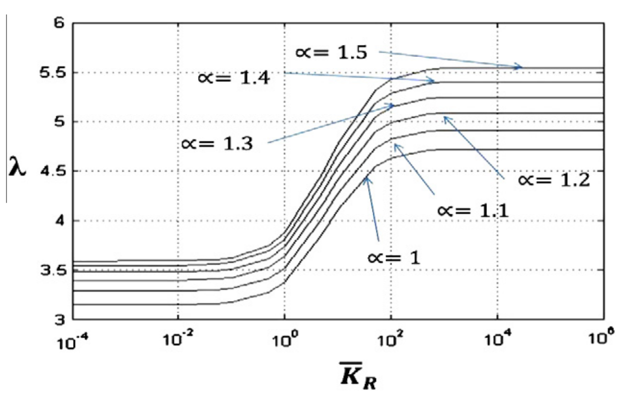

Figure 3 Influence of the rotational stiffness parameter $\bar{k}_R$ at both ends on the frequency parameter λ ($\bar{k}_{T0} = \bar{k}_{TL} = 10^5$, $\bar{k}_{R0} = \bar{k}_{RL} = \bar{k}_R$ and $\overline{P}_o = 0$).

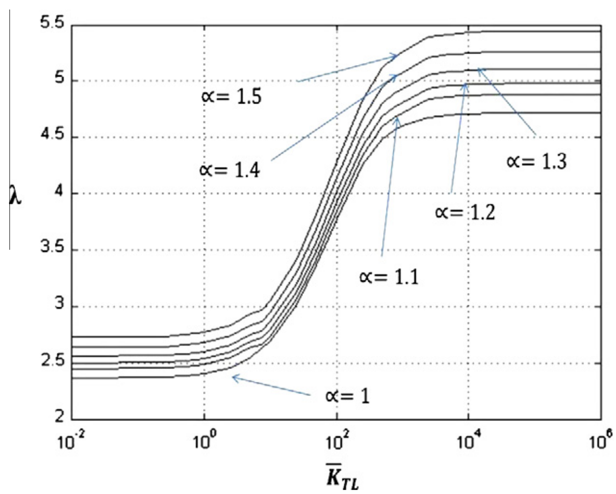

Figure 4 Influence of the lateral displacement stiffness parameter $\bar{k}_{TL}$at one end on the frequency parameter λ ($\bar{k}_{T0} = 10^5$, $\bar{k}_{R0} = \bar{k}_{RL} = 10^3$ and $\overline{P}_o = 0$).

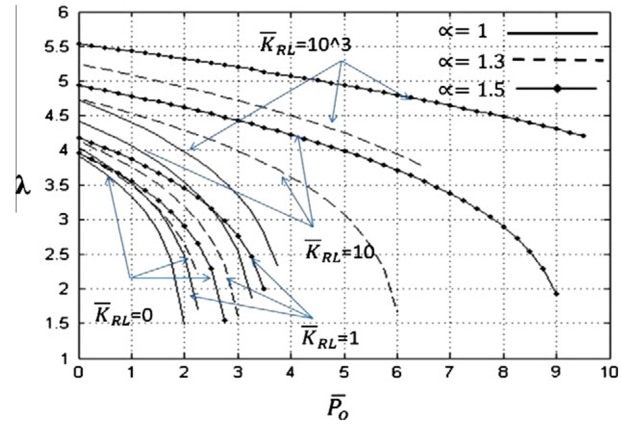

Figure 5 Influence of the load parameter $\overline{P}_o$ on the frequency parameter λ ($\bar{k}_{T0} = \bar{k}_{TL} = 10^5$ and $\bar{k}_{R0} = 10^3$).

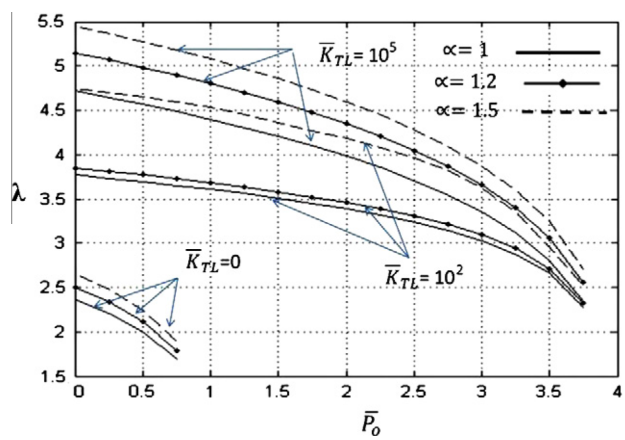

Figure 6 Influence of the load parameter $\overline{P}_o$ on the frequency parameter λ ($\bar{k}_{R0} = \bar{k}_{RL} = 10^3$ and $\bar{k}_{T0} = 10^5$).

as the translation stiffness increases due to the increase in the system stiffness. Furthermore, the effect of the tapering ratio α on the frequency parameter is more noticeable in case of the stiff end restraint than in the case of flexible one.

The effects of the axial-load parameter $\overline{P}_o$ on the frequency parameter λ are depicted in Figs. 5 and 6 for different boundary conditions. Fig. 5 shows the effect of load parameter $\overline{P}_o$ on the frequency parameter for different values of the rotation stiffness at one end while at the other end is assumed clamped. In Fig. 6, the influence of the load parameter $\overline{P}_o$ on the frequency parameter λ for different values of the lateral translation stiffness at one end is shown. It is obvious in all cases that as the load parameter increases, the frequency parameter decreases.

5. Conclusions

The stability and vibrational behavior of axially-loaded tapered columns elastically retrained at both ends by translation and rotation restraints are investigated using the differential quadrature method (DQM). The governing differential equations with variable coefficients are derived and discretized at sampling points. The boundary conditions are discretized and substituted into the discretized governing equation yielding a system of $N - 4$ algebraic equations in functional values of lateral displacement. The condition of non-trivial solution transforms the problem into a two parameter eigenvalue problem (P_o and ω). The eigenvalue problem is solved by assigning a proper value for one of the two parameters to obtain the value of the other. It is found that the frequency parameter as well as the stability parameter (λ, λ_b) for the taper columns increase as the stiffness of elastic end restraints increase and as the tapering ratio α increases. The gain in the two parameters (λ, λ_b) for large values of α is greater than the gain for smaller values. As the axial load increases, the frequency parameter of the system decreases.

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