WORK AND ENERGY
IN RIGEID BODY PLANE MOTION

1) INTRODUCTION

In case of complex mechanical systems, the dynamics of the system can be studied using the energy principles in an easier way than using the equations of motion method. In this chapter the concepts of using work and energy rules in solving such problems are introduced. Examples are given to show how these principles can be used to obtain solutions of many complex rigid body dynamical systems.

2) THE WORK

The work done by a constant force $\underline{F}$ acting on a particle during its displacement $\underline{D}$ is defined by the scalar product (dot product):

$$W = \underline{F} \cdot \underline{D} = F(AB)\cos\theta \qquad (3.1)$$

If: θ=0, W=$F(AB)$ ($\underline{F}$ in direction of $\underline{D}$)

If: θ=90, W=0 ($\underline{F}$ in direction ⊥ of $\underline{D}$)

If: θ=0, W=-$F(AB)$ ($\underline{F}$ opposite to $\underline{D}$ as friction)

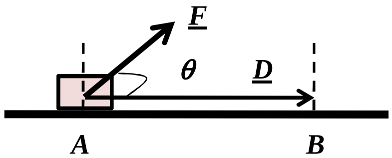


Fig. 3.1

To obtain the work of a variable force $\underline{F}$ acting on a RB during its motion form position (1) to position (2), assume the force $\underline{F}$ is constant during an infinitesimal time **dt** and assume the infinitesimal displacement of its point of action is defined by $d\underline{r}$ as shown in Fig. 3.2, then the infinitesimal work of the force is given by:

$$dW = \underline{F} \cdot \underline{dr} \qquad (3.2)$$

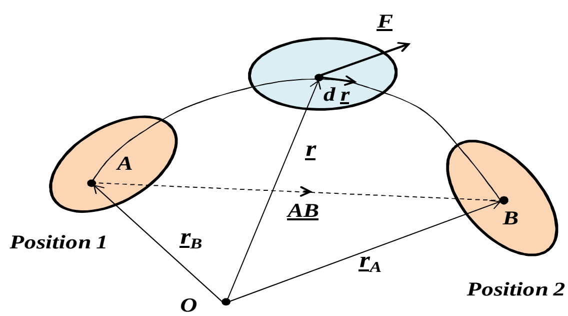


Fig. 3.2

Then, the work done by the force $\underline{F}$ on the RB during its motion from position (1) to position (2), is given by the integral:

$$W_{1\to2} = \int_1^2 dW = \int_1^2 \underline{F} \cdot \underline{dr} \qquad (3.3)$$

Subsitituting expressions of the force $\underline{F}$ and $d\underline{r}$ in the above equation, the work of the varaible force $\underline{F}$ can be obtained.

2.1) Constant force:

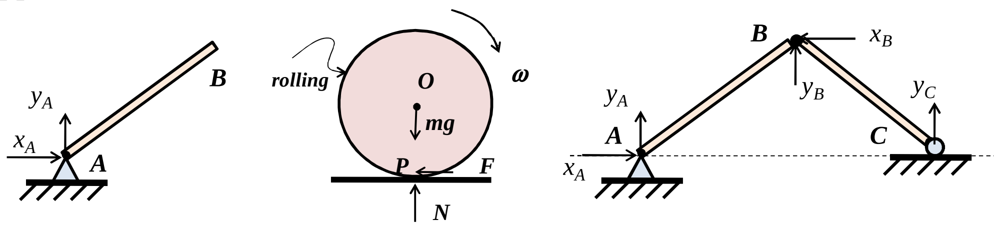
In the case of constant force, the work of the force may be expressed as follow:

$$W_{1\to2}\Big|_{\text{constant } \underline{F}} = \underline{F}\cdot\int_A^B d\underline{r} = \underline{F}\cdot\underline{r}\Big|_A^B = \underline{F}\cdot(\underline{r}_B-\underline{r}_A) = \underline{F}\cdot\underline{AB} = F(AB)\cos\theta \qquad (3.4)$$

Hence, the work of the constant force is independent of the path of the point of its action on the RB during the motion from position (1) to position (2). It is only depend on the locations of the point of its action in positions (1) and position (2).

2.2) Forces do no work:

If a force acts at a fixed point, or it is always acting normal to the displacement of its point of application, then this force do no work.



(a) *(b)* *(c)*

Fig. 3.3

The above figures show examples of forces that do no work. In case (*a*), the reactions (x_A and y_A) acting at fixed point *A*, *i.e*. the displacement is zero, then the work is zero. In case (*b*), as the disk rolls, then the displacement of point *P* is zero, then the forces (*F* and *N*) do no work. Also the gravity force of the disc do no work as its direction is normal to the displacement of its point of action (θ=90^o). In case (*c*), the work of the internal reactions at the intermediate hinge *B* cancels each other.

2.3) The Work of the gravity force (mg)of a RB:

Since the force *mg* is constant, then its work is given by:

$$W_{1\to2}\Big|_{\text{constant}} = F(AB)\cos\theta = mgh \qquad (3.5)$$

where *h* is the vertical downward displacement moved by the mass center *G* of the body during the motion of the body from position (1) to position (2). This work is independent of the path between the two positions (1) and (2).

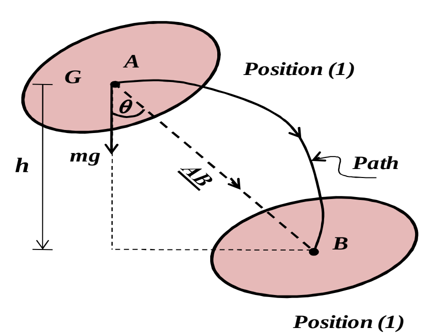


Fig. 3.4

2.4) The Work done by a Couple Q:

Assuming a couple ***Q*** acting on a **RB** as in Fig 3.5 is replaced by two forces of the same magnitude ***F*** such that its moment (magnitude) ***M = Fh*** where ***h*** is the perpendicular distance between their lines of action. During an infinitesimal time ***dt***, the movement of the RB may be considered as the sum of an infinitesimal translation ***dx*** plus infinitesimal rotation $d\theta$, the resultant work done by the forces during the infinitesimal translation is zero, while the work during infinitesimal rotation $d\theta$ is given by:

$$dW = F(hd\theta) = Md\theta$$

then the work done by *M* during a motion of the RB between two positions (1) and (2) is given by:

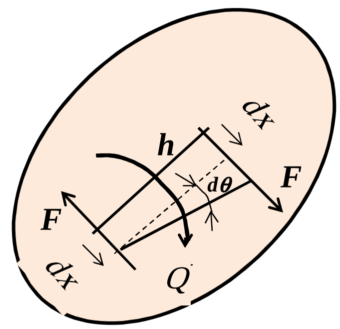


Fig. 3.5

$$W_{1\to2}\big|_M = \int_1^2 Md\theta \qquad (3.6)$$

If *M* is constant, then:

$$W_{1\to2}\big|_{\text{constant } M} = M\theta \qquad (3.7)$$

where θ is the rotation angle in radiant of the RB between the two positions (1) and (2). If *M* acts in opposite sense of the RB rotation, then its work done will be negative.

2.5) The Work done by a the Force in a Spring:

a) Linear spring:

To obtain the work done by a linear spring connected to a RB, when the RB moves between two positions, during infinitesimal time *dt*, the point of action of spring force moves from A_1 to A_2, assume *x* and *y* axes are as shown, then:

$\underline{F} = -kx\,\underline{i}$ and $d\underline{r} = dx\,\underline{i} + dy\,\underline{j}$

$$W_{1\to2} = \int_{X_1}^{X_2} \underline{F}\cdot d\underline{r} = \int_{X_1}^{X_2} -kx.dx = -\frac{1}{2}kx^2\Big|_{x_1}^{x_2}$$

$$\therefore \quad W_{1\to2} = \frac{1}{2}k\left(x_1^2 - x_2^2\right) \qquad (3.8)$$

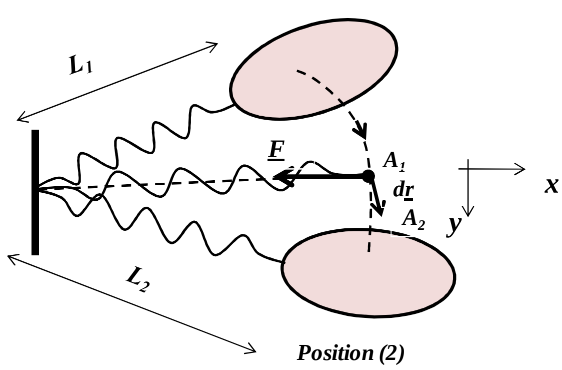


Fig. 3.6

Where:

$$x_1 = L_1 - L_o \quad and \quad x_2 = L_2 - L_o$$

b) Rotational spring:

Similarly, in the case of a RB connected with a rotational spring, the work done by the spring during an infinitesimal time ***dt*** is given by:

$$dW = -(k\theta)d\theta$$

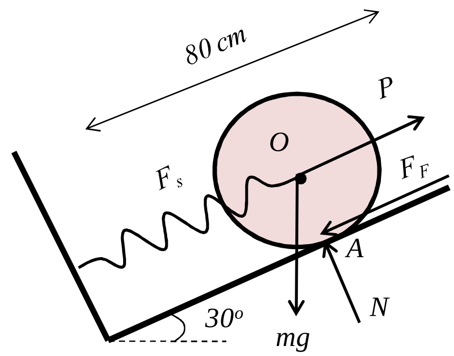
Then, the total work done by the rotational spring when the RB moves between two positions is given by:

$$W_{1\to2}\Big|_{sp} = -\int_{\theta_1}^{\theta_2} k\theta d\theta = \frac{1}{2}k\left(\theta_1^2 - \theta_2^2\right) \qquad (3.9)$$

where θ_1 is the rotation in the spring in position (1) and θ_2 is the rotation in the spring in position (2) in radiant.

EXAMPLE (3.1):

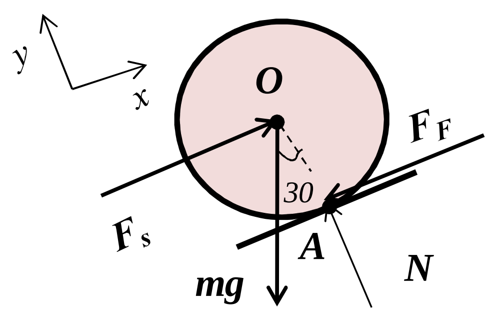
In the shown figure, a disc (*m* = 20 kg) connected with spring of constant k = 200 N/cm is at rest in a vertical plane. If a force *P* (1.2 kN) parallel to the line of greatest slop is applied to role the disc on the fixed inclined rough plane as shown. Calculate the initial compression in the spring and the work of all forces acting on the disc during moving 80 cm.



SOLUTION:

Let position (1) is the initial position and position (2) is the position reached after the disc has moved a distance of 80 cm upward the plane.

When the disc is in pos.(1) the disc is in equilibrium under the action of the forces shown in the opposite figure, then:



$$\sum M_A = 0 \quad , \therefore F_s(r) - mg\sin 30(r) = 0$$

$$F_s = kx_1, \quad \therefore kx_1 = 0.5mg, \; sub. \; k = 200 \; , \; m = 20kg$$

$$\therefore x_1 = 0.5$$

Where: $\boldsymbol{F}_s$ is the force in spring.

where x_1 is a compression in the spring in position (1). When the disc reaches pos.(2) there will be an extension in the spring of 30 cm (= 50 + 30=80 cm).

The forces acting on the disc during its motion from position (1) to position (2) are:

- The weight mg : does work given by:

  $$W_{1\to2}\big|_{mg} = -mgL\sin 30 = -(20)(9.8)(0.8\sin 30^\circ) = -78.4 \text{ Joule}$$

- The force P : does work given by:

  $$W_{1\to2}\big|_{P} = PL = (1200)(0.8) = 960 \text{ Joule}$$

- The force in the linear spring : does work given by:

  $$W_{1\to2}\big|_{sp} = \frac{1}{2}k\left(x_1^2 - x_2^2\right) = \frac{1}{2}(200)\left[(-0.5)^2 - (0.3)^2\right] = 16 \text{ Joule}$$

  $x_1 = -0.50$ m = the compression in the spring at pos. (1)

  $x_2 = 0.8 - 0.5 = 0.3$ m = the extension in the spring at pos. (2)

- The normal reaction N: does no work. ($\perp$ to the displacement of its point of action)
- The friction force $\underline{\boldsymbol{F}}_F$: does no work (rolling without slipping).

Then, the total work done by all external forces acting on the disc during the above motion is:

$$W_{1\to2} = -78.4 + 960 + 16 = 897.6 \quad Joule$$

3) THE KINETIC ENERGY:

3.1) For a particle:

The kinetic energy (denoted by T or $K.E$) of a particle of mass m and moving with velocity $\underline{v}$ is given by:

$$T = \text{K.E of the particle} = \frac{1}{2}mv^2 = \frac{1}{2}m\underline{v}\cdot\underline{v} \qquad (3.10)$$

3.2) For a Rigid Body:

For a RB, the kinetic energy may be obtained as:

$$T = \sum \frac{1}{2}m_i v_i^2 \qquad (3.11)$$

Case (1) : Translational motion

All points of the RB have the same velocity v, then:

$$T = v^2 \sum \frac{1}{2} m_i = \frac{1}{2} m v^2 \qquad (3.12)$$

where v is the velocity of any point in the RB

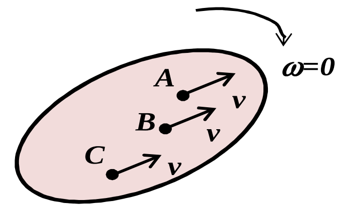


Fig. 3.7

Case (2) : Rotational motion about O

All points of the RB have a velocity $v = \omega r_i$, then:

$$T = \sum \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2} \omega^2 \sum m_i r_i^2$$

$$\because \; I_O = \sum \frac{1}{2} m_i \, r_i^2$$

$$\therefore T = \frac{1}{2} I_O \omega^2 \qquad (3.13)$$

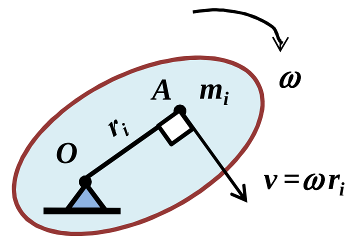


Fig. 3.8

Case (3) : General plane motion

For a **RB** in general plane motion with known instantaneous center (*I.C.*) , the velocity of the particle i is given by: $v_i = \omega \, r_i$

Then, the kinetic energy of the **RB** is given by:

$$T = \sum \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2} \omega^2 \sum m_i r_i^2$$

$$\because \; I_{I.C} = \sum m_i r_i^2$$

$$\therefore T = \frac{1}{2} I_{I.C} \omega^2 \qquad (3.14)$$

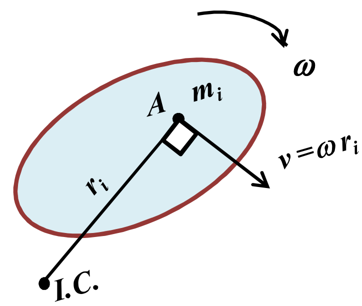


Fig. 3.9

Another expression for the kinetic energy of a **RB** in general plane motion can be derived as follows:

$$\because I_{I.C} = I_G + m r_G^2$$

where r_G is the distance between G and $I.C$ and m is the mass of the RB. Substituting into (3.14), the kinetic energy T may be expressed as:

$$T = \frac{1}{2}\left(I_G + m r_G^2\right)\omega^2 \quad = \frac{1}{2} I_G \omega^2 + \frac{1}{2} m \left(\omega r_G\right)^2 = \frac{1}{2} I_G \omega^2 + \frac{1}{2} m v_G^{\,2} \qquad (3.15)$$

Noting that: $\omega \, r_G = v_G$ = the velocity of G.

Note that :

1. For a RB in translational motion (ω=0) with speed v, Eq. (3.15) becomes:

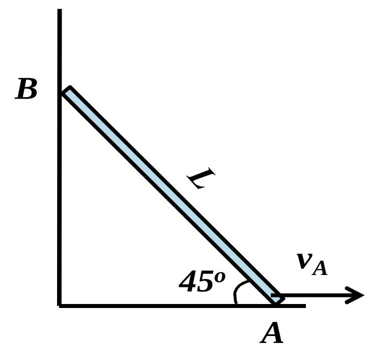
$$T = \frac{1}{2}mv_G^2 = \frac{1}{2}mv^2$$ as expressed in Eq. (3.12)

2. For a RB in rotational plane motion about a point O, Eq. (3.15) becomes:

$$T = \frac{1}{2}I_O\omega^2$$ as expressed in Eq. (3.13)

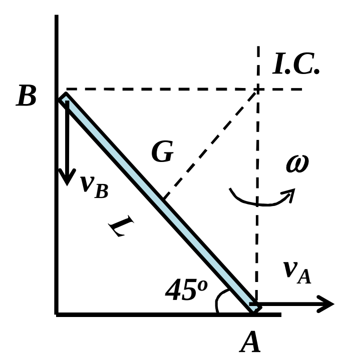
EXAMPLE(3.2):

The figure shows a uniform rod AB (m= 60 kg, L= 2m) moving in a vertical plane such that its end A slides horizontally and its end B slides vertically. At the instant shown v_A= 5 m/s. Find at this instant the kinetic energy of the rod.



SOLUTION:

The position of the I.C. of the rod AB is the point of intersection of the two normals drawn from A and B to $\underline{v}_A$ and $\underline{v}_B$ respectively. The kinetic energy of the rod AB at the shown position is given by:



$$T = \frac{1}{2}I_{I.C}\omega^2$$

$$\omega = \frac{v_A}{AI} = \frac{5}{\sqrt{2}} \text{ rad/s (c.w)}$$

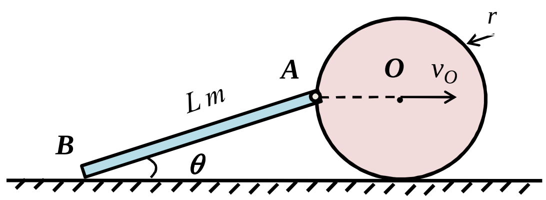
$$I_G = \frac{1}{12}ml^2 = 20 \text{ kg.m}^2$$

$$I_{I.C} = I_G + m(IG)^2 = 20 + 60(1)^2 = 80 \text{ kg.m}^2$$

$$\therefore \quad T = \frac{1}{2}(80)\left(\frac{5}{\sqrt{2}}\right)^2 = 500 \text{ Joule}$$

EXAMPLE(3.3):

The figure shows a uniform disc O (m_d=20 kg, r = 1.0 m) rolls on a rough fixed horizontal plane.

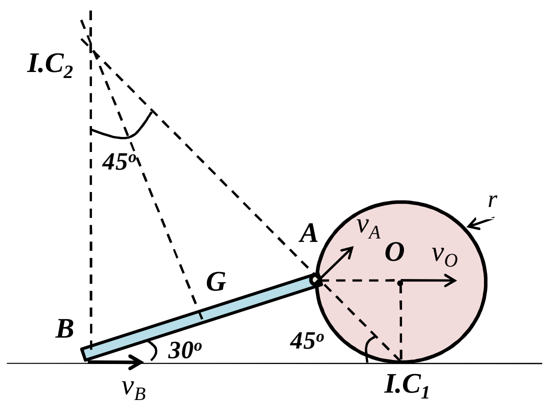


A uniform rod AB (m_r=15 kg, L=2m) is hinged to the disc at A while its end B slides on the horizontal plane. At the instant shown, v_O= 3 m/sec. Find at this instant the kinetic energy of the system.

SOLUTION: In this example we have a system of two bodies; the disc O and the rod AB. The kinetic energy of the system is the sum of the kinetic energies of the two bodies.

The disc O:

As the disc rolls, then the point of contact P is its $I.C$, hence:



$$T_d = \frac{1}{2} I_{I.C} \omega_d^2$$

$$\omega_D = \frac{v_O}{r} = \frac{4}{1} = 4 \text{ rad/s (c.w)}$$

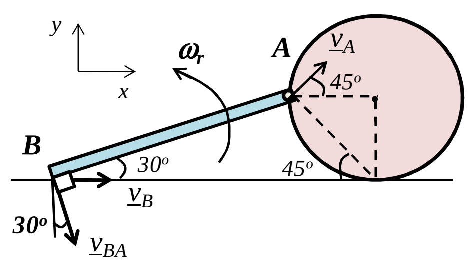
$$I_{I.C.} = I_G + m_d (r)^2 = \frac{1}{2} m_d r^2 + m_d r^2$$

$$= \frac{3}{2} m_d r^2 = 30 \text{ kg.m}^2$$

$$\therefore\ T_d = \frac{1}{2}(30) * (4)^2 = 240\ Joule$$

The rod AB :

We can use the instantaneous center law to calculate T_r, but we must calculate the lengths of AD, BD and GD using trigonometry.



$$\sin\theta = \frac{1}{2} \quad , \quad \therefore \theta = 30^o$$

It simple to use the low of G to calculate T_r, as:

$$T_r = \frac{1}{2} I_G \omega_r^2 + \frac{1}{2} m_r v_G{}^2$$

From the disc:

$$\underline{v}_A = \omega_d (AI_1)(\cos 45\underline{i} + \sin 45\underline{j}) = 4\underline{i} + 4\underline{j}$$

$$\underline{v}_B = v_B \underline{i}$$

Substitute in the velocity relation on the rod *AB*:

$$\underline{v}_B = \underline{v}_A + \underline{v}_{BA}$$

$$v_B \underline{i} = 4\underline{i} + 4\underline{j} + \omega_r (AB)(\sin 30\underline{i} - \cos 30\underline{j}) = 4\underline{i} + 4\underline{j} + \omega_r \underline{i} - \omega_r \sqrt{3}\underline{j}$$

$$y-comp.:\ 0 = 4 - \omega_r\sqrt{3} \quad \therefore \omega_r = \frac{4}{\sqrt{3}}\ rad/\sec\ (c.c.w.)$$

$$x-comp.:\ v_B = 4 + \omega_r = 4 + \frac{4}{\sqrt{3}} \quad \therefore \underline{v}_B = \left(4 + \frac{4}{\sqrt{3}}\right)\underline{i}$$

$$\underline{v}_G = \frac{1}{2}(\underline{v}_A + \underline{v}_B) = \frac{1}{2}\left((4\underline{i} + 4\underline{j} + (4 + \frac{4}{\sqrt{3}})\underline{i}\right) = (4 + \frac{2}{\sqrt{3}})\underline{i} + 2\underline{j}$$

$$v_G = \sqrt{(4 + \frac{2}{\sqrt{3}})^2 + 4} = 5.529\ \ m/\sec$$

$$I_G = \frac{1}{12} m_r L^2 = \frac{(15)*(2)^2}{12} = 5\ kg.m^2$$

$$T_r = \frac{1}{2} I_G \omega_r^2 + \frac{1}{2} m_r v_G{}^2 = \frac{1}{2}(5)\left(\frac{4}{\sqrt{3}}\right)^2 + \frac{1}{2}(15)*(5.529)^2 = 242.607\ \ Joule$$

$$\therefore\ \ T = T_d + T_r = 240 + 242.607 = 482.607\ \ Joule$$

4) PRINCIPLE OF WORK AND KINETIC ENERGY

The principle of work and kinetic energy states that:

“*The work done by all external forces acting on a mechanical system during its motion from position (1) to position (2) is equal to the change in its kinetic energy.”*

$$i.e.:\quad W_{1\to2}\big|_{all} = T_2 - T_1 \qquad (3.16)$$

Proof:

During the motion from position (1) to position (2), the work done $W_{1\to2}$ by all forces acting on the system is:

$$W_{1\to 2} = \sum_{i=1}^{N} \int_1^2 \underline{F}_i \cdot d\underline{r}_i \quad (1)$$

where $d\underline{r}_i$ is the infinitesimal displacement vector of point of application of $\underline{F}_i$ acting on the system and N is the number of the external forces. From Newton's 2nd law for a particle:

$$\underline{F}_i = m_i \left(\frac{d\underline{v}_i}{dt} \right) \quad (2)$$

$$\because \ \underline{v}_i = \frac{d\underline{r}_i}{dt} \quad , \quad \therefore \ d\underline{r}_i = \underline{v}_i dt \quad (3)$$

Substituting Eqs. (2) and (3) into Eq. (1), one gets:

$$W_{1\to 2} = \sum_{i=1}^{n} \left(\int_1^2 m_i \left(\frac{d\underline{v}_i}{dt} \right) \cdot (\underline{v}_i dt) \right) = \sum_{i=1}^{n} \left(\int_1^2 m_i \left(\underline{v}_i \cdot d\underline{v}_i \right) \right) = \sum_{i=1}^{n} \left(\int_1^2 m_i \left(v_i dv_i \right) \right)$$

$$\because \quad \underline{v} \cdot \underline{v} = v^2 \quad \textit{by differentiation} \ \therefore \ \underline{v} \cdot d\underline{v} = v dv$$

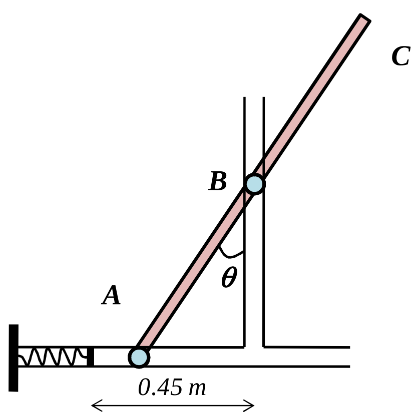
Therefore:

$$W_{1\to 2} = \sum_{i=1}^{n} m_i \left(\int_1^2 v_i dv_i \right) = \sum_{i=1}^{n} \tfrac{1}{2} m_i v_i^2 \Big|_{(2)} - \sum_{i=1}^{n} \tfrac{1}{2} m_i v_i^2 \Big|_{(1)} = T_2 - T_1$$

i.e., $W_{1\to 2} = T_2 - T_1$

EXAMPLE (3.4)

In the opposite figure, a rod ABC (m=20 kg, L=1.2 m), its midpoint is B. The rod is released from rest at θ=0, where B moves in a smooth vertical guide while end A moves in a smooth horizontal guide to compress a spring (stiffness k = 5 kN/m) as the rod falls. Neglecting the masses of the rollers at A and B, determine :

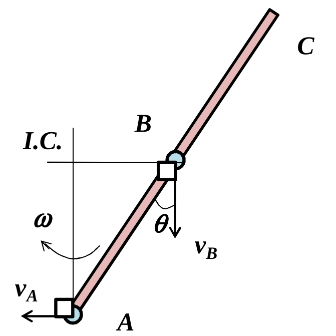


(a) The angular velocity of the bar at $\theta = 30^\circ$.

(b) The velocity with which B when strikes the horizontal surface.

SOLUTION:

The *I.C. (I)* of the rod is obtained by draw perpendiculars to v_A and v_B. The polar moment of inertia $I_{I.C}$ is given by :

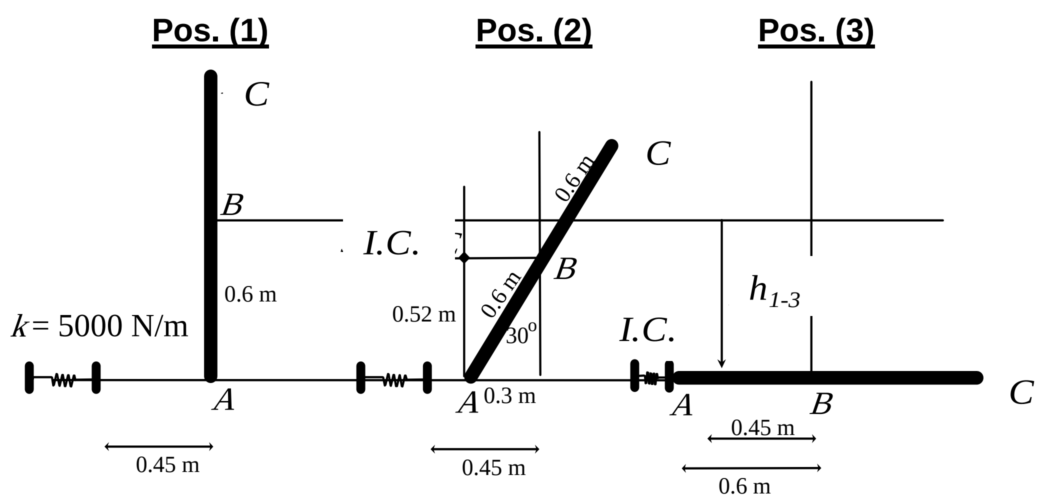


$$I_{i.c} = I_B + m(BI)^2 = \frac{1}{12} mL^2 + m(BI)^2 \qquad (1)$$

The initial vertical position is denoted by pos. (1), the position $\theta = 30^\circ$ is denoted by pos. (2), and the final horizontal position is denoted by pos. (3).

Case (1): Applying the principle of work and kinetic energy between pos. (1) and (2):

$$W_{1\to2}\big|_{all} = T_2 - T_1 \qquad (2)$$



The external forces acting of the bar are:

- The gravity force mg = 196 N (does work) :

$$W_{1\to2}\big|_{mg} = mgh_{1-2} = 200\ (0.6)(1 - Cos30) = 16.08 \quad Joule$$

- The normal reactions at *A* and *B* (do no work).

The kinetic Energy

T_1 =0 (the rod starts motion from rest)

$$T_2 = \tfrac{1}{2} I_C \omega_2^2$$

$I_C = \frac{1}{12}(20)(1.2)^2 + 20(0.3)^2 = 4.2 \text{ kg.m}^2$

$T_2 = \frac{1}{2}(4.2)\omega_2^2 = 2.1\omega_2^2$

Substituting into (2):

$16.08 = 2.1\omega_2^2 - 0 \quad , \quad \therefore \ \omega_2 = 2.767 \ \ rad/\sec$

Case (2): Applying the principle of work and kinetic energy between pos. (1) and (3):

$W_{1\to3}\big|_{all} = T_3 - T_1$ (3)

The external forces acting of the bar are:

- The gravity force mg = 196 N (does work) :

  $W_{1\to3}\big|_{mg} = mgh_{1-3} = 200(0.6) = 120 \ \ Joule$

- The normal reactions at A and B (do no work).
- The spring force when *A* strikes the spring given by:

  $W_{1\to3}\big|_{sp} = \frac{1}{2}k\left(x_1^2 - x_3^2\right) = \frac{1}{2}(5000)\left(0^2 - (-0.15)^2\right) = -56.25 \text{ J}$

where $x_1 = 0$, $x_2 = 0.45 - 0.6 = -0.15$ m .

T_1 =0 (the rod starts motion from rest)

The instantaneous center of the rod in position (3) is its end *A*, where the intersection of the perpendiculars to $\underline{\boldsymbol{v}}_A$ and $\underline{\boldsymbol{v}}_B$.

$T_3 = \frac{1}{2}I_C\omega_3^2 = \frac{1}{2}I_A\omega_3^2$

$I_A = \dfrac{mL^2}{3} = \dfrac{20(1.2)^2}{3} = 9.6 \ kg.m^2$

$T_3 = \frac{1}{2}(9.6)\omega_3^2 = 4.8 \ \omega_3^2$

Substituting into (3):

$120 - 56.25 = 4.8\omega_3^2 - 0 \quad \Rightarrow \quad \omega_3 = 3.644 \text{ rad/s}$

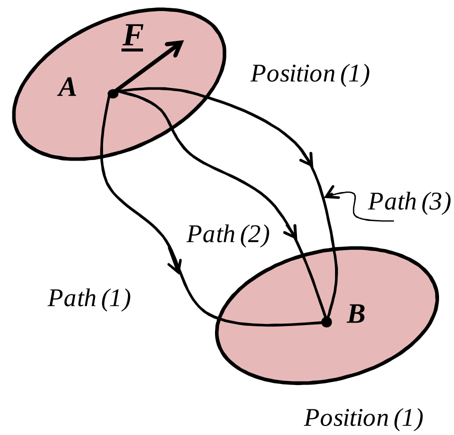
The speed v_B of end *B* at this instant is:

$v_B = \omega_3(BC) = \omega_3(BA) = 0.6\omega_3 = 2.186 \text{ m/s}$

5) CONSERVATIVE FORCE

A force ***F*** is said to be conservative if the work done by ***F*** during the motion between two position (1) and (2) is independent of the path between the two positions, i.e.:

$$W_{1\to2}\big|_{(1)} = W_{1\to2}\big|_{(2)} = W_{1\to2}\big|_{(3)}$$



From this definition we can conclude that:

- Any constant force is conservative.
- The gravity force is a conservative force.
- The force in a spring is a conservative force.

5.1) Potential function of a conservative force(V^F):

Since the work of the conservative force ***F*** during the motion of a RB from position (1) to position (2) is dependent only on the positions of the points *A* and *B*, then this work may be obtained mathematically by a function V^F, such that:

$$W_{1\to2} = V_1^F - V_2^F \tag{3.17}$$

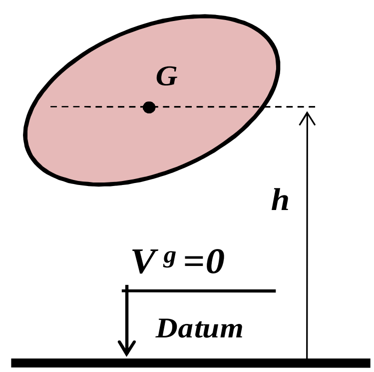
This function is dependent on positions (coordinates) of both *A* and *B* and is called the potential function of the force ***F*** .

Potential function of the gravity force(V^g):

The potential function of the gravity force is given by:

$$V^g = mgh \tag{3.18}$$

where *h* is the vertical distance of the mass center *G* of the RB above an arbitrary horizontal surface called the datum at which $V^g = 0$.



Potential function of a spring(V^S):

In the case of a linear spring, the potential function of the spring force is given by:

$$V^s = \frac{1}{2}kx^2 \tag{3.19}$$

where *x* is the extension in the spring ($x=L-L_o$)and *k* is the spring stiffness.

In the case of a rotational spring, the potential function of the spring is given by:

$$V^s = \frac{1}{2}k\theta^2 \qquad (3.20)$$

where θ is the rotational angle in the spring and k is the spring stiffness.

6) POTENTIAL ENERGY OF A SYSTEM (*V*)

The potential energy V of a mechanical system is defined to be the algebraic sum of the potential functions of all conservative forces acting on the system.

$$V = \sum_{i=1}^{N} V^{F_i}$$

Hence, the work of all conservative forces acting on a mechanical system during its movement from position (1) to position (2) may be expressed as:

$$W_{1\to2}^{C.F.} = V_1 - V_2 \qquad (3.21)$$

where $W_{1\to2}^{C.F}$ is the work done by all conservative forces acting on the mechanical system during the motion from position (1) to position (2) , V_1 is the potential energy of the system at position (1) and V_2 is the potential energy of the system at position (2) .

7) MECHANICAL ENERGY OF A MECHANICAL SYSTEM (*E*)

The mechanical energy ***E*** of a mechanical system is defined by the sum of its kinetic and potential energies. i.e. :

$$E = T + V \qquad (3.22)$$