8) PRINCIPLE OF WORK AND MECHANICAL ENERGY

The principle of work and mechanical energy is: *"The work done by all nonconservative forces acting on a mechanical system during its motion from position (1) to position (2) is equal to the change in its mechanical energy."*

$$W_{1\to2}^{NCF} = E_2 - E_1 \tag{3.23}$$

where:

E_i is the mechanical energy of the system at position (i).

$W_{1\to2}^{NCF}$ the work of nonconservative forces acting on the system during motion from position (1) to position (2).

Proof:

The work done $W_{1\to2}$ by all external forces acting on the system can be divided as:

$$W_{1\to2} = W_{1\to2}^{CF} + W_{1\to2}^{NCF} \tag{1}$$

where $W_{1\to2}^{CF}$ is the work done by all conservative forces and $W_{1\to2}^{NCF}$ is the work done by all nonconservative forces acting on the system.

Using the definition of the potential energy: $W_{1\to2}^{CF} = V_1 - V_2$ and subsisting in (1):

$$W_{1\to2} = (V_1 - V_2) + W_{1\to2}^{NCF} \tag{2}$$

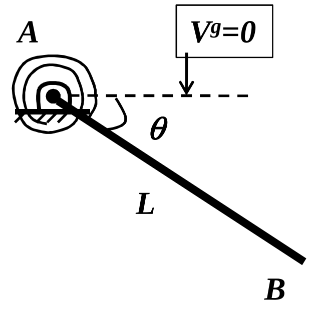
where V_1 is the potential energy of the system at position (1) and V_2 is the potential energy of the system at position (2) . From the principle of work and kinetic energy:

$$T_2 - T_1 = W_{1\to2} = (V_1 - V_2) + W_{1\to2}^{NCF} \tag{3}$$

$$\therefore \quad (T_2 + V_2) - (T_1 + V_1) = W_{1\to2}^{NCF} \tag{4}$$

$$\therefore \quad W_{1\to2}^{NCF} = E_2 - E_1$$

EXAMPLE (3.5):



A rod *AB* (m=20 kg, *L*=3 m) is hinged at rough hinge *A*, and connected with a rotational spring (*k*=150 N.m/rad) untwisted at θ=0. The rod released from rest at θ=0 and at θ=30° its angular velocity ω=2 rad/sec. Calculate:

- The frictional moment at *A*.
- The rod angular velocity at θ=90°.
- The reactions at *A* at θ=90°.

SOLUTION

The external forces acting on the rod during its motion are:

- The gravity force *mg* (conservative force);
- The moment in the spring (conservative force);
- The reactions at *A* (nonconservative force do no work) ;
- The frictional moment at *A* (nonconservative force do wok).

Assume pos. (1) is the initial horizontal position of the rod (at θ=0), pos. (2) at θ=30° and pos. (3) at θ=90°. The rod has rotational motion about *A*, then at any position:

$$T = \frac{1}{2} I_A \omega^2 \ \ , \ \ I_A = \frac{1}{3} mL^2$$

In position (1):

$$T_1 = 0$$

$$V_1 = V_1^g + V_1^S = 0 + 0 = 0$$

$$E_1 = T_1 + V_1 = 0$$

In position (2):

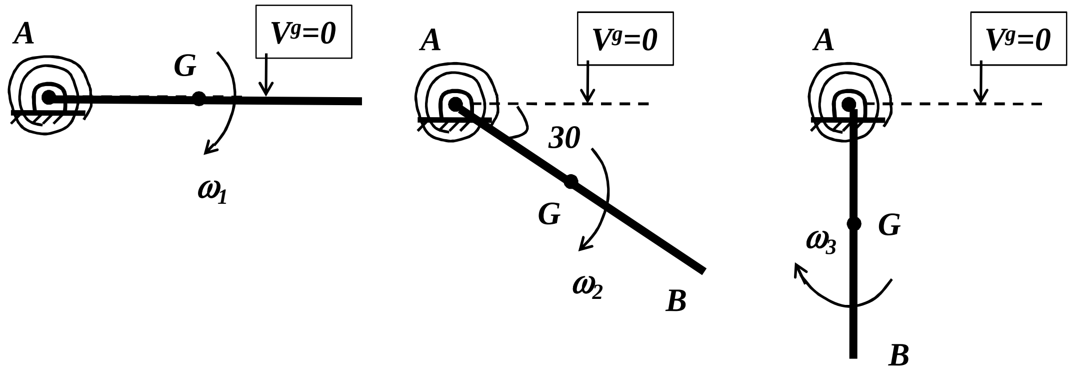
$$T_2 = \frac{1}{2} I_A \omega_2^2 = \frac{1}{2}\left(\frac{1}{3} mL^2\right)\omega_2^2 = \frac{1}{6}(20)*(3)^2*(2)^2 = 120 \ \ Joule$$

$$V_2 = V_2^g + V_2^S = -mgh_2 + \frac{1}{2}k\theta_2^2 = -(20\times 10)(1.5\sin(30)) + \frac{1}{2}(150)*(\frac{\pi}{6})^2 = -129.438 \ Joule$$

$$E_2 = T_2 + V_2 = 120 - 129.438 = -9.438 \ Joule$$

Work of nonconservative forces:

$$W_{1\to 2}^{NCF} = -M_F \theta \quad \Rightarrow \quad W_{1\to 2}^{NCF} = -M_F\left(\frac{\pi}{6}\right)$$



W-E rule between pos. (1) and pos. (2):

$$E_1 + W_{1\to 2}^{NCF} = E_2 \quad \Rightarrow 0 - M_F\left(\frac{\pi}{6}\right) = -9.438 \Rightarrow \therefore \ M_F = 18.03\ N.m/rad$$

In position (3):

$$T_3 = \frac{1}{2} I_A \omega_3^2 = \frac{1}{2}\left(\frac{1}{3} m L^2\right)\omega_3^2 = \frac{1}{6}(20) * (3)^2 * \omega_3^2 = 30\ \omega_3^2 \quad Joule$$

$$V_3 = V_3^g + V_3^S = -mgh_3 + \frac{1}{2} k\,\theta_3^2 = -(20\times 10)(1.5) + \frac{1}{2}(150) * (\frac{\pi}{2})^2 = -114.945\ Joule$$

$$E_3 = T_3 + V_3 = 30\ \omega_3^2 - 114.945 \quad Joule$$

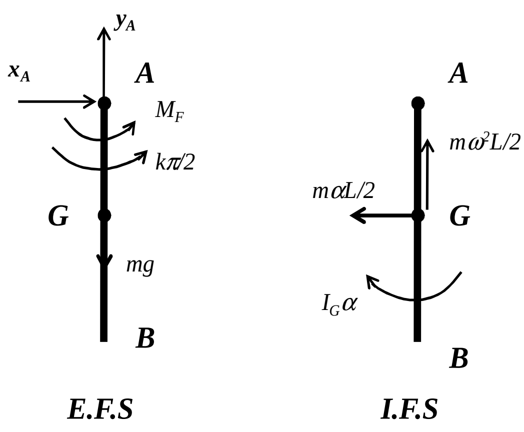
Work of nonconservative forces:

$$W_{1\to 3}^{NCF} = -M_F \theta_3 \quad \Rightarrow W_{1\to 2}^{NCF} = -18.03\left(\frac{\pi}{2}\right) = -28.322 \quad Joule$$

W-E rule between pos. (1) and pos. (3):

$$E_1 + W_{1\to 3}^{NCF} = E_3 \quad \Rightarrow 0 - 28.322 = 30\ \omega_3^2 - 114.945 \ \Rightarrow \therefore \ \omega_3 = 1.7 \ rad/\sec$$

The reactions at A in pos. (3):



Equations of motion:

In x- direction:

$$x_A = -m\alpha\left(\frac{L}{2}\right) \qquad (1)$$

In y- direction:

$$y_A - mg = m\omega^2\left(\frac{L}{2}\right) \quad \Rightarrow \quad y_A = 200 + 20*(1.7)^2*1.5 = 286.7 \quad N \qquad (2)$$

Moment about A:

$$-M_F - k\left(\frac{\pi}{2}\right) = I_G\alpha + \left(\frac{m\alpha L}{2}\right)\left(\frac{L}{2}\right) = I_A\alpha \qquad (3)$$

From (3):

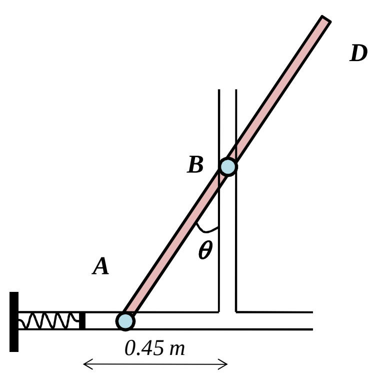
$$-18.03 - 150\left(\frac{\pi}{2}\right) = \left(\frac{20(3)^2}{3}\right)\alpha \quad \Rightarrow \quad \alpha = -4.228 \;\; rad/\sec^2$$

Substitute in (1):

$$\therefore \; x_A = -20(-4.227)(1.5) = 126.825 \;\; N$$

EXAMPLE (3.6)

In the opposite figure, a rod *ABD* (m=20 kg, *L*=1.2 m), its midpoint is *B*. The rod is released from rest at θ=0, where *B* moves in a smooth vertical guide while end *A* moves in a smooth horizontal guide to compress a spring (stiffness *k* = 5 kN/m) as the rod falls. Use the rule of work and mechanical energy to determine:



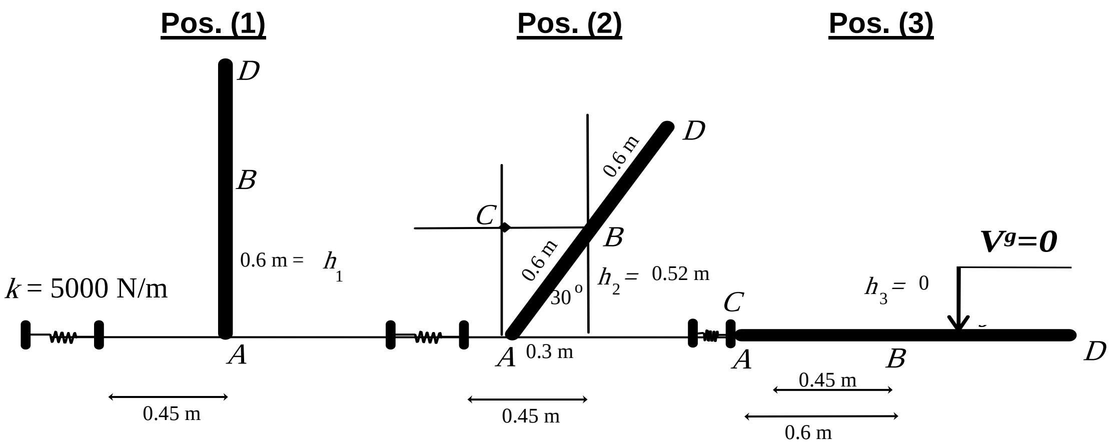
(a) The angular velocity of the rod at $\theta = 30^\circ$.

(b) The velocity with which *B* strikes the horizontal surface.

SOLUTION

The external forces acting on the rod during its motion are:

- The gravity force *mg* (conservative force);
- The force in the spring when *A* strikes the spring (conservative force do work);
- The normal reactions at *A* and *B* (nonconservative forces do no work).



The mechanical energy in different positions:

In position (1):

$T_1 = 0$ (motion starts from rest)

$h_1 = 0.6$ m , $x_1 = 0$

$V_1 = V^g + V^S = mgh_1 + \frac{1}{2}kx_1^2 = 200(0.6) + 0 = 120\ Joule$

$\therefore\ E_1 = V_1 + T_1 = 120\ \ Joule$

In position (2):

If ω_2 is the angular velocity of the rod in pos. (2), then:

$$T_2 = \frac{1}{2} I_C \omega_2^2 = \frac{1}{2}\left[\left(\frac{mL^2}{12}\right) + m\left(\frac{L}{4}\right)^2\right]\omega_2^2 = 2.1\ \omega_2^2$$

$h_2 = 0.52$ m , $x_2 = 0$

$V_2 = V^g + V^S = mgh_2 + \frac{1}{2}kx_2^2 = 20*10*0.6*\cos(30) + 0 = 103.92\ Joule$

$E_2 = V_2 + T_2 = 103.92 + 2.1\omega_2^2\ \ Joule$

Work of nonconservative forces:

Nonconservative forces acting on the system do no work, i.e.:

$\therefore\ W_{1\to2}^{NCF} = W_{1\to2}^{NCF} = 0$

Applying between positions (1) and position (2):

$E_1 = E_2$

$120 = 103.92 + 2.1\omega_2^2 \quad \Rightarrow \quad \therefore \omega_2 = 2.767$ rad/sec

In position (3):

By drawing perpendicular lines on $\underline{v}_A$ (→) and $\underline{v}_B$ (↓), one finds that the point A is *I.C.* of the rod in pos.(3), then:

$$T_3 = \frac{1}{2} I_A \omega_3^2 = \frac{1}{2}\left(\frac{mL^2}{3}\right)\omega_3^2 = 4.8\ \omega_3^2 \quad Joule$$

$$h_3 = 0$$

$$x_3 = -0.15 \text{ m}$$

$$V_3 = V^g + V^S = mgh_3 + \tfrac{1}{2}kx_3^2 = \tfrac{1}{2}(5000)(-0.15)^2 = 56.25 \text{ J}$$

$$\therefore \quad E_3 = V_3 + T_3 = 56.25 + 4.8\omega_3^2 \ \ Joule$$

Applying between positions (1) and position (3):

$$E_1 = E_3$$

$$120 = 56.25 + 4.8\omega_3^2 \quad \Rightarrow \quad \therefore\ \omega_3 = 3644 \text{ rad/sec}$$

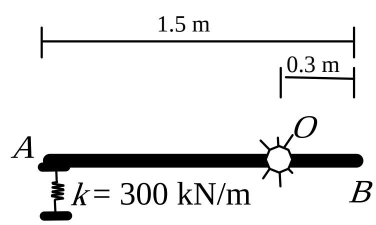
The speed v_B of end B at this instant is :

$$v_B = \omega_3 (BA) = 0.6\omega_3 = 2.187 \ \ m/\sec$$

EXAMPLE (3.7):

A rod AB (m=15 kg, L =1.5 m) is hinged at O which is 0.3 m from its end B. The other end A is pressed against a spring of stiffness k = 300 kN/m until the rod reaches the horizontal position and the spring is compressed by 25 mm, then the rod is released. In the vertical position, calculate:

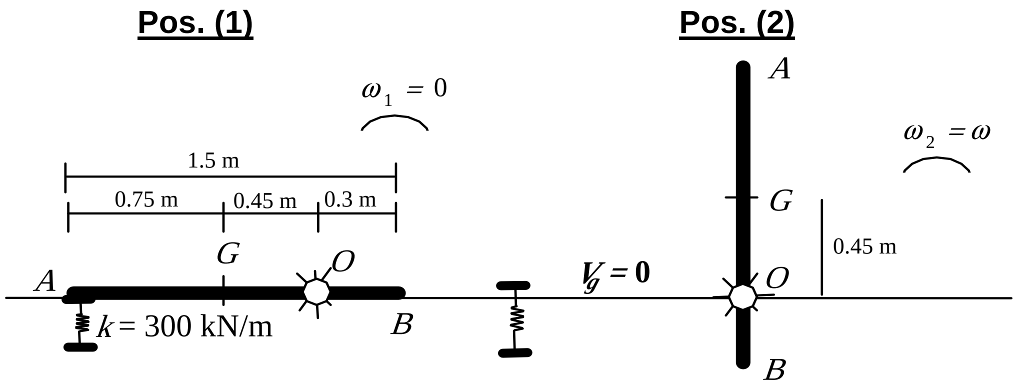
- The rod angular velocity.
- The reaction at the hinge O.



SOLUTION

The external forces acting on the rod during its motion are:

- The gravity force *mg* (conservative force);
- The force in the spring (conservative force);
- The reactions at O (nonconservative force do no work).



Assume the initial horizontal position of the rod is denoted by pos. (1) and vertical position is denoted by pos. (2).

The rod has a rotational motion about O, therefore O is its $I.C$ at any position. The given data and the polar moment of inertia I_O can be summarized as follows:

$$\omega_1 = 0 \qquad \omega_2 = ?$$
$$x_1 = -25 \text{ mm} = -0.025 \text{ m} \qquad x_2 = 0$$
$$h_1 = 0 \qquad h_2 = 0.45 \text{ m}$$

$$I_G = \frac{mL^2}{12} = \frac{(15)(1.5)^2}{12} = 2.8125 \text{ kg.m}^2$$
$$I_O = I_G + m(OG)^2 = 2.8125 + (15)(0.45)^2 = 5.85 \text{ kg.m}^2$$

In position (1):

$$T_1 = 0$$
$$V_1 = V_1^g + V_1^S = mgh_1 + 0.5kx_1^2 = (0) + 0.5(300000)(-0.025)^2 = 93.75 \quad Joule$$

In position (2):

$$T_2 = 0.5\, I_O \omega_2^2 = 0.5\,(5.85)\omega_2^2 = 2.925\, \omega_2^2$$
$$V_2 = V_2^g + V_2^S = mgh_2 + 0.5kx_2^2 = (15\times 10)(0.45) + 0 = 67.50\, Joule$$

Work of nonconservative forces:

Nonconservative forces acting on the system do no work, *i.e.*:

$$E_1 = E_2 \quad \Rightarrow T_1 + V_1 = T_2 + V_2 \qquad (1)$$

Substituting into (1):

$$0 + 93.75 = 2.925\omega_2^2 + 67.5 \quad \Rightarrow \quad \omega_2 = 2.996 \ \ rad\ /\sec$$

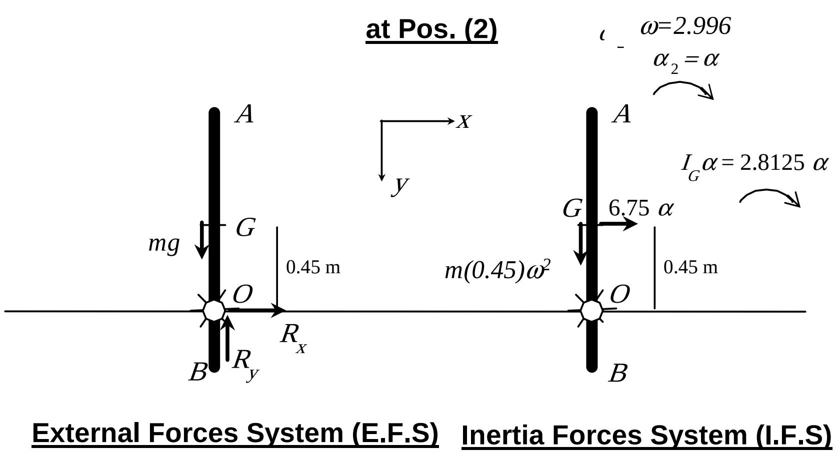
The reactions at O:

To obtain the reactions at *O* at position (2), the equations of motion must be used. The rod has a rotational motion about *O*. Since we know the angular velocity of the rod ω at this position, then there are 3-unkowns and the equations of motion of the RB at this position are enough to determine the required unknowns (i.e. no need for general position solution).

The external forces acting on the rod are:

- The gravity force *mg*;
- The reactions at O.

Draw the external force system and inertia force system



The acceleration of the rod G is:

$$\underline{a}_G = \underline{a}_O + \underline{a}_{GO}^n + \underline{a}_{GO}^t = \underline{0} + \omega^2(GO)\,\underline{j} + \alpha(GO)\,\underline{i} = 0.45\alpha\,\underline{i} + 4.04\,\underline{j}$$

The inertia forces of the rod are:

$$m\underline{a}_G = 15\left(0.45\alpha\,\underline{i} + 4.031\,\underline{j}\right) = 6.75\alpha\,\underline{i} + 60.588\ \underline{j}\ \text{N}$$

Equations of motion of the rod at position (2) :

Moment equation about O:

$$0 = 2.8125\alpha + 6.75\alpha\,(0.45) \quad \Rightarrow \quad 5.85\alpha = 0 \quad \Rightarrow \quad \alpha = 0$$

In the x direction:

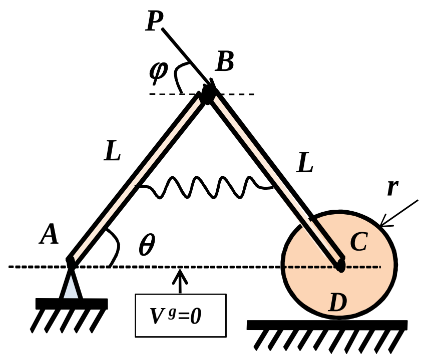
$$R_x = 5.75\alpha \quad \Rightarrow \quad \boxed{R_x = 0}$$

In the y direction :

$$150 - R_y = 60.588 \quad \Rightarrow \quad R_y = 89.412 \;\; N$$

EXAMPLE (3.8) :

A constant force $P=0.435\ kN$ ($\varphi=45^o$) is applied to the shown mechanism which composed of two similar rods AB and BC (m_1 =30 kg and L=3.0m) and a disc C (m_2=10 kg, r=1.0 m) which rolls on a horizontal surface as shown. There is a spring (k, L_O=2.0 m) joins the midpoints of the two rods. If the system starts its motion from rest at $\theta=60^o$ and at θ=30°, the velocity $\underline{v}_C$ becomes 6 m/sec, Calculate:



- The spring constant k.
- The velocity of point B at $\theta = 0$.

SOLUTION

The external forces acting on the system are:

- The gravity force m_1g of the rods (conservative force do work);
- The gravity force m_2g of the disc (conservative force do no work);
- The spring force (conservative force do work);
- The reactions at A and D (nonconservative force do no work);
- The external force P (nonconservative forces do work).

In position (1) (θ=60°):

$T_1 = 0$ (motion starts from Rest)

$$V_1 = V_1^g\Big|_{AB} + V_1^g\Big|_{BC} + V_1^S = m_1 g\left(\frac{L}{2}\sin 60\right) + m_1 g\left(\frac{L}{2}\sin 60\right) + 0.5\ k x_1^2$$

$$x_1 = L_1 - L_O = 2*\left(\frac{L}{2}\sin 30\right) - 2 = 1.5 - 2 = -0.5\ m$$

Substitute:

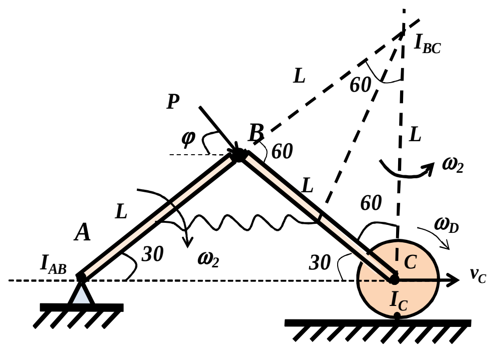
$$\therefore V_1 = (300)*(3)*\sin 60 + 0.5k\,(-0.5)^2 = 779.423 + 0.125\,k \quad Joule$$

$$\therefore E_1 = 779.423 + 0.125k \quad Joule$$

In position (2) (θ=30^o):

Since the two rods are similar, then they have the same angular position, the same angular velocity and the same angular accelerations, *i.e.*:

$$\because \theta_{AB} = -\theta_{BC} \;\Rightarrow\; \therefore\; \omega_{AB} = -\omega_{BC} \quad \text{and} \quad \alpha_{AB} = -\alpha_{BC}$$



Using the instantaneous center method, the hinge *A* is the *I.C.* of the rod *AB*, then using *I.C.* of *AB*, the direction of $\underline{\mathbf{v}}_B$ will be normal to *AB*. Draw perpendicular lines to the directions of the $\underline{\mathbf{v}}_B$ and $\underline{\mathbf{v}}_C$ which moves on a horizontal line passing through *A*, then one can obtain the *I.C.* of the rod *BC***.** Since the disc rolls, then the point *D* is the *I.C.* of the disc.

$$I_A^{AB} = \frac{m_1 L^2}{3} = \frac{(30)(3)^2}{3} = 90\ \text{kg.m}^2$$

Using trigonometric relations:

$$\because \angle ABC = 120^o \quad \therefore \angle CBI_{BC} = 60^o \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

$$\text{The triangle } BCI_{BC} \text{ is equilateral triangle; then: } BC = BI_{BC} = CI_{BC} = 3m$$

$$G_{BC} I_{BC} = 3\sin(60) = 1.5\sqrt{3}\ m$$

$$I_G^{BC} = \frac{mL^2}{12} = \frac{(30)(3)^2}{12} = 22.5\ \text{kg.m}^2$$

$$I_{I.C}^{BC} = I_G + m\,(GI_2)^2 = 22.5 + (30)(1.5\sqrt{3})^2 = 225\ \text{kg.m}^2$$

Using the instantaneous center of the rod *BC*:

$$v_C = (CI_{BC})\omega_{BC} \;\Rightarrow\; \therefore\; \omega_{BC} = \frac{v_C}{CI_{BC}} = \frac{6}{3} = 2\ \text{rad/sec (c.c.w)}$$

$$T_2 = T_2^{AB} + T_2^{BC} + T_2^{C} = \frac{1}{2} I_A^{AB} \omega_2^2 + \frac{1}{2} I_{I_2}^{BC} \omega_2^2 + \frac{1}{2}(\frac{1}{2} m_2 r^2 + m_2 r^2)(\frac{v_C}{r})^2$$

$$T_2 = \frac{1}{2}(90)(2)^2 + \frac{1}{2}(225)(2)^2 + \frac{3}{4}(10)\left(\frac{6}{1}\right)^2 = 900 \ \ Joule$$

$$V_2 = V_2^g\big|_{AB} + V_2^g\big|_{BC} + V_2^S = 2 m_1 g\left(\frac{1}{2} L \sin 30\right) + \frac{1}{2} k x_2^2$$

$$x_2 = L_2 - L_O = 2*\left(\frac{1}{2} L \sin 60\right) - 2 = 2.598 - 2 = 0.598 \ m$$

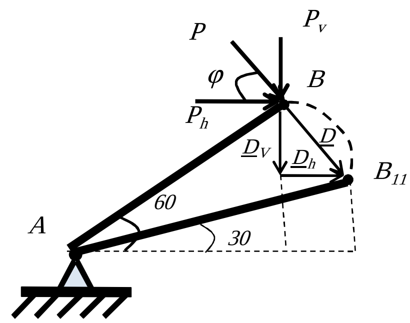
$$\therefore V_2 = 450 + 0.17885 k$$

Substitute:

$$\therefore E_2 = T_2 + V_2 = 1350 + 0.17885 k \quad Joule$$

Work of nonconservative forces:

Since the work of a force is equal to the work of its components, then the work of nonconservative forces acting on the system (*P*) may be calculated as follow:



$$W_{1\to2}^{NCF} = W_{1\to2}\big|_{P_h} + W_{1\to2}\big|_{P_v} = P_h D_h + P_v D_v$$

$$P_h = P\cos\varphi = 435 * \cos(45) = 307.591 = P_v$$

$$D_h = L\left((\cos(30) - \cos(60)\right) = 1.0981 \ \ m$$

$$D_v = L\left((\sin(60) - \sin(30)\right) = 1.0981 \ \ m$$

$$\therefore W_{1\to2}^{NCF} = 307.591(1.0981) * 2 = 675.517 \ Joule$$

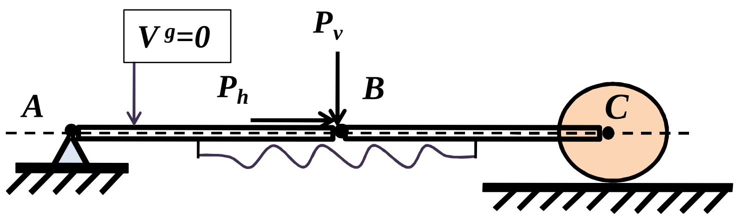
Principle of work and mechanical energy:

$$E_1 + W_{1\to2}^{NCF} = E_2$$

$$\therefore 779.423 + 0.12 k + 675.517 = 1350 + 0.17885 k \quad \Rightarrow \therefore \ k = 1948.75 \ N/m$$

In position (3) (θ=0^o):

In position (3), the point *A* is the *I.C.* of rod *AB*. For the rod *BC*, since the point *B* moves vertically and the point *C* moves horizontally, then by drawing two perpendicular lines on the directions of $\mathbf{v}_B$ and $\mathbf{v}_C$, one finds that the point *C* is the *I.C.* of the rod *BC*. As the disc center is a fixed point (*I.C.* of the rod *BC*) then the disc is at rest instantaneously.



$$T_3 = T_3^{AB} + T_3^{BC} + T_3^{C} = \frac{1}{2} I_A^{AB} \omega_3^2 + \frac{1}{2} I_C^{BC} \omega_3^2 + 0$$

$$T_3 = \frac{1}{2}(90)\omega_3^2 + \frac{1}{2}(90)\omega_3^2 = 90\,\omega_3^2 \ \ Joule$$

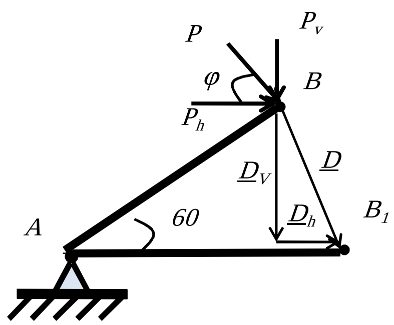
$$V_3 = V_3^g + V_3^S = 0 + \frac{1}{2} k x_3^2$$

$$x_2 = L_3 - L_O = 3 - 2 = 1\ m$$

$$\therefore V_3 = \frac{1}{2}(1948.75)(1)^2 = 974.375\ Joule$$

Work of nonconservative forces:

The work of nonconservative forces acting on the system (P) may be calculated as follow:



$$W_{1\to3}^{NCF} = W_{1\to3}\big|_{P_h} + W_{1\to3}\big|_{P_v} = P_h D_h + P_v D_v$$

$$P_h == P_v = P\cos 45 = 307.591$$

$$D_h = L - L\cos(60) = 1.5\ \ m$$

$$D_v = L\sin(60) = 2.598\ \ m$$

$$\therefore\ W_{1\to3}^{NCF} = 307.591(1.5 + 2.598) = 1260.51\ Joule$$

Principle of work and mechanical energy between pos. (1) and pos. (3):

$$E_1 + W_{1\to3}^{NCF} = E_3$$

$$\therefore 779.423 + 0.125(1948.75) + 1260.531 = 90\omega_3^2 + 974.375$$

$$\therefore\ \omega_3 = 3.814\ rad/\sec\ \ \Rightarrow v_B = L\,\omega_3 = 11.442\ m/\sec$$