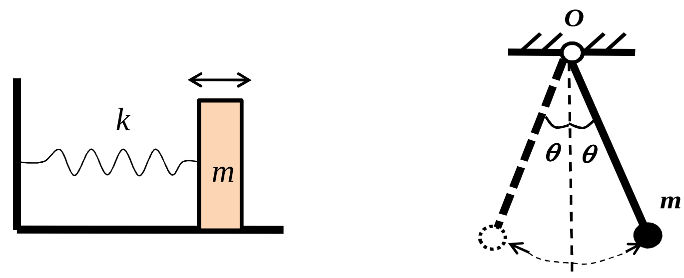
Vibrations of Structural Systems

1) INTRODUCTION

The analysis of vibration is an important subject for engineers and it contains large physical and mathematical knowledge. In this chapter, the study will be limited to simple types of vibrations to describe the main concepts of the subject.

Vibration is defined as the oscillatory motion of a RB or a system about its stable equilibrium position as shown if Fig. (4.1).



Vibration of Mass-spring System   Vibration of Simple Pendulum

Fig. (4.1): Oscillatory Motion

Equilibrium is a mathematical concept means that the forces acting on a system are in equilibrium and equations of equilibrium can be applied. However, stability is a physical concept means that the system will return to its stable equilibrium position if it slightly displaced away from it as in case (a) in Fig. (4.2). However, if the system is slightly displaced from the unstable equilibrium position it will not return to it but will continue to rest in the stable position as the case (b) shown in Fig. (4.2).

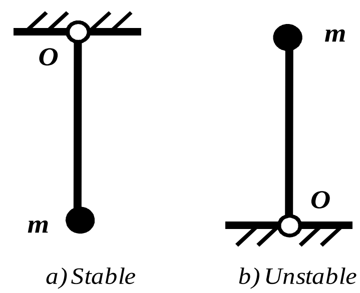


Fig. (4.2): Equilibrium Positions

Vibration is undesirable phenomenon because it increases the level of stresses in structures due to the exerted inertia forces appear when the structure dynamic state is changed. Also, vibration causes energy losses in machinery operation and causes damages for both heavy and sensitive machines. Recently, many high rise buildings and light structures are constructed. These structures are very sensitive to vibration and must be designed to control the vibration amplitude to those levels suitable for the human satisfaction.

2) DEGREES OF FREEDOM

Degrees of freedom of a system are the minimum number of variables required to describe the system motion. As the number of variables required for describing the system motion increases, the mathematical treatment of the system becomes more complicated. Figure (4.3) shows systems with different degrees of freedom.

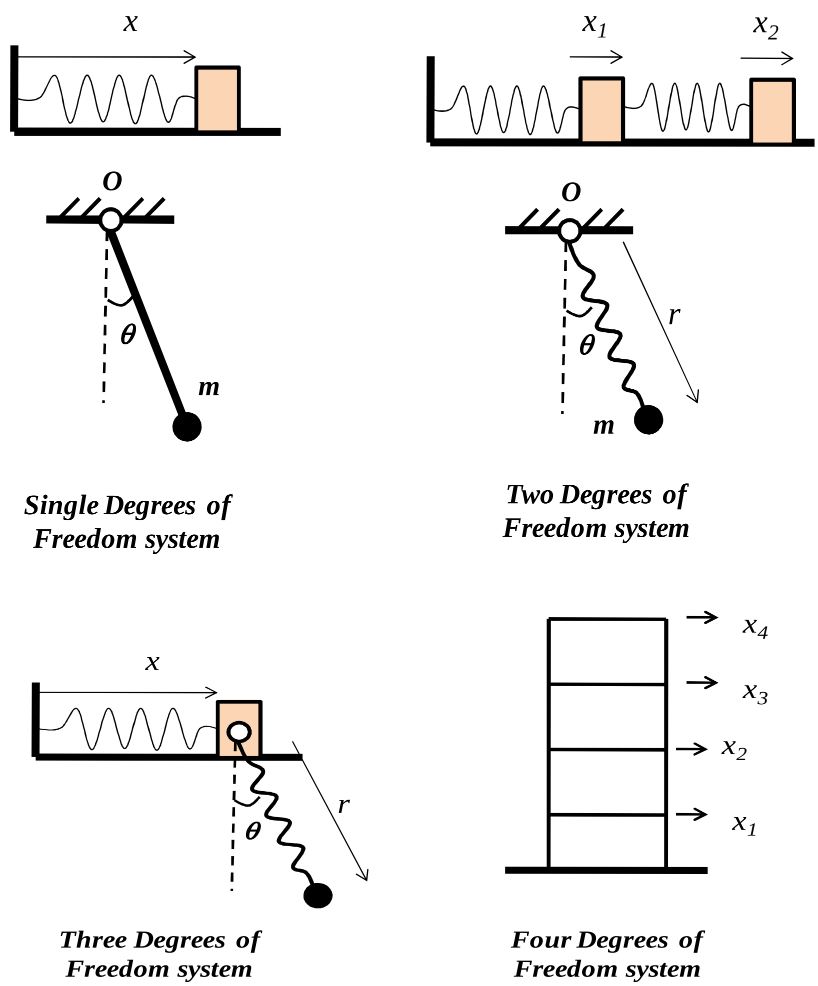


Fig. (4.3): Systems with Different Degrees of Freedom

In single degree of freedom system, only one variable (x, y or θ) is enough to define position, velocity and acceleration of the RB, while in two degrees of freedom system two variables are required and so on. The number of differential equations required to study the system motion is equal to its degrees of freedom. In this study, only single degree of freedom systems will be studied.

3) DYNAMIC LOADS

Dynamic loads (called excitations) are these forces whose magnitude or direction or both varies with time.

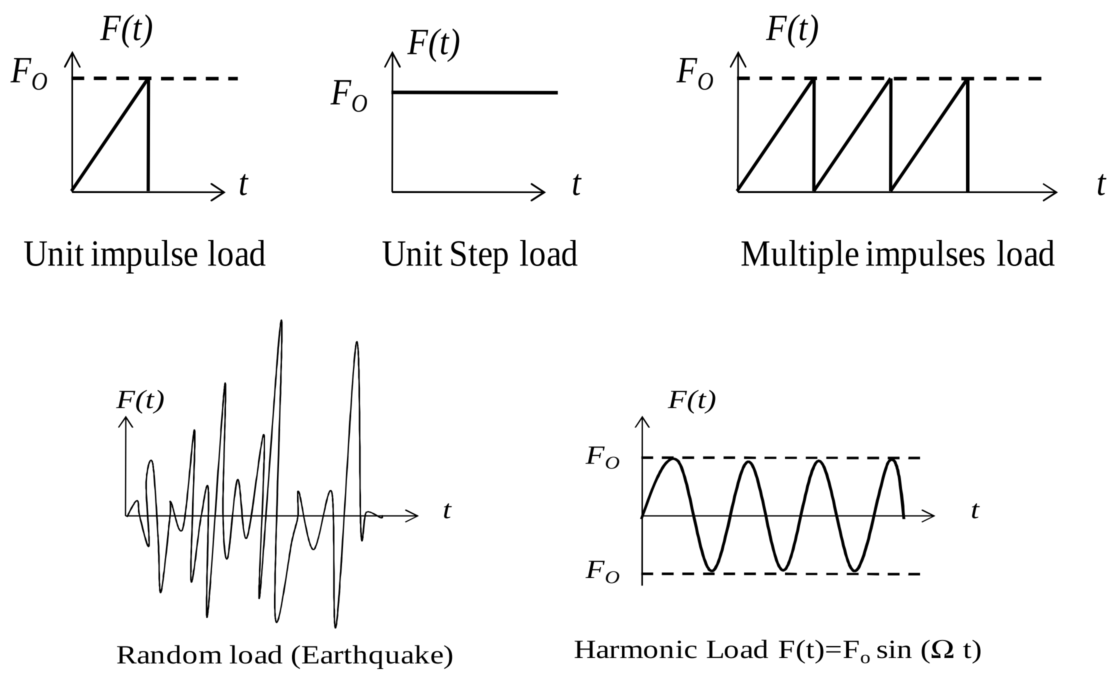


Fig. (4.4): Different Types of Dynamic Loads (Excitations)

Mathematically, the harmonic load is the simplest one to be dealt with in dynamic analysis while other types need difficult treatment. Most of dynamic loads acting on structure are in the form of harmonic load due to motion of the rotating parts in machines and equipments. The harmonic load may be expressed in sine or cosine form and totally described by its ($\boldsymbol{F_O}$) amplitude and frequency ($\boldsymbol{\Omega}$). Actually, as $\boldsymbol{F_O}$ or $\boldsymbol{\Omega}$ increases the effect of the load increases. The harmonic load may be expressed as:

$$F(t) = F_O \sin \Omega t \quad N \tag{4.1}$$

$$F(t) = F_O \cos \Omega t \quad N \tag{4.2}$$

where:

F_O is the load amplitude which is the maximum (or minimum) value of the load.

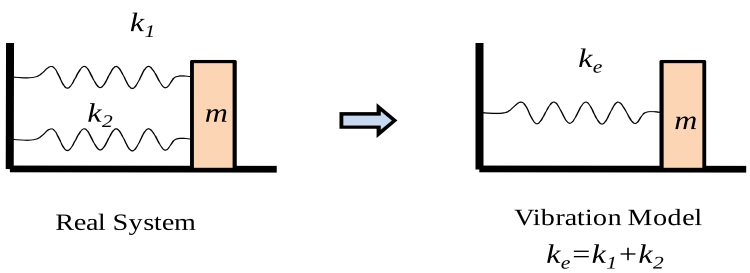
Ω is the circular frequency of the load which represents the variation with time.

The ordinary frequency of load is given by: $F=\Omega/2\pi$ Hz (Hz =cycle per second)

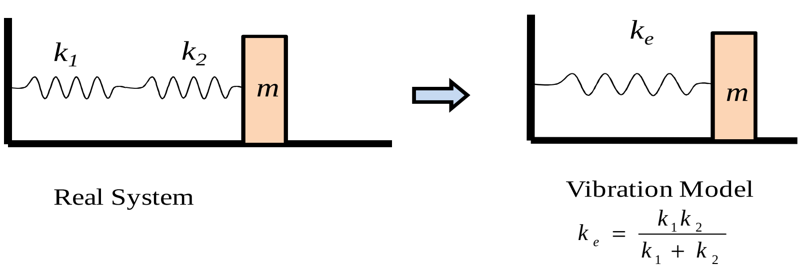
Other types of non-harmonic loading can be transformed to harmonics load by mathematical methods as *Fourier Transform*.

4) MODLING

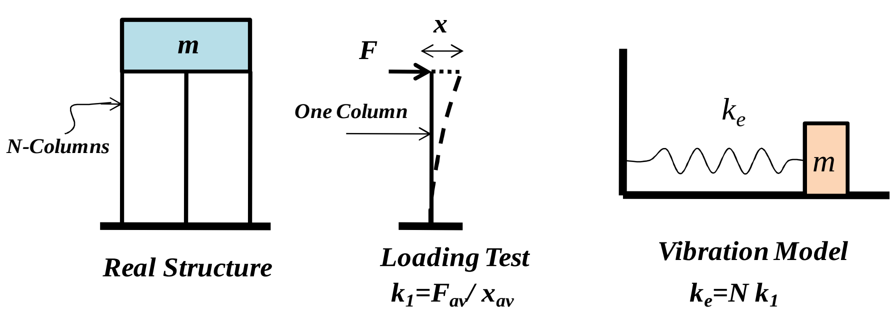
Modeling are important tools widely used to solve complex engineering problems, by transforming a real structure to simple model that behaves as the real structure to simplify the mathematical treatments. The model must have the mechanical properties and dynamic behavior of the real structure. Simple modeling process is that in which many springs in mass-spring system are replaced by one equivalent spring. In vibration analysis, most real structures can be modeled as mass-spring systems. In the following cases, as system is displaced from stable equilibrium position, internal force (called restoring force) of the system will appear and try to return back the system to its stable equilibrium position. This force is proportional to the displacement of the system from its equilibrium position due to the elasticity of the system or gravity force. Following figures in Fig. (4.5) show examples of modeling of different systems.



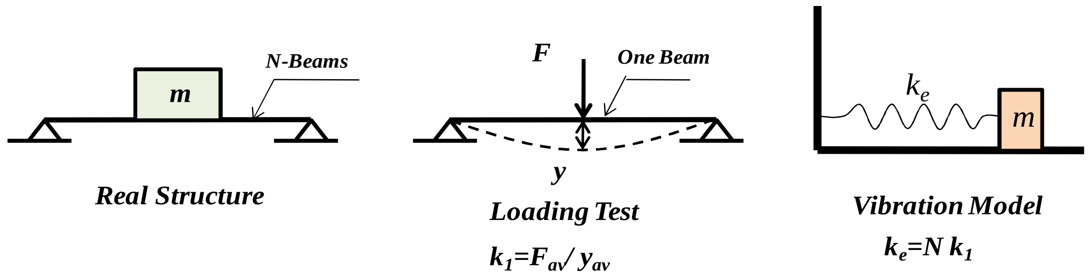
a) Replacing Two Parallel Springs by Equivalent Spring



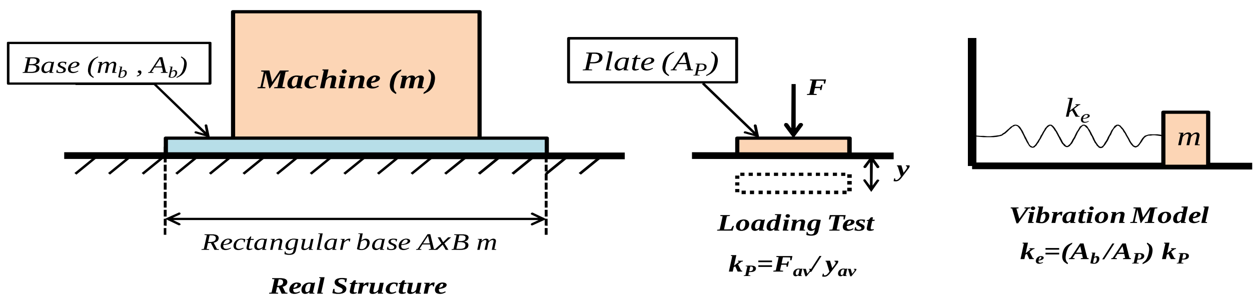
b) Replacing Two Serial Springs by Equivalent Spring



c) Modeling Water Tank by Replacing N-Columns by Equivalent Spring



d) Modeling Machine Resting on N-Beams by Equivalent Spring



e) Modeling Machine Foundation by Equivalent Spring

Fig. (4.5): Modeling Real Structures by Simple Vibration Model

In above examples, real structures of single degree of freedom are represented by a simple mass-spring system. In case (a) and (b) many springs may be represented by one spring of equivalent stiffness. In case (c), water elevated tank or guard tower supported by cantilever *N*-columns can be modeled by a mass-spring system of equivalent stiffness k_e. A loading test is carried on one column to obtain the column stiffness against horizontal displacement. A horizontal force *F* is applied at the top of the column and the displacement *x* is measured many times using different forces, then different values F_i and the corresponding displacement x_i can be obtained. Then, the average force F_{av} and the average displacement x_{av} can be calculated as:

$$F_{av} = \frac{\sum_{i=1}^{N} F_i}{N} \quad , \quad x_{av} = \frac{\sum_{i=1}^{N} x_i}{N} \tag{4.3}$$

Hence, the stiffness of the equivalent spring is obtained as:

$$k_e = \frac{F_{av}}{x_{av}} \tag{4.4}$$

In case (d), machine vibrates in the vertical direction and similar procedures for loading test in case (c) are carried out to obtain the equivalent stiffness of the vibration model.

In case (e), a machine fitted with rectangular base with area (A_b= AxB m) fixed in the ground is represented by mass-spring system. In that case a plate loading test with plate of area A_p is carried out to obtain the plate stiffness k_p. Then, the stiffness of the equivalent spring can be calculated as:

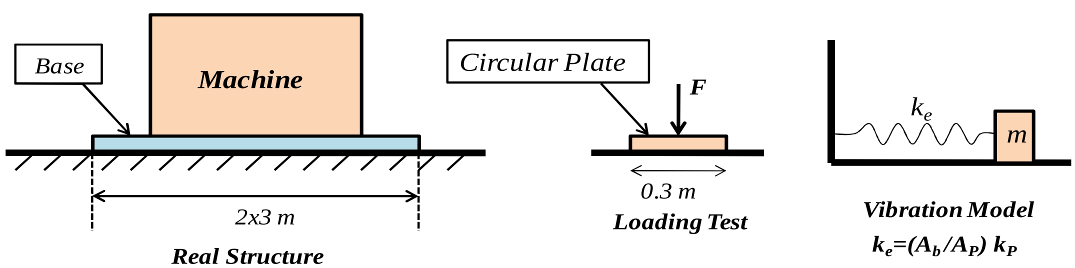
$$k_e = \frac{A_b}{A_p} k_p \qquad (4.5)$$

Example (1):

A machine supported on a rectangular base (2x3 m). The base is fixed in a rocky layer. To measure the system stiffness, a plate loading test is carried out in the site where the machine will be fitted using circular plate of diameter 0.3 m. The data of the load test are:

F (N)	100	150	200	300
x (mm)	1.03	1.48	1.95	3.1

Calculate the system stiffness. Also, if the required stiffness of the system to control vibration amplitude is 7200 kN/m, calculate L if a rectangular base (Lx2L m) to be used.



Solution:

a) The system stiffness

$$k_P = \frac{F_{av}}{x_{av}} = \frac{\left(\sum F_i / N\right)}{\left(\sum x_i / N\right)} = \frac{\sum F_i}{\sum x_i}$$

$$\because \quad \sum F_i = 750 \ \ N \quad , \sum x_i = 7.65 \text{ mm}$$

$$\therefore \quad k_P = \frac{750}{7.65\text{x}10^{-3}} = 98.04\text{x}10^3 N/m = 98.04 \text{ kN/ms}$$

$$k_e = \frac{A_b}{A_P} \text{x } \mathrm{k_P}$$

$$\therefore \ k_e = \frac{(2*3)}{(\pi * 0.3^2/4)} \text{x } 98.04 = 8321.9 \text{ kN/m}$$

b) The new base dimensions

$$k_e = \frac{A_b}{A_P} \text{ x } k_P \Rightarrow \therefore A_b = \frac{k_e}{k_P} \text{ x } A_P$$

$$\therefore A_b = \frac{7200}{98.04} \text{ x } (\pi * 0.3^2 / 4) = 5.2 \text{ m}^2$$

$$A_b = L\text{x}2L = 2L^2 \Rightarrow \therefore L = (A_b / 2)^{0.5} = (5.2/2)^{0.5} = 1.611\, m$$

5) TYPES OF VIBRATIONS

There are two main types of vibration, free vibration and forced vibration:

5.1) Free vibration

In free vibration, the system is disturbed from its equilibrium position by a given initial energy and then vibrates under the action of its internal forces without any external dynamic force. Example of free vibration is the vibration of a mass-spring system when it is given an initial potential energy by displaced it from its equilibrium position and then released. Also, another example of the free vibration is the vibration of a simple pendulum if it is given an initial velocity (kinetic energy) in its equilibrium position, it will vibrate about its equilibrium position.

The system response ***x***(t) in free vibration is depending on the resistance of the system forces to movement (resistance or friction which dissipates the initial energy given to the system). There are two cases, undamped free vibration and damped free vibration.

Undamped Free Vibration

This is a theoretical case to introduce the mathematical treatment in a simple way. If no resistance or friction is considered, then the system will keep the given initial given energy without change (conservative system) and continue to vibrate due to the change between kinetic energy and potential energy, theoretically to an infinite time as shown in Fig. (4.6).

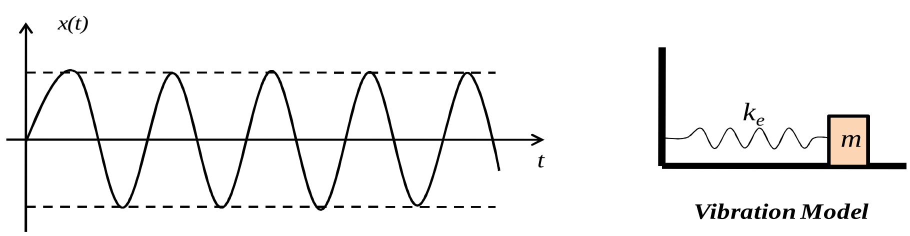


Fig. (4.6): Undamped Free vibration

Damped Free Vibration

This is the real case, where the resistance or friction of the system is usually acting on the system. The resistance or friction will do negative work on the system and cause dissipation of the given initial energy. The system will vibrate with decreasing amplitude until it rests again after a time depending on the value of the resistance of the system to movement as shown in Fig. (4.7).

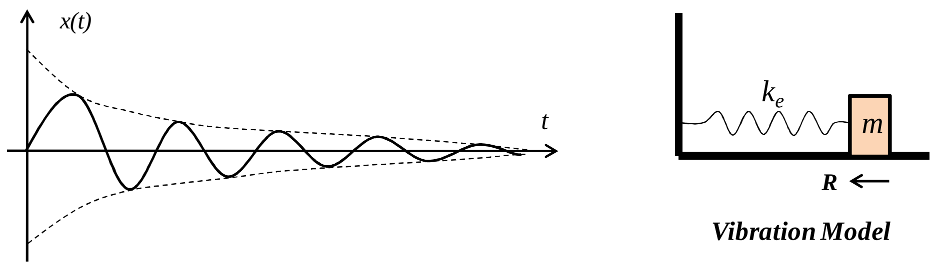


Fig. (4.7): Damped Free Vibration

5.2) Forced vibration

In the forced vibration, there are external dynamic force (excitation) acting on the system and forced it to vibrate. Similar to free vibration, there are two types of forced vibration based on the neglecting or considering the system resistance to movement.

Undamped Forced Vibration

Neglecting the system resistance to movement will simplify the mathematical treatment and produce approximate solution. In this case, the system will vibrate under the action of the external dynamic force (excitation) with constant amplitude as shown in Fig. (4.8).

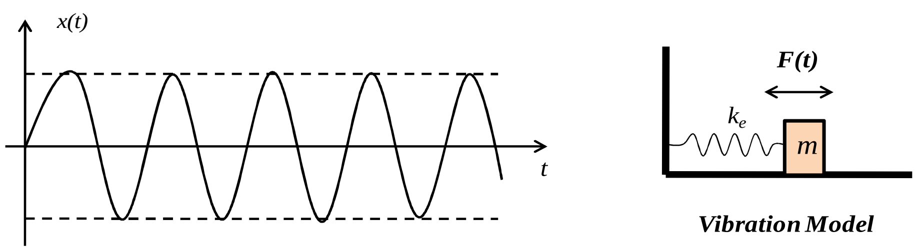


Fig. (4.8): Undamped Forced Vibration

Damped Forced Vibration

In real structures, when the system resistance to motion is considered the mathematical solution becomes more difficult but the obtained solution will be accurate. In this case, the system will vibrate with variable amplitude as shown in Fig. (4.9).

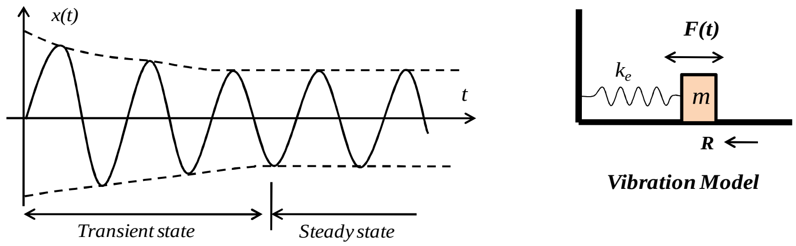


Fig. (4.9): Damped Forced Vibration

6) UNDAMPED FREE VIBRATIONS

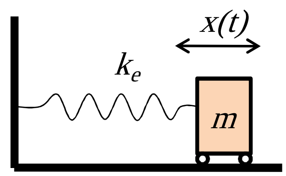


Fig. (4.10): Undamped Free Vibration

The undamped free vibration model is shown in Fig. (4.10). To study the system motion, assume the block is at a distance x from its equilibrium position , derive the equation of motion and solve it to obtain the response $x(t)$, the velocity $\dot{x}(t)$ and the acceleration $\ddot{x}(t)$ of the block.

Equations of motion:

Since the block has a rectilinear translational motion, then its equations of motion reduces to one equation in x-direction. Draw *E.F.S* and *I.F.S* as shown, the equation of motion is:

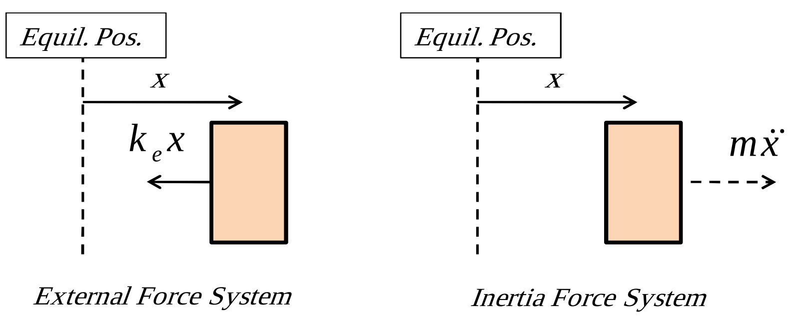


Fig. (4.11): Mechanical Model of Undamped Free Vibration

Equations of motion:

$$\rightarrow -k_e x = m\ddot{x} \tag{4.5}$$

Rearrange the terms and divide by m then;

$$\therefore \ddot{x} + \left(\frac{k_e}{m}\right)x = 0$$

let: $\omega_n^2 = \dfrac{k_e}{m}$ (ω_n is called the natural frequncy)

Substitute in the above equation, the equation of motion can be obtained in the standard form as:

$$\ddot{x} + \omega_n^2 x = 0 \qquad (4.6)$$

According to the theory of differential equations, Eq. (4.6) is a homogeneous linear differential equation (*D.E.*) of the second order and its general solution may be obtained by different methods.

The solution of a *D.E.* means obtaining a function $x(t)$ that satisfies the equation. By inspection of Eq. (4.6), one can think that the sine or cosine function may be a solution the equation, as these functions when differentiating twice give the same function.

Trying sin ($\omega_n t$) or cos ($\omega_n t$) by substituting it into Eq.(4.6), one can find that these functions satisfy the equation, then it may consider a solution. According to Theory of D.E's, the general solution is a superposition of the two functions, then:

$$x(t) = C_1 \sin(\omega_n t) + C_2 \cos(\omega_n t) \qquad (4.7)$$

where C_1 and C_2 are constants obtained from the initial conditions of the motion. The initial condition for the given problem are only two cases:

Case (1):

Displacing the block a distance *A m* from the equilibrium position and then released, then the initial conditions are: at t=0, x=A and v_O=0, substitute into Eq. (4.7):

$$A = C_1 \sin(0) + C_2 \cos(0) \;\Rightarrow \therefore C_2 = A \;\; m$$

Substitute again in Eq.(3):

$$\therefore \; x(t) = C_1 \sin(\omega_n t) + A \cos(\omega_n t) \qquad (4.8)$$

Differentiate Eq. (4.8) with respect to time (t) and substitute by the initial conditions:

$$\dot{x}(t) = C_1 \omega_n \cos(\omega_n t) - A\omega_n \sin(\omega_n t)$$

$$\text{At } \; t = 0 \,,\, \dot{x} = 0 \Rightarrow \therefore 0 = C_1 \omega_n \cos(0) - 0 \qquad (4.9)$$

$$\Rightarrow \; \therefore C_1 \omega_n = 0 \;\Rightarrow \because \omega_n \neq 0 \;\Rightarrow \therefore C_1 = 0$$

Substitute by constants C_1 and C_2 in Eq. (4.7), the displacement of the RB as a function of time (t), the velocity and the acceleration are obtained as:

$$x(t) = A \cos(\omega_n t) \qquad (4.10)$$

$$\dot{x}(t) = -A\omega_n \sin(\omega_n t) \qquad (4.11)$$

$$\ddot{x}(t) = -A\omega_n^2 \cos(\omega_n t) \qquad (4.12)$$

Case (2):

Giving the block an initial velocity v_O from its equilibrium position, then the initial conditions are ; at t=0, $x=0$ and $\dot{x} = v_O$, substitute in Eq. (4.7):

$$0 = C_1 \sin(0) + C_2 \cos(0) \Rightarrow \therefore C_2 = 0\ m$$

Substitute again into Eq.(4.7), the response of the block is obtained as:

$$\therefore\ x(t) = C_1 \sin(\omega_n t) \qquad (4.13)$$

Differentiate Eq. (4.13) with respect to time (t) and substitute by the initial conditions:

$$\dot{x}(t) = C_1\omega_n \cos(\omega_n t)$$

$$v_O = C_1\,\omega_n \cos(0) \Rightarrow \therefore C_1 = \frac{v_O}{\omega_n}$$

Substitute by constants C_1 in Eq. (4.13), the displacement of the RB as a function of time (t), the velocity and the acceleration are obtained as:

$$x(t) = \left(\frac{v_O}{\omega_n}\right)\sin(\omega_n t) \qquad (4.14)$$

$$\dot{x}(t) = -A\,\omega_n \cos(\omega_n t) \qquad (4.15)$$

$$\ddot{x}(t) = -A\,\omega_n^2 \sin(\omega_n t) \qquad (4.16)$$

The characteristics of the machine vibration are defined by:

The natural circular frequncy $\omega_n = \sqrt{\dfrac{k_e}{m}}$

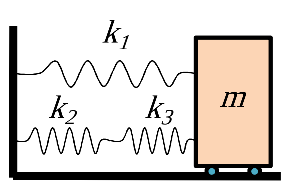
The vibration frequncy: $F = \dfrac{\omega_n}{2\pi} Hz$ (cicle/sec)

The periodical time $T = \dfrac{1}{F} = \dfrac{2\pi}{\omega_n}$ sec/cicle (4.17)

7) EXAMPLES OF UNDAMPED FREE VIBRATIONS

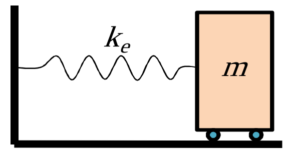
Example (2):

A machine (m=100 kg) supported by connecting it to a wall with three springs k_1= 12 kN/m, k_2= 6 kN/m and k_3= 4 kN/m as shown. The machine is mounted on two smooth rollers to eliminate the friction with the floor. If the machine is displaced by 10 cm to the right and then released, study the subsequent motion.



Real Problem

Solution:



Vibration Model

Since the machine moves only in the horizontal direction, it has one degree of freedom and its motion is totally described by x, $\dot{x}$ and $\ddot{x}$. The machine can be represented dynamically by a mass-spring system (vibration model) as shown, where:

$$k_e = k_1 + \frac{k_2 k_3}{k_2 + k_3} \Rightarrow \therefore k_e = 12000 + \frac{6000 * 4000}{6000 + 4000} = 12000 + 2400$$

$$= 14400 \;\; N/m = 14.4 \; kN/m$$

According to the above analysis, the equation of motion of the machine is given by:

$$\therefore \ddot{x} + 144\,x = 0$$

$$\therefore \;\; \omega_n^2 = \frac{14400}{100} = 144 \;\Rightarrow\; \therefore \omega_n = 12 \text{ sec}^{-1} \qquad (1)$$

<u>According to case (a), Eq. (4.10 - 4.13):</u>

The response of the machine, the velocity and the acceleration as functions of time are given by:

$$x(t) = A\cos(12t) = 0.1\cos(12t) \qquad (2)$$

$$\dot{x}(t) = -1.2\sin(12t) \qquad (3)$$

$$\ddot{x}(t) = -14.4\omega_n^2 \cos(12t) \qquad (4)$$

The characteristics of the machine vibration are defined by:

The natural circular frequncy ω_n = 12 sec^{-1}

The vibration frequncy: $F = \frac{\omega_n}{2\pi} = \frac{12}{2\pi} = 1.91\,Hz \;\; = 1.91\text{cicle/sec}$

The periodical time: $T = \frac{1}{F} = \frac{2\pi}{\omega_n} = 0.524 \;\; \text{sec/cicle}$

Also, the maximum displacement of the machine (vibration amplitude), the maximum velocity and the maximum acceleration are:

x_{max} (the amplitude) = 0.1 m;

$v_{max} = \dot{x}_{max}$ = 1.2 m/sec ;

$\ddot{x}_{max}$ = 14.4 m/sec^2