Chapter (6): Plane Motion of a Rigid Body Due to Impulsive Forces

1) INTRODUCTION

In this chapter, the plane motion of a rigid body due to impulsive forces is illustrated. First, the definitions of momentum impulse and impulsive forces are introduced. Then the principles of momentum and impulse are driven. A group of practical examples are solved to discuss how these principles are used to deal with problems of the plane motion of rigid body which subjected to impulsive forces or sudden huge blows.

2) DEFINITIONS

6.1) Linear Momentum:

Particle: The linear momentum $\underline{P}$ of a particle of mass m and velocity $\underline{v}$ is defined as:

$$\boxed{\underline{P} = m\underline{v}} \qquad (6.1)$$

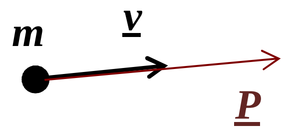


Fig.(6.1)

Rigid Body: The linear momentum $\underline{P}$ of a rigid body is defined as the sum of the linear momentum of its particles, i.e.;

$$\underline{P} = \sum m_i \underline{v}_i$$

From the definition of the mass center "G":

$$\underline{r}_G = \frac{\sum m_i \underline{r}_i}{\sum m_i} = \frac{\sum m_i \underline{r}_i}{m} \rightarrow \therefore\ m\underline{r}_G = \sum m_i \underline{r}_i$$

By differentiation w.r.to time:

$$\therefore\ m\underline{v}_G = \sum m_i \underline{v}_i$$

Sub. into above equation, then:

$$\boxed{\underline{P} = m\underline{v}_G} \qquad (6.2)$$

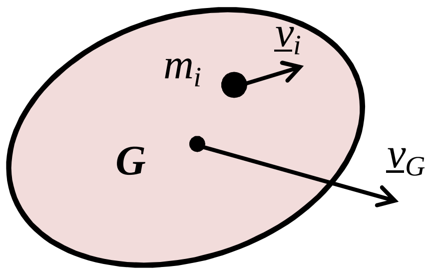


Fig.(6.2)

where m is the mass of the rigid body and $\underline{v}_G$ is the velocity of its mass-center G.

6.1) Angular Momentum:

Particle: The angular momentum of a particle of mass ***m*** and velocity $\underline{v}$ about a point *O* is defined as the moment of its linear momentum about that point, i.e.;

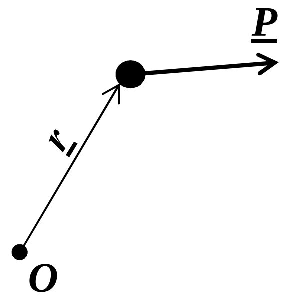


Fig.(6.3)

$$\underline{h}_O = \underline{r} \times \underline{P} = \underline{r} \times m\underline{v} \qquad (6.3)$$

Rigid Body: The angular momentum of a rigid body about a point *O* is defined as the sum of the angular momentum of its particles about *O*, i.e.;

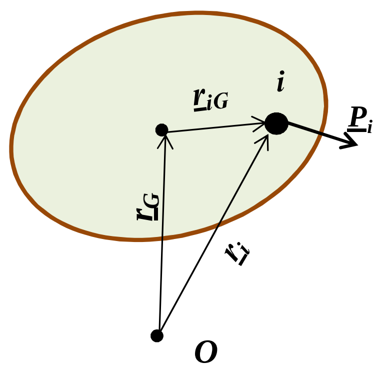


Fig.(6.4)

$$\underline{h}_O = \sum \underline{r}_i \times \underline{P}_i = \sum (\underline{r}_G + \underline{r}_{iG}) \times m_i \underline{v}_i$$

$$= \underline{r}_G \times \sum m_i \underline{v}_i + \sum \underline{r}_{iG} \times m(\underline{v}_G + \underline{v}_{iG})$$

$$= \underline{r}_G \times m\underline{v}_G + \left(\sum m_i \underline{r}_{iG}\right) \times \underline{v}_G + \sum m_i \underline{r}_{iG} \times \underline{v}_{iG}$$

$\because \sum m_i \underline{r}_{iG} = 0$; From the definitions of "*G*":

$|\underline{v}_{iG}| = \omega r_{iG}$ and $\perp to\ \underline{r}_{iG}$

The vectorial product of vectors is defined as:

$$\therefore \underline{r}_{iG} \times \underline{v}_{iG} = (r_{iG})(\omega r_{iG}) \sin(90)\, \underline{k}$$

$$\sum m_i \underline{r}_{iG} \times \underline{v}_{iG} = \sum m_i \omega r_{iG}^2\, \underline{k} = \left(\sum m_i r_{iG}^2\right) \omega \underline{k}$$

From the definitions of moment of inertai:

$$I_G = \sum m_i r_{iG}^2$$

$$\therefore \sum m_i \underline{r}_{iG} \times \underline{v}_{iG} = I_G\, \omega \underline{k}$$

$$\therefore\ \underline{h}_O = \underline{r}_G \times m\underline{v}_G + \omega I_G \underline{k} \qquad (6.4)$$

where ***m*** is the mass of the rigid body, $\underline{v}_G$ is the velocity of its mass-center *G*, I_G is the polar moment of inertia about *G*, ω is the angular velocity of the R.B. and $\underline{\boldsymbol{k}}$ is the unit vector perpendicular to the plane of motion in sense of ω.

Angular Momentum about G:

$$\therefore\ \underline{h}_G = \underline{r}_{GG} \times m\underline{v}_G + \omega I_G \underline{k} = \omega I_G \underline{k} \qquad (6.5)$$

Angular Momentum about I.C.:

$\therefore\ \underline{h}_{IC} = \underline{r}_{GC} \times m\underline{v}_G + \omega I_G \underline{k}$

$|\underline{v}_G| = \omega r_{GC}$ and $\perp \underline{r}_{GC}$, $\therefore \underline{r}_{GC} \times m\underline{v}_G = \omega m r_{GC}^2 \underline{k}$, then:

$\underline{h}_{IC} = (m r_{GC}^2 + I_G)\,\omega \underline{k} = I_{IC}\omega \underline{k}$

$$\boxed{\underline{h}_{IC} = (m r_{GC}^2 + I_G)\,\omega \underline{k} = I_{IC}\omega \underline{k}} \qquad (6.6)$$

Important Notes:

1. The center of gravity of a rigid body "G" is defined as:

$$\underline{r}_G = \frac{\sum m_i \underline{r}_i}{\sum m_i} = \frac{\sum m_i \underline{r}_i}{m} \ \rightarrow \therefore\ \underline{r}_G \sum m_i = \sum m_i \underline{r}_i \ \rightarrow \therefore\ \sum m_i(\underline{r}_i - \underline{r}_G) = 0$$

$$\therefore \sum m_i \underline{r}_{iG} = 0; \quad \sum m_i \underline{v}_{iG} = 0 \quad \text{and} \quad \sum m_i \underline{a}_{iG} = 0$$

2. The vectorial product of two vectors is given by:

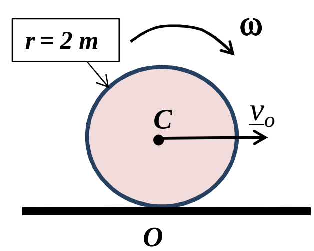
$\underline{c} = \underline{a} \times \underline{b} = ab\sin\theta\ \underline{\hat{n}}$, where $\underline{\hat{n}}$ is a unit vector $\perp$ to the plane of $\underline{a}$ and $\underline{b}$

then: $\underline{r}_{iG} \times \underline{v}_{iG} = |\underline{r}_{iG}||\underline{v}_{iG}|\sin(90)\underline{k} = (r_{iG})(\omega r_{iG})\underline{k} = \omega r_{iG}^2 \underline{k}$

3. The angular momentum is a vectorial quantity in the direction perpendicular to the plane of motion. As its direction is always known, then it may be considered as scalar quantity (as the moment of a force) and its direction is defined by c.w. or c.c.w.

EXAMPLE (6.1):

For the shown disc (m=20 kg, r=2m) which rolls on a horizontal surface with ω=2 rad/sec (c.w.), calculate the linear momentum and angular momentum about *O*.



SOLUTION:

The disc moves with a general plane motion with *I.C.* at *O*.

Linear momentum:

$\underline{P} = m\underline{v}_G = m r \omega \underline{i} = (20)(2)(2)\underline{i} = 80\underline{i} \quad kg.m$

Angular momentum about "*O*":

As "*O*" is the instantaneous center of the disc, then:

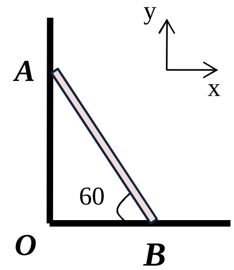
$\underline{h}_O = I_O \omega \underline{k}$ or its magnitude is: $h_O = \omega I_O$ (*c.w.*)

$$h_O = \omega I_O = \omega(\frac{1}{2}mr^2 + mr^2) = \frac{3}{2}mr^2\omega$$

$$= \frac{3}{2}(20)(2)^2(2) = 240 \;\; kg.m^2 \;\; (c.w.)$$

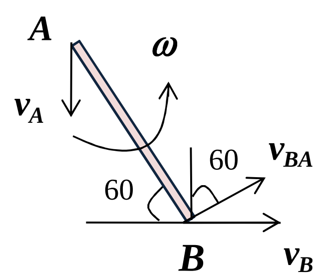
EXAMPLE (6.2):

The shown rod (m=30 kg, L=4m) moves in a vertical plane such that ***A*** slides on a vertical surface while ***B*** moves on a horizontal surface. In the shown instant, the velocity of ***B*** is 3 m/sec to the right. Calculate the rod linear momentum and angular momentum about *O* in the shown instant.



SOLUTION:

The rod moves with a general plane motion, the velocity relation is:



The velocity of *G*:

$\underline{v}_B = \underline{v}_A + \underline{v}_{BA}$,

where :

$\underline{v}_B = 3\underline{i}$, $\underline{v}_A = -v_A \underline{j}$

$\underline{v}_{BA} = \omega(AB)\left(\sin(60)\underline{i} + \cos(60)\underline{j}\right)$

$3\underline{i} = -v_A \underline{j} + \omega(4)\left(\sin(60)\underline{i} + \cos(60)\underline{j}\right)$

$\underline{x - comp.}$: $3 = \omega(4)\sin(60)$

$\therefore \omega = 0.866 \, rad/\sec \; (c.c.w)$

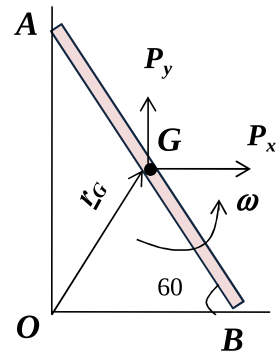
$\underline{y - comp.}$: $0 = -v_A + (0.866)(4)\cos(60)$

$\therefore v_A = 1.732 \;\; m/\sec \; (\downarrow)$

$\underline{v}_B = 3\underline{i}$, $\underline{v}_A = -1.732\,\underline{j} \;\; \rightarrow \underline{v}_G = \left(\dfrac{\underline{v}_A + \underline{v}_B}{2}\right)$

$\therefore \underline{v}_G = 1.5\underline{i} - 0.866\underline{j}$

Linear momentum:



$$\underline{P} = m\underline{v}_G$$

$\therefore\ \underline{P} = 30(1.5\underline{i} - 0.866\underline{j}) = 45\underline{i} - 25.98\underline{j}$

Angular momentum about "*O*":

The angular momentum of the ***AB*** about ***O*** is given by:

$$\therefore\ \underline{h}_O = \underline{r}_G \times m\underline{v}_G + \omega I_G \underline{k}$$

$$\underline{h}_O = \left(\frac{L}{2}\cos(60)\underline{i} + \frac{L}{2}\sin(60)\underline{j}\right) \times (45\underline{i} - 25.98\underline{j}) + 0.866\left(\frac{30*4^2}{12}\right)(\underline{k})$$

$$= -25.98\underline{k} - 77.94\underline{k} + 34.64 = -69.28\underline{k} \quad (-\underline{k} \text{ means c.w.})$$

Another Solution:

The moment of the linear momentum equals the moment of its components about *O*, in addition to the angular momentum of ω-term, then the total angular momentum about *O* in the *c.w.* direction is given by:

$$h_O = -P_x\left(\frac{L}{2}\sin(60)\right) + P_y\left(\frac{L}{2}\cos(60)\right) + 0.866\left(\frac{30*4^2}{12}\right)$$

$$= -45(2)\sin(60) + (-25.98)(2)\cos(60) + 34.64 = -69.28 \ kg.m^2/\sec \ (c.c.w.)$$

3. Linear Impulse:

The linear impulse $\underline{\boldsymbol{S}}_{12}$ of a force $\underline{\boldsymbol{F}}$ over a time interval $[t_1, t_2]$ is defined by:

$$\underline{S}_{12} = \int_{t_1}^{t_2} \underline{F}\ dt \qquad (6.7)$$

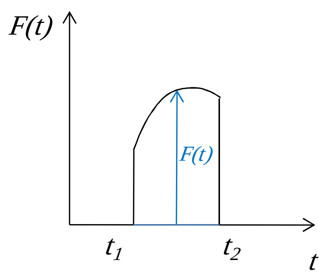


Fig.(6.5)

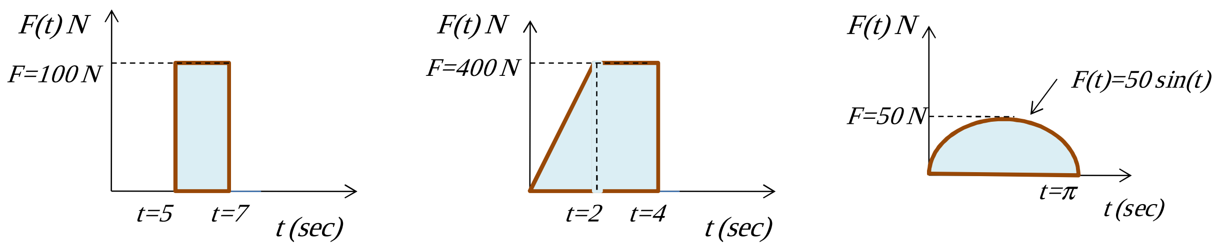
Linear impulse is a vector whose direction is the direction of the force $\underline{\boldsymbol{F}}$, and its magnitude is represented by the area under the $\underline{\boldsymbol{F}}$-*t* curve between t_1, and t_2. The total (resultant) linear impulse $\underline{\boldsymbol{S}}_{12}$ of a set of forces is given as:

$$\underline{S}_{12} = \int_{t_1}^{t_2} \underline{F}_i\ dt = \int_{t_1}^{t_2} \underline{R}\ dt \qquad (6.8)$$

where: $\underline{R} = \sum \underline{F}_i$

EXAMPLE (6.3):

Calculate the magnitude of the linear impulse $\underline{\mathbf{S}}_{12}$ for the following forces:



(a) (b) (c)

SOLUTION:

The magnitude of linear impulse is given by the area under $\underline{\mathbf{F}}$-t curve, then:

Case (a):

$$\left|\underline{S}_{12}\right| = \int_{t_1}^{t_2} F\ dt = \int_5^7 (100)\ dt = 100t\Big|_5^7 = 100(7-5) = 200\ N.sec$$

Case (b):

$$\left|\underline{S}_{12}\right| = \int_{t_1}^{t_2} F\ dt = \int_0^2 200t\ dt + \int_2^4 400\ dt = \frac{1}{2}(200t^2)\Big|_0^2 + 400t\Big|_2^4$$

$$= 400 + (1600 - 800) = 1200\ N.sec$$

Case (c):

$$\left|\underline{S}_{12}\right| = \int_{t_1}^{t_2} F(t)\ dt = \int_0^{\pi} 50 sin(t)\ dt = -50 cos(t)\Big|_0^{\pi} = -50(-1-1) = 100 N.sec$$

4. Angular Impulse:

The angular impulse of a set of forces $F_1, F_2, \ldots F_N$ about a point O over the time interval $[t_1, t_2]$ is defined as:

$$\underline{M}_{12}^{O} = \int_{t_1}^{t_2} \sum_{i=1}^{i=n} \underline{r}_i \times \underline{F}_i\ dt \qquad (6.9)$$

3) THE IMPULSE - MOMENTUM PRINCIPLES:

The principle of Impulse and momentum, which is the integral of equations of motion with respect to time.

3.1) Principle of Impulse and momentum for a particle:

Consider a particle of mass m acted upon by a force $\underline{F}$, Newton's second law may be expressed in the form

$$\underline{F} = \frac{d}{dt}(m\underline{v})$$

Integrating with respect to time from $t = t_1$ to time $t = t_2$:

$$\int_{t_1}^{t_2} \underline{F} dt = \int_{t_1}^{t_2} d(m\underline{v})$$

$$\boxed{\therefore \underline{S}_{12} = m\left[\underline{v}(t_2) - \underline{v}(t_1)\right]} \qquad (6.10)$$

That is:

The linear impulse acting on a particle equals the change in its linear momentum.

3.2) Principle of Impulse and momentum for a Rigid Body:

A) Linear impulse – Linear momentum principle:

The linear momentum of a rigid body is defined as:

$$\underline{P} = m\underline{v}_G$$

The equation of linear motion of a rigid body is given as:

$$\underline{R} = \sum \underline{F} = \frac{d\underline{P}}{dt} = \frac{d}{dt} m\underline{v}_G$$

Integrating with respect to time from $t = t_1$ to time $t = t_2$, then:

$$\int_{t_1}^{t_2} \underline{R} dt = \int_{t_1}^{t_2} d(m\underline{v}_G)$$

$$\boxed{\therefore \underline{S}_{12} = m\left[\underline{v}_G(t_2) - \underline{v}_G(t_1)\right]} \qquad (6.11)$$

That is:

The total linear impulse acting on a rigid body over the interval [t_1, t_2] equals the change in its linear momentum.

B) Angular impulse – Angular momentum principle:

The angular momentum of a rigid body about a point *O* is defined as:

$$\underline{h}_O = \sum \underline{r}_{iO} \times \underline{P}_i = \sum \underline{r}_{iO} \times m_i \underline{v}_i$$

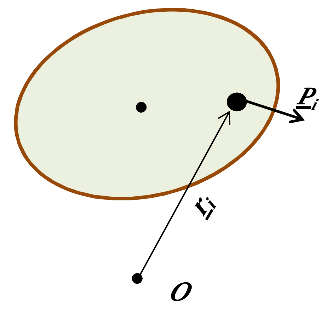


Fig.(6.6)

By differentiation with respect to time:

$$\begin{aligned} \frac{d\,\underline{h}_O}{dt} &= \sum \frac{\underline{r}_{iO}}{dt} \times m_i \underline{v}_i + \sum \underline{r}_{iO} \times m_i \frac{\underline{v}_i}{dt} \\ &= \sum \underline{v}_{iO} \times m_i \underline{v}_i + \sum \underline{r}_{iO} \times m_i \underline{a}_i \\ &= \sum (\underline{v}_i - \underline{v}_O) \times m_i \underline{v}_i + \sum \underline{r}_{iO} \times m_i \underline{a}_i \\ &= \sum \underline{v}_i \times m_i \underline{v}_i - \sum \underline{v}_O \times m_i \underline{v}_i + \sum \underline{r}_{iO} \times m_i \underline{a}_i \\ &= \sum \underline{v}_i \times m_i \underline{v}_i - \underline{v}_O \times \sum m_i \underline{v}_i + \sum \underline{r}_{iO} \times m_i \underline{a}_i \end{aligned}$$

$\because \ \sum \underline{v}_i \times m_i \underline{v}_i = 0$; $\sum m_i \underline{v}_i = m\underline{v}_G$; and $m_i \underline{a}_i = \underline{F}_i$;

$$\therefore \frac{d\,\underline{h}_O}{dt} = 0 + \underline{v}_O \times m\underline{v}_G + \sum \underline{r}_{iO} \times \underline{F}_i$$

$$= \underline{v}_O \times m\underline{v}_G + \sum \underline{r}_{iO} \times \underline{F}_i$$

$$\therefore d\,\underline{h}_O = \left(\underline{v}_O \times m\underline{v}_G + \sum \underline{r}_{iO} \times \underline{F}_i \right) dt$$

By integration the two sides over the interval [t_1,t_2]::

$$\int_{t_1}^{t_2} d\,\underline{h}_O = -\int_{t_1}^{t_2} (\underline{v}_O \times m\underline{v}_G)\,dt + \int_{t_1}^{t_2} \left(\sum \underline{r}_{iO} \times \underline{F}_i \right) dt$$

$$\therefore \underline{h}_O(t_2) - \underline{h}_O(t_1) = \underline{M}_{12}^{O} - \int_{t_1}^{t_2} (\underline{v}_O \times \underline{P})\,dt$$

$$\boxed{\therefore \underline{M}_{12}^{O} = \underline{h}_O(t_2) - \underline{h}_O(t_1) + \int_{t_1}^{t_2} \underline{v}_O \times \underline{P}\,dt} \qquad (6.11)$$

Important Notes:

1) Fixed Point: If "*O*" is a fixed point, $\underline{\mathbf{v}}_O$=0, then the principle of angular impulse - angular momentum about "O" may be expressed as:

$$\underline{M}_{12}^{O} = \underline{h}_O(t_2) - \underline{h}_O(t_1) + \int_{t_1}^{t_2} \underline{0} \times \underline{P} dt = \underline{h}_O(t_2) - \underline{h}_O(t_1)$$

2) Center of Gravity G: By using Eq. (6.11) about center of gravity "*G*", the principle of angular impulse - angular momentum may be expressed as:

$$\underline{M}_{12}^{G} = \underline{h}_G(t_2) - \underline{h}_G(t_1) + \int_{t_1}^{t_2} \underline{v}_G \times \underline{P} dt \; ; \quad \because \int_{t_1}^{t_2} \underline{v}_G \times \underline{P} dt = \int_{t_1}^{t_2} \underline{v}_G \times m\underline{v}_G dt = 0$$

$$\boxed{\therefore \underline{M}_{12}^{G} = \underline{h}_G(t_2) - \underline{h}_G(t_1) = I_G(\omega_2 - \omega_1)(\pm \underline{k})}$$

That is:

The angular impulse about the center of mass G or about a fixed point O over interval [t_1, t_2] is equal to the change of its angular momentum about G or about O respectively

3.3) Summary "Equations of Momentum-Impulse":

Three equations of motion may be written to study the rigid body motion, two equations obtained by summing and equating the *x* and *y* components of the acting forces impulses and linear momentum, and the third by summing and equating the angular impulses and angular momentum of the rigid body about its mass center G (or equivalently about a fixed point O).

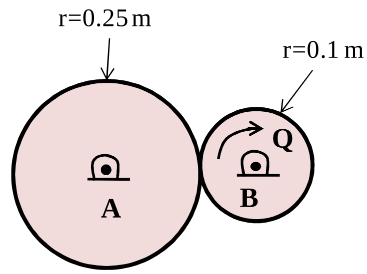
$$S_{12}\big|_x = m\left[v_{Gx}(t_2) - v_{Gx}(t_1)\right] \quad (1)$$

$$S_{12}\big|_y = m\left[v_{Gy}(t_2) - v_{Gy}(t_1)\right] \quad (2)$$

$$M_{12}^{G} = I_G(\omega_2 - \omega_1) \quad (3)$$

Using these equations, the velocity of the mass center of the rigid body and its angular velocity can be calculated at any time, if the forces acting on the rigid body are known.

EXAMPLE (6.4):



In the shown figure, two gears ***A*** (m=10 kg, r=0.25m, k_A=0.2 m) and ***B*** (m=3 kg, r=0.1m, k_A=0.08 m) starting motion from rest under the action of a couple ***Q*** = 6 N.m. Calculate:

a) the time required for ω_B= 600 r.p.m

b) the number of rotation of the gear ***A*** during that time.

SOLUTION:

Kinematical condition:

When the gear ***B*** moves it will cause the gear ***A*** to move at a certain velocity depends on the gear ***B*** velocity. In that case, the two gears are called kinematically dependent. In other words, the velocity of the gear ***A*** may be obtained in terms of the velocity of the gear ***B*** as follow:

The velocity of point of contact is:

$$v_C = \omega_A r_A = \omega_B r_{\mathbf{B}} \qquad (1)$$

Differentiation and integration of Eq. (1) with respect to time, relations between accelerations and rotations of the two gears can be obtained as follow:

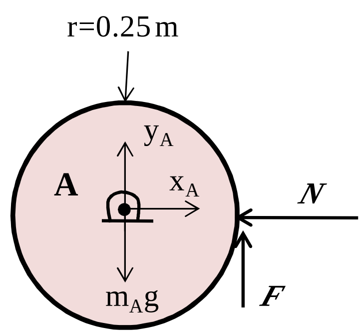
$$\alpha_A r_A = \alpha_B r_{\mathbf{B}} \qquad (2)$$

$$\theta_A r_A = \theta_B r_{\mathbf{B}} \qquad (3)$$

Equations (1), (2) and (3) are called the kinematical conditions relates the motion components of gear ***A*** to the motion components of the gear ***B***.

Momentum - Impulse Equation for gear A:

The angular momentum-impulse equation (c.c.w) :



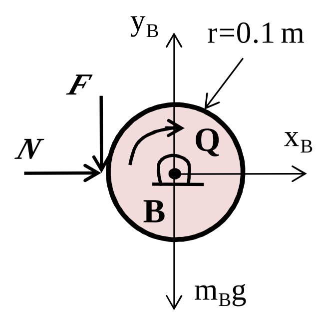
$$M_{12}^{G} = I_G\left(\omega_2 - \omega_1\right)$$

$$\int_{t_1}^{t_2} F(0.25)dt = k_A^2 m_A\left(\omega_A - 0\right) = k_A^2 m_A \omega_A$$

$$\int_{0}^{t_2} Fdt = \frac{(0.2)^2(10)}{0.25}\omega_A = 1.6\omega_A \qquad (4)$$

Momentum - Impulse Equation for gear B:

The angular momentum-impulse equation (*c.w.*):



$$M_{12}^{G} = I_G\left(\omega_2 - \omega_1\right)$$

$$\int_{t_1}^{t_2}\left(Q - F(0.1)\right)dt = k_B^2 m_B\left(\omega_B - 0\right) = k_B^2 m_B \omega_B$$

$$\int_{0}^{t_2}\left(Q - F(0.1)\right)dt = (0.08)^2(3)\omega_B = 0.0192\,\omega_B$$

$$\therefore 6t - 0.1\int_0^{t_2} F dt = 0.0192\,\omega_B \qquad (5)$$

Equations Solving:

Sub. Eq. (4) into Eq. (5):

$$\therefore 6t - 0.1(1.6\omega_A) = 0.0192\,\omega_B \qquad (6)$$

From the given data:

$$\omega_B = 600 r.p.m = \left(\frac{600*2*\pi}{60}\right) = 62.832\ rad/\sec$$

From the kinematical condition:

$$\omega_A = \frac{0.1\omega_B}{0.25} = \frac{0.1(62.832)}{0.25} = 25.133$$

Sub. into Eq. (6), the time is obtained as:

$$\therefore t = \frac{0.0192\,\omega_B + 0.1(1.6\omega_A)}{6} = 0.871\sec$$

The rotations of gear ***A***:

As the gears moves under the action of constant couple, then the accelerations are constant, and the laws of motion with constant accelerations may be used:

$$\omega_A\big|_t = \omega_A\big|_{t=0} + \alpha_A t \;\rightarrow\; \therefore \alpha_A = \frac{\omega_A - 0}{t} = \frac{25.133}{0.871} = 28.85\ rad/\sec^2$$

$$\theta_A\big|_t = \theta_A\big|_{t=0} + \frac{1}{2}\alpha_A t^2 \;\rightarrow\; \therefore \theta_A = 0 + \frac{1}{2}(28.85)(0.871) = 12.563\ rad = 2 \text{ revolutions.}$$