Duality of LP

Lecture 4

Introduction

- One of the most important discoveries in the early development of linear programming was the concept of duality.
- This discovery revealed that every linear programming problem has associated with it another linear programming problem called the **dual**.
- The relationships between the dual problem and the original problem (called the **primal**) prove to be extremely useful in a variety of ways.

Duality problem

Primal Problem	*Dual Problem*
Maximize $Z = \sum_{j=1}^{n} c_j x_j,$	Minimize $W = \sum_{i=1}^{m} b_i y_i,$
subject to	subject to
$\sum_{j=1}^{n} a_{ij} x_j \leq b_i,$ for $i = 1, 2, \ldots, m$	$\sum_{i=1}^{m} a_{ij} y_i \geq c_j,$ for $j = 1, 2, \ldots, n$
and	and
$x_j \geq 0,$ for $j = 1, 2, \ldots, n.$	$y_i \geq 0,$ for $i = 1, 2, \ldots, m.$

1. The coefficients in the objective function of the primal problem are the right-hand sides of the functional constraints in the dual problem.
2. The right-hand sides of the functional constraints in the primal problem are the coefficients in the objective function of the dual problem.
3. The coefficients of a variable in the functional constraints of the primal problem are the coefficients in a functional constraint of the dual problem.

Dual of the dual problem

Dual Problem

Minimize $W = \mathbf{yb}$,

subject to

$$\mathbf{yA} \geq \mathbf{c}$$

and

$$\mathbf{y} \geq \mathbf{0}.$$

→

Converted to Standard Form

Maximize $(-W) = -\mathbf{yb}$,

subject to

$$-\mathbf{yA} \leq -\mathbf{c}$$

and

$$\mathbf{y} \geq \mathbf{0}.$$

↓

Its Dual Problem

Minimize $(-Z) = -\mathbf{cx}$,

subject to

$$-\mathbf{Ax} \geq -\mathbf{b}$$

and

$$\mathbf{x} \geq \mathbf{0}.$$

←

Converted to Standard Form

Maximize $Z = \mathbf{cx}$,

subject to

$$\mathbf{Ax} \leq \mathbf{b}$$

and

$$\mathbf{x} \geq \mathbf{0}.$$

Duality properties

- **Weak duality property**: If x is a feasible solution for the primal problem and y is a feasible solution for the dual problem then

$$\boldsymbol{cx \leq yb}$$

- **Strong duality property**: If x* is an optimal solution for the primal problem and y* is an optimal solution for the dual problem, then

$$\boldsymbol{cx = yb}$$

- Thus, these two properties imply that $\boldsymbol{cx < yb}$ for feasible solutions if one or both of them are not optimal for their respective problems, whereas equality holds when both are optimal.

Duality properties

- **Complementary solutions property:** At each iteration, the simplex method simultaneously identifies a CPF solution x for the primal problem and a complementary solution y for the dual problem, where

$$\boldsymbol{cx} = \boldsymbol{yb}$$

- **Complementary optimal solutions property:** At the final iteration, the simplex method simultaneously identifies an optimal solution x* for the primal problem and a complementary optimal solution y* for the dual problem

$$\boldsymbol{cx}^* = \boldsymbol{y}^*\boldsymbol{b}$$

Duality properties

- **Symmetry property:** For any primal problem and its dual problem, all relationships between them must be symmetric because the dual of this dual problem is this primal problem.