Duality of LP

Lecture 4

Introduction

- One of the most important discoveries in the early development of linear programming was the concept of duality.
- This discovery revealed that every linear programming problem has associated with it another linear programming problem called the **dual**.
- The relationships between the dual problem and the original problem (called the **primal**) prove to be extremely useful in a variety of ways.

Duality problem

Primal Problem

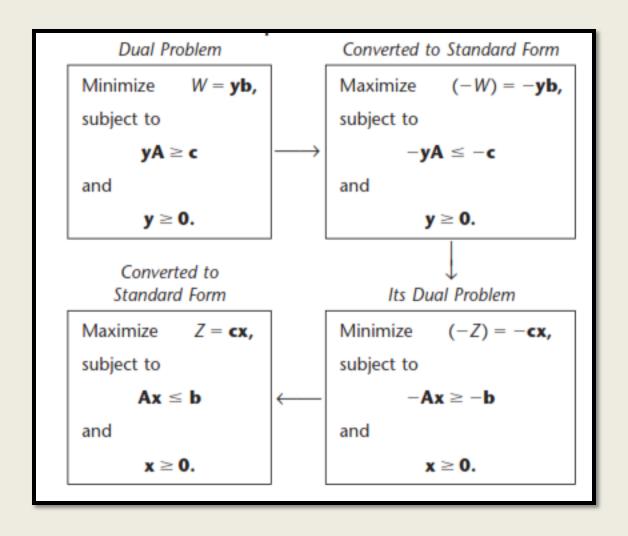
Maximize
$$Z = \sum_{j=1}^{n} c_j x_j$$
, subject to
$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad \text{for } i = 1, 2, \dots, m$$
 and
$$x_j \ge 0, \quad \text{for } j = 1, 2, \dots, n.$$

Dual Problem

Minimize
$$W = \sum_{i=1}^{m} b_i y_i$$
, subject to
$$\sum_{i=1}^{m} a_{ij} y_i \ge c_j, \quad \text{for } j = 1, 2, \dots, n$$
 and
$$y_i \ge 0, \quad \text{for } i = 1, 2, \dots, m.$$

- 1. The coefficients in the objective function of the primal problem are the right-hand sides of the functional constraints in the dual problem.
- 2. The right-hand sides of the functional constraints in the primal problem are the coefficients in the objective function of the dual problem.
- 3. The coefficients of a variable in the functional constraints of the primal problem are the coefficients in a functional constraint of the dual problem.

Dual of the dual problem



Duality properties

• Weak duality property: If x is a feasible solution for the primal problem and y is a feasible solution for the dual problem then

$$cx \leq yb$$

• **Strong duality property**: If x* is an optimal solution for the primal problem and y* is an optimal solution for the dual problem, then

$$cx = yb$$

• Thus, these two properties imply that cx < yb for feasible solutions if one or both of them are not optimal for their respective problems, whereas equality holds when both are optimal.

Duality properties

• Complementary solutions property: At each iteration, the simplex method simultaneously identifies a CPF solution x for the primal problem and a complementary solution y for the dual problem, where

$$cx = yb$$

• Complementary optimal solutions property: At the final iteration, the simplex method simultaneously identifies an optimal solution x* for the primal problem and a complementary optimal solution y* for the dual problem

$$cx^* = v^*b$$

Duality properties

• Symmetry property: For any primal problem and its dual problem, all relationships between them must be symmetric because the dual of this dual problem is this primal problem.