The Simplex method

Lecture 3

Shadow prices

- linear programming problems often can be interpreted as allocating resources to activities.

- b_i values used in the initial model may represent management's tentative initial decision on how much of the organization's resources will be provided to the activities in the model.

Shadow prices

- Some of the b_i values can be increased in a revised model, but only if a sufficiently strong case can be made to management that this revision would be beneficial.
- Information on the economic contribution of the resources to the measure of performance (Z) for the current study often would be extremely useful.
- The simplex method provides this information in the form of **shadow prices** for the respective resources.

Shadow prices

The shadow price for resource i denoted by (y^*) measures the marginal value of this resource, i.e., the rate at which Z could be increased by (slightly) increasing the amount of this resource (b_i) being made available.

Sensitivity analysis

- A main purpose of sensitivity analysis is to identify the sensitive parameters (i.e., those that cannot be changed without changing the optimal solution).
- The sensitive parameters are the parameters that need to be estimated with special care to minimize the risk of obtaining an erroneous optimal solution.

How the sensitive parameter is identified?

- If $y^* > 0$, then the optimal solution changes if the b_i is changed, so b_i is sensitive parameter.
- If $y^* = 0$, implies that the optimal solution is not sensitive to at least small changes in b_i

Matrix form of the LP problem

- Last lecture we worked the basic mechanics of the simplex method using the tabular form.
- We then describe the matrix form of the simplex method

Matrix form of the LP problem

- Using matrices, our standard form for the general linear programming model is

Maximize $Z = \mathbf{cx}$,

subject to

$\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$,

- Where **c** is a row vector $c = [c_1, c_2, \ldots, c_n]$

Matrix form of the LP problem

- The **x**, **b**, and **0** are column vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \qquad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

- The **A** matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Matrix form of the LP problem

- To get the augmented form of the optimization problem, a column of slack variables is included

$$\mathbf{x}_s = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}$$

$$[\mathbf{A}, \mathbf{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b} \quad \text{and} \quad \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} \geq \mathbf{0},$$

Matrix form of the LP problem

- Setting the non-basic variables to be equal to zero

$$\mathbf{B}\mathbf{x}_B = \mathbf{b},$$

where the **vector of basic variables**

$$\mathbf{x}_B = \begin{bmatrix} x_{B1} \\ x_{B2} \\ \vdots \\ x_{Bm} \end{bmatrix}$$

- The simplex method introduces the basic variables such that B is non-singular

$$\mathbf{B}^{-1}\mathbf{B}\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}.$$

Matrix form of the LP problem

- Since $\mathrm{B}^{-1}\mathrm{B} = \mathrm{I}$, the solution for the basic variables

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}.$$

- The value of the objective function

$$Z = \mathbf{c}_B\mathbf{x}_B = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}.$$

- Let c is the vector whose elements are the coefficients of the objective function (with zeros of the slack variables) for the corresponding elements of the X_B